APPROXIMATIONS TO THE POWER OF RANK TESTS

By Chia Kuei Tsao

Wayne State University

Summary. Proposed is a method for approximating the distribution of the ranks, which is the basis for evaluating the power of an arbitrary rank test (see definition of "rank test" in Section 2 below). The method involves, in essence, a transformation of the original distributions, by means of interpolating polynomials, into distributions defined on the unit interval (0, 1). A somewhat detailed discussion is given to the problem of testing the hypothesis that two populations are identical against the alternative hypothesis that they have two specified (non-identical) distributions. Explicit formulas for approximating the distributions of the ranks under the alternative hypothesis are given. A few tables are computed for the case where both distributions are normal with the same variance but different means.

The last section is devoted to the investigation of the asymptotic power efficiency of certain rank tests.

1. Introduction. A number of rank tests [8] have been proposed for testing the hypothesis that several populations are identical. The power of such (and other) rank tests is determined by the distribution $P(R)$ of the ranks under the alternative hypothesis. In [5] Hoeffding gives a simple formula for $P(R)$ for an arbitrary alternative. However, the difficulty in evaluating this formula makes it hard to obtain the exact power results, except in special cases. Fortunately, this difficulty has been partially overcome by other means. As examples, we mention here a few known results. Terry [10] investigated empirically the power of Hoeffding's $c_1(R)$ test [5] against normal alternatives for the two-sample case. Lehmann [6] investigated the power of several two-sample rank tests against some non-parametric alternatives. Dixon [1] obtained, by numerical methods, some power curves of several two-sample rank tests for normal alternatives. Recently, Teichroew [9] obtained a few empirical power curves for the same alternatives.

We shall in this paper investigate the power of an arbitrary rank test (see definition of "rank test" in Section 2) against an arbitrary alternative hypothesis. Since no method has yet been found for evaluating analytically the distribution $P(R)$ of the ranks, we shall here propose an approximation procedure which, as we shall see, appears to be quite satisfactory in certain cases. The computations are carried through for a few examples. However, in order to make practical uses of the method, much more systematic computation is required.

For convenience we shall here make the assumption, to hold throughout this paper, that all distribution functions that we consider are to be continuous. Furthermore,

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all definitions and notations given in the following sections will remain the same throughout the discussion.

2. Rank tests and power of rank tests. The term "rank test" is used here in a rather broad sense. For testing the hypothesis that several populations are identical, a test will be called a "rank test," if it is based entirely on the ranks of the random variables. The following consideration will illustrate the point.

Let

$$F_0, \cdots, F_k \tag{2.1}$$

be $k + 1$ continuous univariate cumulative distribution functions (cdf's) defined over the infinite interval $(-\infty, \infty)$ (or over a finite or half-infinite interval (a, b)). Let

$$Z = (Z_0, \cdots, Z_k) = (Z_{01}, \cdots, Z_{0m_0}, \cdots, Z_{k1}, \cdots, Z_{km_k}), \tag{2.2}$$

where

$$Z_i = (Z_{i1}, \cdots, Z_{im_i}) \tag{2.3}$$

are the ordered values of m_i independent random variables distributed according to $F_i(z)$, $i = 0, \cdots, k$; that is, for each $i (i = 0, \cdots, k)$, we have

$$Z_{i1} < \cdots < Z_{im_i}. \tag{2.4}$$

Let

$$\theta = (\theta_0, \cdots, \theta_k) = (\theta_{01}, \cdots, \theta_{0m_0}, \cdots, \theta_{k1}, \cdots, \theta_{km_k}), \tag{2.5}$$

where

$$\theta_i = (\theta_{i1}, \cdots, \theta_{im_i}) \tag{2.6}$$

are the ranks of the m_i variables Z_i in Z; obviously, for each i $(i = 0, \cdots, k)$, we always have

$$\theta_{i1} < \cdots < \theta_{im_i}. \tag{2.7}$$

Let

$$R = (r_{01}, \cdots, r_{0m_0}, \cdots, r_{k1}, \cdots, r_{km_k}) \tag{2.8}$$

be a permutation of the first $M = m_0 + \cdots + m_k$ positive integers $(1, \cdots, M)$ such that for each i $(i = 0, \cdots, k)$, we have

$$r_{i1} < \cdots < r_{im_i}. \tag{2.9}$$

It is evident that there are $\eta = M!/\prod m_i!$ permutations R. Let Ω denote the class of all possible sets ω of such permutations R. Then, θ is a random variable over Ω.

Now, suppose the hypothesis

$$H_0: F_0 = \cdots = F_k \tag{2.10}$$

is to be tested against the alternative hypothesis

$$H_1 : F_i = F_i^*, \qquad i = 0, \cdots, k, \tag{2.11}$$

where $F_0^*, \cdots, F_k^*$ are $k + 1$ given cdf's. We shall denote by $h(R; H_i)$ the probability distribution function (pdf) of θ under the hypothesis H_i; that is,

$$h(R; H_i) = \Pr(\theta = R \mid H_i), \qquad i = 0, 1. \tag{2.12}$$

Let us denote the η permutations R by $R_1, \cdots, R_\eta$ in such a way that

$$h(R_{i+1}; H_1) \leqq h(R_i; H_1), \qquad i = 1, \cdots, \eta - 1. \tag{2.13}$$

Let $a_1, \cdots, a_t$ be t positive integers such that

$$a_1 + \cdots + a_t = \eta. \tag{2.14}$$

Let

$$\begin{aligned} S_1 &= R_1 \cup \cdots \cup R_{a_1}, \\ S_2 &= R_{a_1+1} \cup \cdots \cup R_{a_1+a_2}, \\ &\cdots \\ S_t &= R_{a_1+\cdots+a_{t-1}+1} \cup \cdots \cup R_\eta. \end{aligned} \tag{2.15}$$

Let X be a random variable derived from the random variable θ such that

$$X = i \quad \text{if} \quad \theta \in S_i, \qquad i = 1, \cdots, t. \tag{2.16}$$

In other words, the group of $k + 1$ random samples of sizes $(m_0, \cdots, m_k)$ will be designated by i $(i = 1, \cdots, t)$ if the rank set θ of the sample values is in S_i. Clearly, under $H_i (i = 0, 1)$, the random variable X is distributed according to

$$\Pr(X = x \mid H_i) = p_{ix}, \qquad x = 1, \cdots, t; i = 0, 1, \tag{2.17}$$

where

$$\begin{aligned} p_{i1} &= h(R_1; H_i) + \cdots + h(R_{a_1}; H_i), \\ p_{i2} &= h(R_{a_1+1}; H_i) + \cdots + h(R_{a_1+a_2}; H_i), \\ &\cdots \\ p_{it} &= h(R_{a_1+\cdots+a_{t-1}+1}; H_i) + \cdots + h(R_\eta; H_i). \end{aligned} \tag{2.18}$$

Now, according to the definition, there are many possible rank tests for testing H_0 against H_1, among which we mention the following three classes:

(a) Univariate rank (UR) tests. This class includes all tests which are based on a single random variable X having the pdf (2.17).

(b) Multivariate rank (MR) tests. This class includes all tests which are based on $g (g \geqq 2)$ independent and identically distributed random variables $X_1, \cdots, X_g$ having the common pdf (2.17).

(c) Sequential rank (SR) tests. This class includes all tests which are based on a sequence of successive independent, identically distributed random variables $X_1, X_2, \cdots$, each having the pdf (2.17).

Many rank tests have been proposed in the past [8]. While most of them are UR tests, a few (e.g., the sign test [3] and the sequential probability ratio test [15, Ch. 6]) are MR and SR tests. Other MR and SR tests can be obtained by the use of various goodness of fit criteria, such as the rank sum and the sequential rank sum tests introduced in [11] and [13].

Since the asymptotic efficiency of the rank sum test will be investigated in Section 6, it is convenient at this point to describe this test in some detail.

To employ the rank sum test, we shall choose the constants $a_1, \cdots, a_t$ in such a way that they are all equal and use

$$s = X_1 + \cdots + X_g \tag{2.19}$$

as a test criterion. It is shown in [12] that under H_i the probability distribution function of s, denoted by $q(y; p_i, g)$, satisfies the following recursion formula

$$q(y; p_i, n) = \sum_{j=1}^{t} p_{ij}\, q(y - j; p_i, n - 1), \qquad i = 0, 1, \tag{2.20}$$

with the initial conditions

$$q(y; p_i, 1) = \begin{cases} p_{iy}, & y = 1, \cdots, t, \\ 0, & \text{otherwise}, \end{cases} \tag{2.21}$$

where

$$p_i = (p_{i1}, \cdots, p_{it}), \qquad i = 0, 1. \tag{2.22}$$

Using this criterion, a critical region of a one-sided test would consist of g, $g + 1, \cdots, g + r$, where r is a non-negative integer so determined that

$$Q(g + r; p_0, g) = \sum_{j=g}^{g+r} q(j; p_0, g) = \alpha \tag{2.23}$$

for a predetermined level of significance α. The power of this test is, of course, given by

$$Q(g + r; p_1, g) = \sum_{j=g}^{g+r} q(j; p_1, g). \tag{2.24}$$

For an equal tailed two-sided test, a critical region would consist of $g, g + 1, \cdots, g + r^*, gt - r^*, \cdots, gt$, where r^* is a non-negative integer so determined that (2.23) holds with r replaced by r^* and α by $\alpha/2$. The power of such a two-sided test is given by

$$1 + Q(g + r^*; p_1, g) - Q(gt - r^* - 1; p_1, g). \tag{2.25}$$

We note that when $t = 2$, these tests reduce to the binomial tests and that for $3 \leqq t \leqq 6$, the function $Q(y; p_0, g)$ is tabulated in [12] for $1 \leqq g \leqq 20$.

Having indicated the scope of rank tests, we shall be concerned mainly with the evaluation of the power of such tests. Clearly, in order to evaluate the power of any rank test, it is necessary to find first the probability distribution function $h(R; H_1)$. The main purpose of this paper is, therefore, to develop an approximation procedure for evaluating $h(R; H_1)$. This procedure is quite effective when η is small. For large η, the approximation of the individual terms $h(R; H_1)$ becomes tedious, since there are too many entries involved. For example, in the two-sample case, there are 252 entries when $m_0 = m_1 = 5$, and the number of entries rises to 12870 when $m_0 = m_1 = 8$; in the three-sample case, even when $m_0 = m_1 = m_2 = 3$, the number of entries is already 1680. Consequently, when the distance between H_0 and H_1 (in a suitable sense) is so small that no sensitive UR test based on small η can be found, a MR or SR test based on a large number of groups of small samples may be recommendable, especially when it has certain other preferable properties.

3. Rank preserving transformations and equivalent tests. Our first step in evaluating the pdf $h(R; H_1)$ is to transform the original distributions $F_0^*, \cdots, F_k^*$ into some new distributions defined over the unit interval (0, 1).

Let $T(z)$ be a continuous, strictly increasing function defined over the infinite interval $(-\infty, \infty)$ such that $T(-\infty) = 0$, $T(\infty) = 1$. Let

$$V = (V_0, \cdots, V_k) = (V_{01}, \cdots, V_{0m_0}, \cdots, V_{k1}, \cdots, V_{km_k}), \tag{3.1}$$

where

$$V_i = (V_{i1}, \cdots, V_{im_i}) = (T(Z_{i1}), \cdots, T(Z_{im_i})) \tag{3.2}$$

are the ordered values of m_i independent random variables distributed according to

$$\psi_i(v) = \psi_i(T(z)) = F_i(z), \qquad i = 0, \cdots, k. \tag{3.3}$$

Let

$$\delta = (\delta_0, \cdots, \delta_k) = (\delta_{01}, \cdots, \delta_{0m_0}, \cdots, \delta_{k1}, \cdots, \delta_{km_k}), \tag{3.4}$$

where

$$\delta_i = (\delta_{i1}, \cdots, \delta_{im_i}) \tag{3.5}$$

are the ranks of V_i in V. Then, clearly, δ is also a random variable over Ω, and has the same distribution as θ([6], Theorem 8.1). It follows that a rank test for testing H_0 against H_1 is equivalent to the same rank test for testing the hypothesis

$$H_0' : \psi_0(v) = \cdots = \psi_k(v) = v \tag{3.6}$$

against the alternative hypothesis

$$H_1' : \psi_i(v) = \Phi_i(v), \qquad i = 0, \cdots, k, \tag{3.7}$$

where

$$\Phi_i(v) = \Phi_i(T(z)) = F_i^*(z), \qquad i = 0, \cdots, k. \tag{3.8}$$

We shall assume that the functions $\Phi_0(v), \cdots, \Phi_k(v)$ are differentiable on (0, 1) with continuous derivatives $\varphi_0(v), \cdots, \varphi_k(v)$ respectively. That is,

$$\varphi_i(v) = \Phi_i'(v), \qquad i = 0, \cdots, k. \tag{3.9}$$

4. Polynomial approximation of a cumulative distribution function. Our next step is to find polynomial approximations for $\Phi_0(v), \cdots, \Phi_k(v)$. That is, for a given cdf $F^*(z)$ and a given transformation $T(z)$, an interpolating polynomial of the function

$$\Phi(v) = \Phi(T(z)) = F^*(z) \tag{4.1}$$

is to be found. Now, letting $0 = v_0 < v_1 < \cdots < v_h = 1$ be $h + 1$ chosen points on the unit interval (0, 1), we shall find a polynomial $P(v)$ of degree h such that

$$P(v_r) = \Phi(v_r), \qquad r = 0, \cdots, h. \tag{4.2}$$

Since, in this case, the function $\Phi(v)$ is a cdf so that we always have

$$\Phi(0) = 0, \qquad \Phi(1) = 1, \tag{4.3}$$

then, $P(v)$ must have the following form

$$P(v) = a_1 v + a_2 v^2 + \cdots + a_h v^h \tag{4.4}$$

with real coefficients $a_1, \cdots, a_h$. The derivative of $P(v)$ will be written as

$$p(v) = b_0 + b_1 v + \cdots + b_q v^q, \tag{4.5}$$

where, of course,

$$q = h - 1, \qquad b_i = (i + 1)a_{i+1}, \qquad i = 0, \cdots, q. \tag{4.6}$$

In the following, we shall derive some formulae for determining the coefficients $a_1, \cdots, a_h$ in terms of a given set of values $(v_0, \Phi(v_0)), \cdots, (v_h, \Phi(v_h))$. Our derivations are based on the Lagrange's interpolating polynomials. It should be pointed out, however, that equivalent formulae can be obtained by other methods (e.g., by means of Newton's interpolating polynomials [14]).

Suppose we let

$$\pi_h(v) = (v - v_0)(v - v_1) \cdots (v - v_h) \tag{4.7}$$

and let $\pi_h'(v)$ be the derivative of $\pi_h(v)$, we then obtain

THEOREM 4.1. *For any given set of values $(v_0, \Phi(v_0)), \cdots, (v_h, \Phi(v_h))$, the coefficients $a_1, \cdots, a_h$ of the polynomial $P(v)$ in (4.4) can be written as*

$$a_r = \sum_{s=1}^{h} \frac{c_{rs}\Phi(v_s)}{\pi_h'(v_s)}, \qquad r = 1, \cdots, h, \tag{4.8}$$

where c_{rs} is the coefficient of v^r in the expansion of

$$\pi_h(v) \,/\, (v - v_s), \qquad s, r = 1, \cdots, h. \tag{4.9}$$

PROOF. From [7, p. 83], the Lagrange's interpolating polynomial is given by

$$P(v) = \sum_{s=0}^{h} \Phi(v_s) L_s^{(h)}(v), \tag{4.10}$$

where

$$L_s^{(h)}(v) = \pi_h(v) \,/\, [(v - v_s)\pi_h'(v_s)], \qquad s = 0, \cdots, h. \tag{4.11}$$

But this can be expanded into polynomial form as

$$L_s^{(h)}(v) = \left[\sum_{r=1}^{h} c_{rs} v^r\right] / \, \pi_h'(v_s), \qquad s = 1, \cdots, h. \tag{4.12}$$

By substituting (4.12) into (4.10), it is easily seen that the coefficients $a_1, \cdots, a_h$ are given by (4.8).

COROLLARY 4.1. *If* $(v_0, \cdots, v_h)$ *are equally spaced, the coefficients* $a_1, \cdots, a_h$ *can be written as*

$$a_r = \frac{h^r}{h!} \sum_{s=1}^{h} (-1)^{h-s} \binom{h}{s} c_{rs}' \Phi(v_s), \qquad r = 1, \cdots, h, \tag{4.13}$$

where c_{rs}' *is the coefficient of* y^r *in the expansion of*

$$\prod_{j=0}^{h} (y - j) \,/\, (y - s), \qquad r, s = 1, \cdots, h. \tag{4.14}$$

PROOF. Since, by assumption, we have

$$v_r = r/h, \qquad r = 0, \cdots, h, \tag{4.15}$$

this corollary follows directly from Theorem 4.1.

Consider now a simple application of Corollary 4.1 to the normal alternatives. Let $N(z; A, B)$ denote the cumulative normal distribution with mean A and standard deviation B. Let the transformation be given by

$$v = T(z) = N(z; \mu, \sigma). \tag{4.16}$$

Some approximating polynomials of degree five are obtained for the normal alternatives

$$\Phi(v) = \Phi(N(z; \mu, \sigma)) = N(z; \mu + d\sigma, \sigma). \tag{4.17}$$

Table 1 gives the coefficients $a_1, \cdots, a_5$ for the cases $d = 0, 0.25, 0.50, 0.75, 1.00, 1.25, 1.50$.

5. Two-sample problem. Applying the results of Sections 3 and 4 to Hoeffding's formula ([5] p. 88), we shall now find the approximate values of $h(R; H_1)$ for the two-sample case ($k = 1$). Since, in the two-sample case, the complete set of ranks θ is determined by the ranks of $Z_{11}, \cdots, Z_{1m_1}$ alone, we need consider only the distribution of the ranks

$$\theta_1 = (\theta_{11}, \cdots, \theta_{1m_1}). \tag{5.1}$$

Consequently, if we let

$$S = (s_1, \cdots, s_{m_1}) \tag{5.2}$$

TABLE 1

Coefficients of Approximating Polynomials of Degree Five for Normal Alternatives $N(z; \mu + d\sigma, \sigma)$

d	a_1	a_2	a_3	a_4	a_5
0	1	0	0	0	0
.25	.590656	.512090	−.083386	−.345181	.325821
.50	.355264	.307583	1.195227	−2.174861	1.316787
.75	.256026	−.409675	3.578142	−5.416004	2.991511
1.00	.255337	−1.475861	6.873877	−9.976240	5.322887
1.25	.319949	−2.759978	10.879441	−15.632358	8.192946
1.50	.422091	−4.146164	15.319144	−21.979847	11.384776

be the permutation of m_1 out of the first $m_0 + m_1$ positive integers such that

$$s_1 < \cdots < s_{m_1}, \tag{5.3}$$

then, our sample space Ω may be considered as being made up of all subsets ω of such permutations S.

To approximate $h(R; H_1)$, we shall employ the convenient transformation

$$v = F_0^*(z); \tag{5.4}$$

that is, we may assume $\Phi_0(v) = v$.

THEOREM 5.1. *Let the interpolating polynomial $P(v)$ of the function*

$$\Phi_1(v) = \Phi_1(F_0^*(z)) = F_1^*(z) \tag{5.5}$$

and the derivative $p(v)$ be given by (4.4) *and* (4.5).

Then, the approximate value of $h(R; H_1)$ is given by

$$h(R; H_1) \cong \binom{m_0 + m_1}{m_0}^{-1} \sum b_0^{n_0} b_1^{n_1} \cdots b_q^{n_q} C(S; n_0, n_1, \cdots, n_q), \tag{5.6}$$

where the sum $\sum$ is taken over all possible $(n_0, \cdots, n_q)$ such that

$$n_0 + \cdots + n_q = m_1,$$

and where

$$C(S; n_0, \cdots, n_q) = \frac{\Gamma(s_{m_1+1})}{\Gamma(s_1)} \sum{}' \prod_{j=1}^{m_1} \frac{\Gamma(s_j + i_1 + \cdots + i_j)}{\Gamma(s_{j+1} + i_1 + \cdots + i_j)},$$

in which the sum $\sum'$ is extended over all possible $(i_0, \cdots, i_{m_1})$ such that n_i of them are equal to i, $i = 0, \cdots, q$ and $s_{m_1+1} = m_0 + m_1 + 1$.

PROOF. By the same argument as in Section 4 of [6], it can be shown that

$$h(R; H_1) = \binom{m_0 + m_1}{m_0}^{-1} \int_0^1 \cdots \int_0^1 \frac{(m_0 + m_1)!}{\prod_{i=0}^{m_1} (s_{i+1} - s_i - 1)!} \prod_{j=1}^{m_1} z_j^{s_j - 1}(1 - z_j)^{s_{j+1} - s_j - 1}$$

$$\times \prod_{k=1}^{m_1} \varphi_1(z_k \cdots z_{m_1})\, dz_1 \cdots dz_{m_1},$$

where $s_0 = 0$. Since

$$\prod_{j=1}^{m_1} \varphi_1(z_j \cdots z_{m_1}) \cong \prod_{j=1}^{m_1} (b_0 + b_1 z_j \cdots z_{m_1} + \cdots + b_q z_j^q \cdots z_{m_1}^q)$$
$$= \sum b_0^{n_0} b_1^{n_1} \cdots b_q^{n_q} \sum{}' (z_1 \cdots z_{m_1})^{i_1} (z_2 \cdots z_{m_1})^{i_2} \cdots (z_{m_1})^{i_{m_1}}$$
$$= \sum b_0^{n_0} b_1^{n_1} \cdots b_q^{n_q} \sum{}' z_1^{i_1} z_2^{i_1+i_2} \cdots z_{m_1}^{i_1+i_2+\cdots+i_{m_1}},$$

we then have

$$h(R; H_1) \cong \binom{m_0 + m_1}{m_0}^{-1} \int_0^1 \cdots \int_0^1 \frac{(m_0 + m_1)!}{\prod_{j=0}^{m_1} (s_{j+1} - s_j - 1)!} \sum b_0^{n_0} b_1^{n_1} \cdots b_q^{n_q}$$
$$\times \sum{}' \prod_{j=1}^{m_1} z_j^{s_j + i_1 + \cdots + i_j - 1} (1 - z_j)^{s_{j+1} - s_j - 1} \, dz_1 \cdots dz_{m_1}$$
$$= \binom{m_0 + m_1}{m_0}^{-1} \sum b_0^{n_0} b_1^{n_1} \cdots b_q^{n_q} \frac{\Gamma(s_{m_1+1})}{\Gamma(s_1)} \sum{}' \prod_{j=1}^{m_1} \frac{\Gamma(s_j + i_1 + \cdots + i_j)}{\Gamma(s_{j+1} + i_1 + \cdots + i_j)}.$$

This completes the proof of THEOREM 5.1.

COROLLARY 5.1. *Under the conditions of Theorem* 5.1, *the approximate values of* $p_{11}, \cdots, p_{1t}$ *are given by*

$$(5.7) \quad p_{1j} \cong \binom{m_0 + m_1}{m_0}^{-1} \sum b_0^{n_0} b_1^{n_1} \cdots b_q^{n_q} K_j(n_0, n_1, \cdots, n_q), \quad j = 1, \cdots, t,$$

where

$$(5.8) \qquad K_j(n_0, n_1, \cdots, n_q) = \sum_{S \in S_j} C(S; n_0, n_1, \cdots, n_q).$$

This corollary is a direct result of Theorem 5.1.

As an application of Theorem 5.1, we shall now consider the problem of testing the hypothesis H_0 that two populations are identical against the alternative hypothesis H_1 that they are two normal populations with same variance but different means. More precisely, we have under H_1,

$$(5.9) \qquad \begin{aligned} F_0^*(z) &= N(z; \mu, \sigma), \\ F_1^*(z) &= N(z; \mu + d\sigma, \sigma). \end{aligned}$$

Since, in this case, the function $h(R; H_1)$ depends only on the parameter d, we shall write $h(R; d)$ in place of $h(R; H_1)$. Table 2 gives the approximate values of $h(R; d)$ for the cases $(m_0, m_1) = (2,2), (3,3)$, where the ranks R of the two samples have been replaced by 0's and 1's, following the conventional notation; that is, 0's represent Z_0's and 1's represent Z_1's. The interpolating polynomials used are those given in Table 1. The ordering is made according to the $c_1(R)$ criterion, that is, the optimum rank order criterion for small d proposed by Hoeffding [5] and studied in detail by Terry [10]. In cases where the c_1 value is the same for two or more rankings, however, the order is by increasing probability for the case $d = .25$. We note that, in Table 2 we have defined R' as the ranks of the sample values in the decreasing order; that is, R' is obtained from R by interchanging 0's and 1's.

TABLE 2

Approximate Values of $h(R;\ d)$ and $h(R';\ -d)$

Ranking	d = .25	.50	.75	1.00	1.25	1.50
			$m_0 = m_1 = 2$			
1100	.11727	.07917	.05136	.03218	.01970	.01206
1010	.13603	.10586	.07893	.05676	.03969	.02725
1001	.16223	.14964	.13087	.10871	.08607	.06541
0110	.16307	.15308	.13881	.12293	.10791	.09534
0101	.19401	.21413	.22391	.22206	.20961	.18958
0011	.22739	.29812	.37612	.45736	.53702	.61036
			$m_0 = m_1 = 3$			
111000	.02860	.01536	.00775	.00370	.00170	.00078
110100	.03157	.01859	.01025	.00535	.00267	.00132
101100	.03527	.02313	.01424	.00836	.00479	.00274
110010	.03532	.02323	.01427	.00823	.00448	.00233
101010	.03945	.02883	.01968	.01266	.00776	.00459
110001	.04069	.03049	.02099	.01326	.00770	.00416
011100	.04098	.03144	.02274	.01582	.01093	.00780
100110	.04382	.03579	.02759	.02034	.01453	.01018
101001	.04541	.03771	.02869	.02006	.01296	.00778
011010	.04580	.03906	.03117	.02350	.01704	.01220
100101	.05037	.04652	.03957	.03125	.02320	.01645
010110	.05081	.04828	.04320	.03678	.03004	.02368
011001	.05267	.05088	.04494	.03635	.02719	.01924
100011	.05669	.05944	.05782	.05245	.04484	.03680
001110	.05714	.06166	.06368	.06402	.06364	.06325
010101	.05836	.06248	.06132	.05528	.04608	.03584
001101	.06553	.07945	.08934	.09410	.09429	.09174
010011	.06563	.07955	.08870	.09056	.08427	.07106
001011	.07367	.10081	.12793	.15056	.16471	.16851
000111	.08219	.12730	.18613	.25737	.33718	.41955

Although no theoretical results are given as a basis to reveal the error of our approximations, the values for some of the extreme cases in Table 2, that is, for the cases $R = (000111)$ and $R = (111000)$, can be compared with the results of Dixon [1] and Teichroew [9]. These comparisons show that our approximate values and the exact figures agree to two or three decimal places for $d \leqq 1.25$. This may be considered as quite satisfactory for many practical purposes. If, however, one wishes to attain more accurate approximations, polynomials of degree higher than five must be used. This, of course, would result in more extensive computations.

6. Asymptotic power efficiency of the rank sum test. The purpose of this section is to investigate the asymptotic behavior of the rank sum criterion described in Section 2. As a representative, we shall consider only the problem of testing the hypothesis H_0 that two normal populations are identical against the alternative hypothesis H_1 that they have the same variance σ^2 but different means ν_0 and ν_1.

Let $n(z; A, B)$ be the normal density with mean A and standard deviation B. Let

$$x = (x_{01}, \cdots, x_{0m_0}, x_{11}, \cdots, x_{1m_1}) \tag{6.1}$$

be a point of a $m_0 + m_1$ dimensional Euclidean space and let

$$f(x; d, m_0, m_1) = \prod_{i=0}^{1} \prod_{j=1}^{m_i} n(x_{ij}; \nu_i, \sigma). \qquad -\infty < x_{ij} < \infty, \tag{6.2}$$

where $\nu_1 = \nu_0 + d\sigma$. Then, $h(R; d)$ can be written as

$$\begin{aligned} h(R; d) &= h(R; d, m_0, m_1) \\ &= \binom{m_0 + m_1}{m_0}^{-1} \int_{-\infty<x_{01}<\cdots<x_{1m_1}<\infty} \cdots \int (m_0 + m_1)!\, f(x_R; d, m_0, m_1)\, dx, \end{aligned} \tag{6.3}$$

where

$$x_R = (x_{01}^{(R)}, \cdots, x_{0m_0}^{(R)}, x_{11}^{(R)}, \cdots, x_{1m_1}^{(R)}) \tag{6.4}$$

is the rearrangement of x according to R (in the obvious manner).

For deriving the power efficiency of a rank sum test, it will also be convenient to write $p_{11}, p_{12}, \cdots, p_{1t}$ as

$$\begin{aligned} p_{11}(d; m_0, m_1, a) &= h(R_1; d, m_0, m_1) + \cdots + h(R_a; d, m_0, m_1), \\ &\cdots \\ p_{1t}(d; m_0, m_1, a) &= h(R_{at-a+1}; d, m_0, m_1) + \cdots + h(R_{at}; d, m_0, m_1), \end{aligned} \tag{6.5}$$

where $t \geqq 2$ and $a \geqq 1$ are two integers such that

$$at = \eta = \binom{m_0 + m_1}{m_0}. \tag{6.6}$$

Consider a rank sum test which is based on g groups of two samples of sizes (m_0, m_1) and which is to have the strength (α_0, α_1) for testing H_0 against H_1 (that is, the probability of rejecting H_0 under H_i is α_i, $i = 0, 1$). If under H_i, the mean and variance of the rank sum statistic s are designated by μ_i and σ_i^2 respectively ($i = 0, 1$), then, from (2.4) of [11], we have

$$\begin{aligned} \mu_0 &= g(t + 1) / 2, \\ \sigma_0^2 &= g(t^2 - 1) / 12, \\ \mu_1 &= g \sum_{j=1}^{t} j p_{1j}(d; m_0, m_1, a) \\ \sigma_1^2 &= g \left[\sum_{j=1}^{t} j^2 p_{1j}(d; m_0, m_1, a) - \left(\sum_{j=1}^{t} j p_{1j}(d; m_0, m_1, a) \right)^2 \right]. \end{aligned} \tag{6.7}$$

Now, suppose d is small so that g is large; then the distribution of s is approximately normal. Consequently, for a two-sided test, g is to be determined so that

$$N(\mu_0 + z_0\sigma_0 ; \mu_i , \sigma_i) - N(\mu_0 - z_0\sigma_0 ; \mu_i , \sigma_i) \cong 1 - \alpha_i , \qquad i = 0, 1, \tag{6.8}$$

from which we obtain

$$g \cong [\beta_2(d; m_0 , m_1 , t, a) \,/\, \beta_1(d; m_0 , m_1 , t, a)]^2 \tag{6.9}$$

where

$$\begin{aligned} \beta_1(d; m_0, m_1, t, a) &= (t+1)/2 - \sum_{j=1}^{t} j p_{1j}(d; m_0, m_1, a), \\ \beta_2(d; m_0, m_1, t, a) & \\ &= z_1 \sqrt{\sum_{j=1}^{t} j^2 p_{1j}(d; m_0, m_1, a) - \left(\sum_{j=1}^{t} j p_{1j}(d; m_0, m_1, a)\right)^2} \\ &\qquad - z_0 \sqrt{\frac{t^2-1}{12}}, \end{aligned} \tag{6.10}$$

and where z_0 and z_1 are two constants determined by α_0 and α_1 respectively.

If $m_0 = m_1 = 1$, then $t = 2$ and $a = 1$, and the test reduces to the sign test. It is well known that the asymptotic efficiency of the sign test (as compared with the most powerful t-test) is $2/\pi$. Consequently, we may first find the asymptotic power efficiency of a rank sum test as compared with the sign test, and then the corresponding power efficiency as compared with the t-test by multiplying the former by the constant $2/\pi$.

The large sample power efficiency of a rank sum test as compared with the sign test is now defined as

$$\mathcal{E}(d) = 2g' \,/\, (m_0 + m_1)g'', \tag{6.11}$$

where g' is the number of pairs of observations required by the sign test and g'' is the number of groups of samples required by the rank sum test. It is then readily seen that

$$\mathcal{E}(d) = \frac{2}{m_0 + m_1}\left(\frac{\beta_1(d; m_0, m_1, t, a)\, \beta_2(d; 1, 1, 2, 1)}{\beta_1(d; 1, 1, 2, 1)\, \beta_2(d; m_0, m_1, t, a)}\right)^2. \tag{6.12}$$

If we now let

$$E(x(r_{1j} \mid m_0 + m_1)) \tag{6.13}$$

be the mean of the r_{1j}th order statistic in a sample of size $(m_0 + m_1)$ from a population with cdf $N(y; 0, 1)$ and let

$$\begin{aligned} c_1(R \mid m_0 + m_1) &= \sum_{j=1}^{m_1} E(x(r_{1j} \mid m_0 + m_1)), \\ A_j(m_0, m_1, a) &= \sum_{i=(j-1)a+1}^{ja} c_1(R_i \mid m_0 + m_1), \qquad j = 1, \cdots, t, \end{aligned} \tag{6.14}$$

then, the asymptotic power efficiency is readily seen to be

$$(6.15)\qquad \mathcal{E}(0) = \lim_{d\to 0} \mathcal{E}(d) = \frac{24\left[\sum_{j=1}^{t} jA_j(m_0, m_1, a)\right]^2}{(t^2-1)(m_0+m_1)\binom{m_0+m_1}{m_0}^2\left[\sum_{j=1}^{2} jA_j(1,1,1)\right]^2}.$$

By the use of the mean values of the order statistics computed by Godwin [4], some of the efficiencies are computed and displayed in Table 3; here again the ordering is made according to the $c_1(R)$ criterion. The values $2\mathcal{E}(0)/\pi$ in Table 3 are to be interpreted as the asymptotic power efficiencies of the rank sum tests as compared with the t-tests.

In conclusion, we remark that the asymptotic power efficiency of one-sided rank sum tests can be obtained similarly according to formula (6.1) in [11]. This can be shown to be equivalent to (6.15) above.

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TABLE 3

Asymptotic power efficiencies of the rank sum tests

m_0	m_1	t	a	$\mathcal{E}(0)$	$2\mathcal{E}(0)/\pi$
2	1	3	1	1.0000	.6366
2	2	2	3	.7397	.4709
2	2	3	2	1.1096	.7064
2	2	6	1	1 1662	.7424
3	1	2	2	.6909	.4398
3	1	4	1	.9000	.5730
3	2	2	5	.7978	.5079
3	2	5	2	1.1638	.7409
3	2	10	1	1.1846	.7541
3	3	4	5	1.1713	.7457
3	3	5	4	1.2160	.7741
3	3	10	2	1.2613	.8030
3	3	20	1	1.2679	.8072
4	1	5	1	.8000	.5093
4	2	3	5	.9761	.6214
4	2	5	3	1.0849	.6907
4	2	15	1	1.1282	.7182
4	3	5	7	1.1911	.7583
4	3	7	5	1.2289	.7823
4	3	35	1	1.2705	.8088
4	4	2	35	.8915	.5675
4	4	7	10	1.2804	.8151
4	4	10	7	1.2946	.8242
4	4	14	5	1.3073	.8322
4	4	35	2	1.3150	.8372
4	4	70	1	1.3158	.8377

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