If this value be denoted χ_0^2, then we take $X = \sqrt{(\frac{1}{2}n)} \cdot \ln(\chi_0^2/n)$, so that we are effectively transforming χ^2 by first forming the ratio of χ^2 to its mean, raised to the power of its standard deviation, and then taking one-half the natural logarithm of this quantity. The expansion for the probability may be obtained from (7) and (8), or from (9), by putting $N = 2n$, $(n_2 - n_1)/(n_1 + n_2) = 1$ and $N / (n_1 + n_2) = 0$, or from (10) with $\sigma = \delta = n^{-1}$. It has been developed from first principles by the author in [7].

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THE MIXTURE OF NORMAL DISTRIBUTIONS WITH DIFFERENT VARIANCES[1]

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1. Introduction. In some practical problems, the observed variable may have a normal distribution whose variance varies from one observation to the next. The purpose of this note is to give the formula for the marginal distribution when the variances are assumed to be distributed according to the Gamma distribution.

2. The distribution in the general case. We assume that the conditional density of X, given σ^2, is

$$f(x/\sigma^2) = \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} e^{-x^2/2\sigma^2} \qquad -\infty < x < \infty, \qquad \sigma^2 > 0,$$

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and that the density function of the variance is

$$g(\sigma^2) = \frac{\alpha^\lambda}{\Gamma(\lambda)} e^{-\alpha\sigma^2}(\sigma^2)^{\lambda-1} \qquad \alpha > 0, \qquad \lambda > 0.$$

Multiplying these two densities together and integrating immediately yields the marginal density function of X in the form

$$f(x) = \frac{\alpha^\lambda}{\Gamma(\lambda)(2\pi)^{1/2}} \int_0^\infty \exp\{-[\alpha\sigma^2 + (x^2/2\sigma^2)]\}\, (\sigma^2)^{\lambda-3/2}\, d\sigma^2,$$

which, using a formula for the modified Hankel function [3], p. 39, gives

$$f(x) = \frac{\alpha^{1/2}(x\sqrt{2\alpha})^{\lambda-1/2} k_{\lambda-1/2}(x\sqrt{2\alpha})}{\sqrt{\pi}2^{\lambda-1}\Gamma(\lambda)}.$$

The distribution function of X could be obtained by integrating the density function or by evaluating two hypergeometric functions, for, by the Paul Lévy inversion formula ([4], p. 93, Eq. (10.3.1)) the well-known relation between sin x and $J_{1/2}(x)$, and Formula 1 of [2] (p. 434), we have

$$F(x) = \tfrac{1}{2} + \frac{2(2\alpha x^2)}{\sqrt{2\pi}} \left[\frac{\Gamma(\frac{1}{2})\Gamma(\lambda - \frac{1}{2})}{(2\alpha x^2)^{\lambda-1/2}2^{3/2}\Gamma(\lambda)\Gamma(\frac{3}{2})}\, {}_1F_2\left(\frac{1}{2}, \frac{3}{2} - \lambda, \frac{3}{2}, \frac{\alpha x^2}{2}\right) \right.$$
$$\left. + \frac{\Gamma(\frac{1}{2} - \lambda)}{2^{2\lambda+1/2}\Gamma(\lambda + 1)}\, {}_1F_2\left(\lambda, \lambda + 1, \lambda + \frac{1}{2}, \frac{\alpha x^2}{2}\right)\right],$$

where ${}_1F_2$ denotes a generalized hypergeometric function defined as

$${}_1F_2(\beta_1, \gamma_1, \gamma_2; z) = \sum_{n=0}^{\infty} \frac{(\beta_1)_n}{(\gamma_1)_n(\gamma_2)_n} z^n,$$

where $(\beta)_n = \beta(\beta + 1) \cdots (\beta + n - 1)$; $(\beta)_0 = 1$.

The density and distribution function can also be obtained from the characteristic function which is

$$\phi(t) = \frac{1}{(1 + t^2/2\alpha)^\lambda}.$$

3. The distribution when λ is an integer. For $\lambda = n$, an integer, from [1], p. 40 and [1], p. 128, No. 67b, we get

$$f(x) = \frac{\sqrt{2\alpha}}{(n-1)!} \frac{e^{-\alpha x\sqrt{2}}}{2^{2n-1}} \sum_{v=0}^{n-1} \frac{(2n - v\, 2)!(2\cdot\alpha x\sqrt{2})^v}{v!(n - z - 1)!}.$$

The distribution function can also be expressed in closed form if $\lambda = n$ an integer by the following formula ([1], p. 127, No. 66c)

$$\int_0^\infty \frac{\sin xt}{(a^2 + t^2)^n} \frac{dt}{t} = \frac{\pi}{2a^{2n}} \left[1 - \frac{e^{-ax}}{2^{n-1}(n-1)!} F_{n-1}(ax)\right],$$

where $F_0(z) = 1$, $F_1(z) = z + 2$, and $F_n(z) = (z + 2n) F_{n-1}(z) - zF'_{n-1}(z)$, for $a > 0; x \geqq 0; n = 1, 2, 3 \cdots$. These recurrence relations could be used to compute a table of the distribution function.

4. Moments. The moments are obtainable directly from the expansion of the characteristic function

$$\frac{1}{\left(1 + \frac{t^2}{2\alpha}\right)^\lambda} = 1 - \frac{\lambda}{\alpha}\frac{t^2}{2} + \frac{\lambda(\lambda + 1)}{\alpha^2}\frac{t^4}{2!4} - \frac{\lambda(\lambda + 1)(\lambda + 2)}{\alpha^3}\frac{t^6}{3!8}.$$

We have

$$\mu_1' = 0 \qquad\qquad 0 = \mu_3' = \mu_5' = \mu_7' = \cdots$$

$$\mu_2 = \mu_2' = \frac{\lambda}{\alpha}$$

$$\mu_4 = \frac{3\lambda(\lambda + 1)}{\alpha^2}$$

$$\beta_1 = 0, \beta_2 = \frac{\mu_4}{\mu_2^2} = 3\left(1 + \frac{1}{\lambda}\right).$$

As one would expect, the variance of X increases as λ increases. It is interesting to note that β_2 is always greater than 3.

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METRICS AND NORMS ON SPACES OF RANDOM VARIABLES

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1. Introduction and summary. Let $\mathfrak{X}$ be the space of random variables defined on an abstract probability space $(\Omega, \mathfrak{A}, P)$ where we consider any two elements of $\mathfrak{X}$ which are equal a.s. (almost surely) as the same. Fréchet [2] exhibited a metric on $\mathfrak{X}$ (for example, $E[| X - Y |/(1 + | X - Y |)]$) with the property that con-

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