ON SOME CHARACTERIZATION PROBLEMS CONNECTED WITH LINEAR STRUCTURAL RELATIONS

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1. Introduction. The problems concerned with the characterizations of the distribution laws of random variables when they are connected by a linear structural relation seem to originate from the stimulating problem first proposed by Ragnar Frisch before the Oxford Conference of the Econometric Society in 1936. His problem may be stated as follows. Let x_0 and x_1 be two observable random variables connected by a linear structural set up,

$$x_0 = a_0\xi + \eta_0,$$

$$x_1 = a_1\xi + \eta_1,$$

where ξ, η_0 and η_1 are mutually independent random variables, and a_0 and a_1 are some unknown constants. What are the conditions on the distribution laws of the random variables ξ, η_0 and η_1 under which the regression of x_0 on x_1 and also that of x_1 on x_0 is linear, irrespective of the values of the constants a_0 and a_1?

A partial solution to the problem of Ragnar Frisch was given by Allen [1] by proving that if the first two moments of η_0 and all the moments of ξ and η_1 exist, then a necessary and sufficient condition for the regression of x_0 on x_1 to be linear irrespective of the values of the constants a_0 and a_1 is that both ξ and η_1 are normally distributed. A more general theorem was proved later independently by Rao [10], [11] and Fix [4] as a complete solution to the problem of Ragnar Frisch. Rao-Fix's theorem may be stated as follows: Let ξ, η_0 and η_1 be three mutually independent proper random variables each having a finite expectation. Then a necessary and sufficient condition for the regression of $x_0 = a_0\xi + \eta_0$ on $x_1 = a_1\xi + \eta_1$ to be linear for some $a_0 \neq 0$ and for all a_1 contained in a closed interval is that both ξ and η_1 should belong to a class of stable laws with finite expectation.

Recently the author [6] has obtained a generalization of Rao-Fix's theorem in a new direction, replacing the condition of stochastic independence of η_0 and η_1 by the weaker assumption that the regression of η_0 on η_1 is linear. The author [5] has also obtained a characterization of the normal law from the consequence of the linearity of multiple regression of one random variable on several others, when the variables are connected by a linear structural relation as in the case of the bifactor theory of Spearman. Several analogous characterization problems connected with linear structural relations have also been solved recently by Ferguson [3]. In the present paper we shall consider some generalizations of these problems in various directions. In Section 4, some theorems on

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the dependent error variables (Theorems 4.1 and 4.2) are deduced and a general theorem on a higher dimensional linear structure is proved in the subsequent section (Theorem 5.1).

2. Definitions and assumptions. Throughout the present investigation we shall confine our attention to a set of observable random variables $x_0, x_1, \cdots, x_n$ having the following linear structural set up:

$$(2.1)\qquad \begin{cases} x_0 = a_{01}\xi_1 + a_{02}\xi_2 + \cdots + a_{0p}\xi_p + \eta_0, \\ x_1 = a_{11}\xi_1 + a_{12}\xi_2 + \cdots + a_{1p}\xi_p + \eta_1, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n = a_{n1}\xi_1 + a_{n2}\xi_2 + \cdots + a_{np}\xi_p + \eta_n, \end{cases}$$

where $\xi_1, \xi_2, \cdots, \xi_p$ are usually called the latent or hypothetical variables and $\eta_0, \eta_1, \cdots, \eta_n$ the error variables of the linear structural relation, and a_{ij}'s are a certain fixed set of constants.

Now using the notations of vectors and matrices, the equation (2.1) may be rewritten as

$$(2.2)\qquad (x_0 : x) = \xi(\alpha_0' : A') + (\eta_0 : \eta)$$

where

$$x = (x_1, x_2, \cdots, x_n), \qquad \xi = (\xi_1, \xi_2, \cdots, \xi_p),$$

$$\alpha_0 = (a_{01}, a_{02}, \cdots, a_{0p}), \qquad \eta = (\eta_1, \eta_2, \cdots, \eta_n),$$

and $A = (a_{ij})_{i=1,2,\cdots n;\, j=1,2,\cdots p}$ and α_0' and A' denote respectively the transposes of α_0 and A.

We shall now make the following assumptions on the distributions of the random variables concerned.

Assumption 1. The conditional distribution of x_0 for fixed $x_1, x_2, \cdots, x_n$ is assumed to exist, wherever needed.

Assumption 2. The set of random variables $\xi_1, \xi_2, \cdots, \xi_p$ is distributed independently of the set of error variables $\eta_0, \eta_1, \cdots, \eta_n$.

Assumption 3. The latent variables $\xi_1, \xi_2, \cdots, \xi_p$ are mutually independent proper random variables each having a finite expectation (which is assumed to be zero without any loss of generality in the proof) as well as a finite variance $\sigma_j^2, j = 1, 2, \cdots, p$. Similarly the error variables $\eta_0, \eta_1, \cdots, \eta_n$ are mutually independent (not necessarily proper) random variables each having a finite expectation (which is also assumed to be zero) as well as a finite variance δ_k^2, $k = 0, 1, \cdots, n$. Let the dispersion matrices of the random vectors ξ and η be denoted by Σ and Δ respectively such that

$$(2.3)\qquad \Sigma = \begin{pmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_p^2 \end{pmatrix} \quad\text{and}\quad \Delta = \begin{pmatrix} \delta_1^2 & & & \\ & \delta_2^2 & & \\ & & \ddots & \\ & & & \delta_n^2 \end{pmatrix}.$$

Now it should be noted that some or all of the elements of the matrix Δ may be zero.

But in Section 4, where some results concerned with dependent error variables are obtained for the special case of the above structure with $p = 1$ and $n \geqq 2$, it is assumed that all the random variables are proper and have only finite expectations and further the multiple regression of η_0 on $\eta_1, \eta_2, \cdots, \eta_n$ is linear.

The role of these assumptions is to ensure the existence of the expectation and the variance of the conditional distribution of x_0 for fixed $x_1, x_2, \cdots, x_n$ which we denote by $E(x_0 \mid x_1, x_2, \cdots, x_n)$ and $V(x_0 \mid x_1, x_2, \cdots, x_n)$ respectively.

3. Some lemmas. We give below some lemmas which are useful in proving the theorems in the subsequent sections.

LEMMA 3.1. *Let $x_0, x_1, \cdots, x_n$ be a set of $n + 1$ proper random variables each having a finite expectation (which is assumed to be zero without any loss of generality) as well as a finite variance. Then the necessary and sufficient conditions for*

$$\begin{cases} E(x_0 \mid x_1, x_2, \cdots, x_n) = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n, \\ V(x_0 \mid x_1, x_2, \cdots, x_n) = \sigma_0^2 \quad \text{a.e.}, \end{cases} \tag{3.1}$$

are that the equations

$$\begin{cases} \left.\dfrac{\partial \varphi(t_0, t_1, \cdots, t_n)}{\partial t_0}\right]_{t_0=0} = \displaystyle\sum_{j=1}^{n} \beta_j \frac{\partial \varphi(0, t_1, \cdots, t_n)}{\partial t_j} \\ \left.\dfrac{\partial^2 \varphi(t_0, t_1, \cdots, t_n)}{\partial t_0^2}\right]_{t_0=0} = -\sigma_0^2 \varphi(0, t_1, \cdots, t_n) + \displaystyle\sum_{j,k=1}^{n} \beta_j \beta_k \frac{\partial^2 \varphi(0, t_1, \cdots, t_n)}{\partial t_j \partial t_k} \end{cases} \tag{3.2}$$

are to be satisfied for all real $t_1, t_2, \cdots, t_n$ where $\varphi(t_0, t_1, \cdots, t_n)$ and $\varphi(0, t_1, \cdots, t_n)$ represent respectively the characteristic functions of the distributions of $(x_0, x_1, \cdots, x_n)$ and $(x_1, x_2, \cdots, x_n)$ and further $\beta_1, \beta_2, \cdots, \beta_n$ and $\sigma_0^2 > 0$ are arbitrary constants.

When the random variables $x_0, x_1, \cdots, x_n$ satisfy the relations in (3.1), we say that the multiple regression of x_0 on $x_1, x_2, \cdots, x_n$ is linear and that the conditional distribution of x_0 for fixed $x_1, x_2, \cdots, x_n$ is homoscedastic.

PROOF: To prove that the conditions are necessary, we can easily verify that

$$\begin{aligned} \left.\frac{\partial \varphi(t_0, t_1, \cdots, t_n)}{\partial t_0}\right]_{t_0=0} &= E\left\{iE(x_0 \mid x_1, x_2, \cdots, x_n) \exp\left(i \sum_{j=1}^{n} t_j x_j\right)\right\} \\ &= \sum_{j=1}^{n} \beta_j E\left\{i x_j \exp\left(i \sum_{j=1}^{n} t_j x_j\right)\right\} \\ &= \sum_{j=1}^{n} \beta_j \frac{\partial \varphi(0, t_1, \cdots, t_n)}{\partial t_j}. \end{aligned}$$

Similarly

$$\left.\frac{\partial^2\varphi(t_0, t_1, \cdots, t_n)}{\partial t_0^2}\right]_{t_0=0} = -E\left\{E(x_0^2 \mid x_1, x_2, \cdots, x_n) \exp\left(i \sum_{j=1}^{n} t_j x_j\right)\right\}$$

$$= -E\left\{\left(\sigma_0^2 + \sum_{j,k=1}^{n} \beta_j \beta_k x_j x_k\right) \exp\left(i \sum_{j=1}^{n} t_j x_j\right)\right\}$$

$$= -\sigma_0^2 \varphi(0, t_1, \cdots, t_n) + \sum_{j,k=1}^{n} \beta_j \beta_k \frac{\partial^2 \varphi(0, t_1, \cdots, t_n)}{\partial t_j \partial t_k}.$$

To prove the sufficiency of the conditions, we note simply that (3.2) may be rewritten as

$$E\left[\left\{E(x_0 \mid x_1, x_2, \cdots, x_n) - \sum_{j=1}^{n} \beta_j x_j\right\} \exp\left(i \sum_{j=1}^{n} t_j x_j\right)\right] = 0$$

and

$$E\left[\left\{E(x_0^2 \mid x_1, x_2, \cdots, x_n) - \sigma_0^2 - \sum_{j,k=1}^{n} \beta_j \beta_k x_j x_k\right\} \exp\left(i \sum_{j=1}^{n} t_j x_j\right)\right] = 0.$$

Then from the uniqueness theorem of Fourier transforms of functions of bounded variation, (3.1) follows immediately.

For the special case of $n = 1$, this reduces to the lemma proved independently by Rao [9] and Rothschild and Mourier [12].

LEMMA 3.2. *Let $x_0, x_1, \cdots, x_n$ be a set of $n + 1$ proper random variables each having a finite expectation (which is assumed to be zero). Then the necessary and sufficient condition for*

$$E(x_0 \mid x_1, x_2, \cdots, x_n) = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n \quad \text{a.e.}$$

is that the equation

$$\left.\frac{\partial\varphi(t_0, t_1, \cdots, t_n)}{\partial t_0}\right]_{t_0=0} = \sum_{j=1}^{n} \beta_j \frac{\partial\varphi(0, t_1, \cdots, t_n)}{\partial t_j}$$

is to be satisfied for all real values of $t_1, t_2, \cdots, t_n$.

This lemma has been already proved independently by the author [5] and Ferguson [3].

LEMMA 3.3. *Let $x_1, x_2, \cdots, x_n$ be n independent proper random variables and let further $\varphi_j(t)$ denote the characteristic function of the distribution of x_j, $j = 1, 2, \cdots, n$. If now the functions $\varphi_j(t)$ satisfy the equation*

$$\prod_{j=1}^{n} \{\varphi_j(t)\}^{\alpha_j} = e^{Q(t)},$$

for all real t in a certain neighbourhood of the origin $|t| < \delta(\delta > 0)$, where α_j's are some positive numbers and $Q(t)$ a quadratic polynomial in t, then each x_j follows normal distribution.[1]

[1] The proof of this lemma is given in A. A. Zinger and Yu. V. Linnik [13].

This lemma may be regarded as an analytical extension of Cramér's theorem on the normal law and has been proved by Linnik [8]. The proof of this lemma has been given by the author in [7].

4. Some results for the case of dependent error variables. We shall now obtain some results connected with dependent error variables for the special case of the above linear structure when $p = 1$ and $n \geqq 2$.

THEOREM 4.1. *Let the observable random variables* $x_j (j = 0, 1, \cdots, n)$ *have the linear structural set-up* $x_j = a_j\xi + \eta_j$ *where the* a_j*'s are fixed nonzero constants and further* ξ *and* $\eta_0, \eta_1, \cdots, \eta_n$ *are proper random variables each having a finite expectation (which is assumed to be zero without any loss of generality) such that*

(i) ξ *is distributed independently of* $(\eta_0, \eta_1, \cdots, \eta_n)$

(ii) $E(\eta_0 \mid \eta_1, \eta_2, \cdots, \eta_n) = \sum_{j=1}^n \beta_j'\eta_j$, *the* β_j'*'s being a set of constants, then the multiple regression of* x_0 *on* $x_1, x_2, \cdots, x_n$ *is always linear, whenever the relation* $a_0 = \sum_{j=1}^n a_j\beta_j'$ *is satisfied.*

PROOF. Let $\varphi(t_0, t_1, \cdots, t_n)$; $\varphi_0(t_0, t_1, \cdots, t_n)$ and $\Phi(t)$ represent the characteristic functions of the distributions of $(x_0, x_1, \cdots, x_n)$; $(\eta_0, \eta_1, \cdots, \eta_n)$ and ξ respectively.

Then we can write

$$\begin{aligned} \varphi(t_0, t_1, \cdots t_n) &= E[\exp (i\textstyle\sum_{j=0}^n t_jx_j)] \\ &= \Phi(\textstyle\sum_{j=0}^n a_jt_j)\varphi_0(t_0, t_1, \cdots, t_n). \end{aligned} \tag{4.1}$$

Again since it is given that $E(\eta_0 \mid \eta_1, \eta_2, \cdots, \eta_n) = \sum_{j=1}^n \beta_j'\eta_j$, by applying Lemma 3.2, we get easily

$$\left.\frac{\partial\varphi_0(t_0, t_1, \cdots, t_n)}{\partial t_0}\right]_{t_0=0} = \sum_{j=1}^n \beta_j' \frac{\partial\varphi_0(0, t_1, \cdots, t_n)}{\partial t_j}. \tag{4.2}$$

Now differentiating both sides of the equation (4.1) with respect to t_0 and then putting $t_0 = 0$ and finally using the equation (4.2), we have

$$\begin{aligned} \left.\frac{\partial\varphi(t_0, t_1, \cdots, t_n)}{\partial t_0}\right]_{t_0=0} &= a_0\Phi'\left(\sum_{j=1}^n a_jt_j\right)\varphi_0(0, t_1, \cdots, t_n) \\ &\quad + \sum_{j=1}^n \beta_j'\Phi\left(\sum_{j=1}^n a_jt_j\right)\frac{\partial\varphi_0(0, t_1, \cdots, t_n)}{\partial t_j}. \end{aligned} \tag{4.3}$$

Again putting $t_0 = 0$ on both sides of the equation (4.1) and then differentiating both sides with respect to t_j, $j = 1, 2, \cdots n$ we get

$$\begin{aligned} \frac{\partial\varphi(0, t_1, \cdots, t_n)}{\partial t_j} &= a_j\Phi'\left(\sum_{j=1}^n a_jt_j\right)\varphi_0(0, t_1, \cdots, t_n) \\ &\quad + \Phi\left(\sum_{j=1}^n a_jt_j\right)\frac{\partial\varphi_0(0, t_1, \cdots, t_n)}{\partial t_j} \qquad j = 1, 2, \cdots n. \end{aligned} \tag{4.4}$$

Now it is given that $a_0 = \sum_{j=1}^n a_j\beta_j'$; hence substituting this value of a_0 in

(4.3) and finally comparing with (4.4), it is easy to obtain

$$\left.\frac{\partial\varphi(t_0, t_1, \cdots, t_n)}{\partial t_0}\right]_{t_0=0} = \sum_{j=1}^{n} \beta_j' \frac{\partial\varphi(0, t_1, \cdots, t_n)}{\partial t_j}. \tag{4.5}$$

Then the proof follows at once using Lemma 3.2 to the equation (4.5).

THEOREM 4.2. *With the same notations and assumptions as used in Theorem* 4.1 *together with the additional assumptions*

(iii) *the variables* $\eta_1, \eta_2, \cdots, \eta_n$ *are mutually independent.*

(iv) *the constants* a_j *satisfy the relation* $a_0 \neq \sum_{j=1}^{n} a_j\beta_j'$,

the necessary and sufficient condition for the multiple regression of x_0 *on* $x_1, x_2, \cdots, x_n$ *to be linear* ($n \geqq 2$) *is that* ξ *and each of* $\eta_1, \eta_2, \cdots, \eta_n$ *is normally distributed.*

PROOF.

Necessity. Let us suppose that $E(x_0 \mid x_1, x_2, \cdots, x_n) = \sum_{j=1}^{n} \beta_j x_j$. Then using Lemma 3.2, we have

$$\left.\frac{\partial\varphi(t_0, t_1, \cdots, t_n)}{\partial t_0}\right]_{t_0=0} = \sum_{j=1}^{n} \beta_j \frac{\partial\varphi(0, t_1, \cdots, t_n)}{\partial t_j}. \tag{4.6}$$

Next using the equations (4.3), (4.4) and (4.6) together and noting that $\eta_1, \eta_2, \cdots, \eta_n$ are mutually independent random variables, we get after a little algebraic simplification,

$$\begin{aligned}(a_0 - \textstyle\sum_{j=1}^{n} a_j\beta_j)\Phi'(\sum_{j=1}^{n} a_jt_j)\prod_{j=1}^{n} \varphi_j(t_j)& \\ = \textstyle\sum_{j=1}^{n} (\beta_j - \beta_j')\cdot\Phi(\sum_{j=1}^{n} a_jt_j)\varphi_j'(t_j)\cdot\prod_{k\neq j} \varphi_k(t_k),&\end{aligned} \tag{4.7}$$

where $\varphi_j(t_j)$ represents the characteristic function of the distribution of η_j; $j = 1, 2, \cdots, n$.

It can be easily shown that under the conditions of the theorem, neither $a_0 - \sum_{j=1}^{n} a_j\beta_j$ nor any of $\beta_j - \beta_j'$ $j = 1, 2, \cdots n$ in the equation (4.7) can be equal to zero. Putting $t_k = 0$ for all $k \neq j$ in (4.7) and noting that $\varphi_j'(t_j)]_{t_j=0} = 0$ for $j = 1, 2, \cdots, n$ we get

$$(a_0 - \textstyle\sum_{j=1}^{n} a_j\beta_j)\Phi'(a_jt_j)\varphi_j(t_j) = (\beta_j - \beta_j')\Phi(a_jt_j)\varphi_j'(t_j), \quad j = 1, 2, \cdots, n. \tag{4.8}$$

Let us now suppose that $\beta_j - \beta_j' = 0$ for some j, but $a_0 - \sum_{j=1}^{n} a_j\beta_j \neq 0$. In this case the equation (4.8) gives

$$\Phi'(a_jt_j)\varphi_j(t_j) = 0. \tag{4.9}$$

But since the characteristic function $\varphi(t)$ is continuous for all real t and equal to unity at the origin, in a suitably chosen neighbourhood of the origin, we have always $\varphi_j(t_j) \neq 0$. Thus it follows that for all t_j in that neighbourhood of the origin $\Phi'(a_jt_j) = 0$, leading to the conclusion that the distribution of ξ is improper.

Proceeding in the same way it can be shown that if $\beta_j - \beta_j' \neq 0$ for any j, whereas $a_0 - \sum_{j=1}^{n} a_j\beta_j = 0$, the distribution of the corresponding η_j is improper. Thus both these cases contradict the conditions of the theorem. Now the only alternative left is when $a_0 - \sum_{j=1}^{n} a_j\beta_j$ and each of $\beta_j - \beta_j' j = 1, 2, \cdots, n$ vanish simultaneously. But in this case we have $a_0 = \sum_{j=1}^{n} a_j\beta_j'$, which is also contrary to the conditions of the theorem.

Now restricting the values of $t_1, t_2, \cdots, t_n$ to a suitably chosen neighbourhood of the origin such that each of the factors occurring in the product

$$\Phi\left(\sum_{j=1}^{n} a_j t_j\right)\prod_{j=1}^{n} \varphi_j(t_j)$$

is different from zero, we divide both sides of the equation (4.7) by this expression and thus obtain,

$$(a_0 - \sum_{j=1}^{n} a_j\beta_j)\theta'\left(\sum_{j=1}^{n} a_j t_j\right) = \sum_{j=1}^{n} (\beta_j - \beta_j')\theta_j'(t_j) \tag{4.10}$$

where

$$\theta(t) = \ln \Phi(t) \qquad \text{and} \qquad \theta_j(t) = \ln \varphi_j(t), \quad j = 1, 2, \cdots, n.$$

Next putting $t_3 = t_4 = \cdots = t_n = 0$ in (4.10), we get

$$\begin{aligned}(a_0 - \sum_{j=1}^{n} a_j\beta_j)\,\theta'(a_1t_1 + a_2t_2) \\ = (\beta_1 - \beta_1')\theta_1'(t_1) + (\beta_2 - \beta_2')\theta_2'(t_2).\end{aligned} \tag{4.11}$$

Then putting successively $t_1 = 0$ and $t_2 = 0$ in (4.11) and noting that

$$a_0 - \sum_{j=1}^{n} a_j\beta_j \neq 0,$$

we get easily

$$\theta'(a_1t_1 + a_2t_2) = \theta'(a_1t_1) + \theta'(a_2t_2). \tag{4.12}$$

But $\theta'(t)$ being continuous in t, it at once follows from the equation (4.12) that $\theta'(t)$ is a linear function of t and hence $\theta(t)$ is a quadratic polynomial in t, thus establishing the normality of the variable ξ. Then the normality of the remaining variables η_j ; $j = 1, 2, \cdots, n$ follows simply from the equation (4.8).

Sufficiency. Let σ^2 denote the variance of the random variable ξ and δ_j^2 that for $\eta_j\, j = 0, 1, 2, \cdots, n$.

Under the conditions of the theorem, we get on using the equation (4.3)

$$\left.\frac{\partial\varphi(t_0, t_1, \cdots, t_n)}{\partial t_0}\right]_{t_0=0} = -\left[\sum_{j=1}^{n} (a_0 a_j \sigma^2 + \beta_j'\delta_j^2)t_j\right] \Phi\left(\sum_{j=1}^{n} a_j t_j\right)\prod_{j=1}^{n} \varphi_j(t_j) \tag{4.13}$$

where

$$\Phi(t) = e^{-\sigma^2 t^2/2} \qquad \text{and} \qquad \varphi_j(t_j) = e^{-\delta_j^2 t_j^2/2};$$

$$j = 1, 2, \cdots, n.$$

Similarly we get, on using the equation (4.4)

$$(4.14)\quad \frac{\partial\varphi(0, t_1, \cdots, t_n)}{\partial t_j} = -\left[a_j\left(\sum_{k=1}^{n} a_k t_k\right)\sigma^2 + \delta_j^2 t_j\right]\Phi\left(\sum_{j=1}^{n} a_j t_j\right)\prod_{j=1}^{n}\varphi_j(t_j),$$
$$j = 1, 2, \cdots n.$$

Thus using (4.13) and (4.14) together, we may write

$$(4.15)\quad \left.\frac{\partial\varphi(t_0, t_1, \cdots, t_n)}{\partial t_0}\right]_{t_0=0} = \sum_{j=1}^{n}\beta_j\frac{\partial\varphi(0, t_1, \cdots, t_n)}{\partial t_j},$$

where the constants β_j are to be determined from the system of equations

$$\beta_1(a_1a_j\sigma^2) + \cdots + \beta_j(a_j^2\sigma^2 + \delta_j^2) + \cdots + \beta_n(a_na_j\sigma^2)$$
$$= a_0a_j\sigma^2 + \beta_j'\delta_j^2, \qquad j = 1, 2, \cdots n.$$

The proof follows at once using Lemma 3.2 to the equation (4.15).

The following corollary can be easily deduced.

COROLLARY 4.1. *Let the observable random variables* $x_j (j = 0, 1, \cdots, n)$ *have the linear structural set up* $x_j = a_j\xi + \eta_j$, *where the* a_j*'s are a set of non-zero constants and the* ξ *and* η_j*'s are mutually independent proper random variables each having a finite expectation. Then the necessary and sufficient condition for the multiple regression of* x_0 *on* $x_1, x_2, \cdots, x_n$ *to be linear (when* $n \geqq 2$*) is that* ξ *and* $\eta_1, \eta_2, \cdots, \eta_n$ *are normally distributed.*

This corollary has been proved earlier independently by the author [5] and Ferguson [3].

5. A theorem in general linear structure. We shall now consider a theorem on characterisation connected with the general linear structural set up already defined by the equation (2.1) in Section 2. In this direction, Ferguson [3] has obtained some necessary and sufficient conditions for the multiple regression of x_0 on $x_1, x_2, \cdots, x_n$ to be linear irrespective of the values of the constants a_{ij}. In the case of the higher dimensional structure, no result has yet been obtained, assuming the regression to be linear for just one set of values of the constants a_{ij}. We shall now show that it is possible to obtain some result for the case of the general linear structural relation for only one set of the values of the constants a_{ij} (with some restrictions upon their values) under the additional assumption that the conditional distribution of x_0 given $x_1, x_2, \cdots, x_n$ is homoscedastic and all the random variables concerned have finite variances.

We are now in a position to prove the following theorem:

THEOREM 5.1. *In the general linear structural set up* (2.1) *and under the Assumptions* 1, 2 *and* 3, *if the constants* a_{ij}*'s are subject to the following restrictions*

(i) *the vector* $\alpha_j = (a_{1j}, a_{2j}, \cdots, a_{nj})$ *has at least one non-zero element for each* $j = 1, 2, \cdots, p$,

(ii) *the matrix* $(A\Sigma A' + \Delta)$ *is non-singular, that is the determinant*

$$|A\Sigma A' + \Delta| \neq 0,$$

(iii) *each of the elements of the vectors* $\alpha_0 \Sigma A'(A\Sigma A' + \Delta)^{-1}$ *and*

$$\alpha_0[I - \Sigma A'(A\Sigma A' + \Delta)^{-1}A]$$

is different from zero,
then the necessary and sufficient condition for

$$\begin{cases} E(x_0|x_1, x_2, \cdots, x_n) = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n, \\ V(x_0|x_1, x_2, \cdots, x_n) = \sigma_0^2, \end{cases}$$

is that each of $\xi_1, \xi_2, \cdots, \xi_p$ *and each of the proper random variables amongst* $\eta_1, \eta_2, \cdots, \eta_n$ *are normally distributed.*

PROOF.

Necessity. Let $\varphi(t_0, t_1, \cdots, t_n)$ denote the characteristic function of the distribution of $(x_0, x_1, \cdots, x_n)$; $\Phi_j(t)$ that for the distribution of $\xi_j (j = 1, 2, \cdots, p)$ and $\varphi_k(t)$ that for the distribution of $\eta_k (k = 0, 1, 2, \cdots, n)$.

Then it is easy to obtain

$$\begin{aligned} \varphi(t_0, t_1, \cdots, t_n) &= E[\exp(i \textstyle\sum_{k=0}^n t_k x_k)] \\ &= \textstyle\prod_{j=1}^p \Phi_j(\sum_{k=0}^n a_{kj} t_k) \prod_{k=0}^n \varphi_k(t_k). \end{aligned} \tag{5.1}$$

Now under the assumptions of the theorem and applying the equation (3.2) in Lemma 3.1 to (5.1) above, we get after some laborious algebraic computations, proceeding in the same way as in Section 4,

$$\textstyle\sum_{j=1}^p (a_{0j} - \sum_{k=1}^n \beta_k a_{kj}) \Theta_j'(\sum_{k=1}^n a_{kj} t_k) = \sum_{k=1}^n \beta_k \theta_k'(t_k), \tag{5.2}$$

$$\textstyle\sum_{j=1}^p \{a_{0j}^2 - (\sum_{k=1}^n \beta_k a_{kj})^2\} \Theta_j''(\sum_{k=1}^n a_{kj} t_k) = -(\sigma_0^2 - \delta_0^2) + \sum_{k=1}^n \beta_k^2 \theta_k''(t_k), \tag{5.3}$$

holding for all real $t_1, t_2, \cdots, t_n$ in a suitably chosen neighbourhood of the origin, where

$$\Theta_j(t) = \ln \Phi_j(t), \qquad j = 1, 2, \cdots p;$$

$$\theta_k(t) = \ln \varphi_k(t), \qquad k = 1, 2, \cdots n.$$

Under the assumption that each of the random variables concerned has a finite second moment, we may again differentiate both sides of the equation (5.2) with respect to $t_l (l = 1, 2, \cdots, n)$ and thus obtain the set of equations

$$\textstyle\sum_{j=1}^p a_{lj}(a_{0j} - \sum_{k=1}^n \beta_k a_{kj}) \Theta_j''(\sum_{k=1}^n a_{kj} t_k) = \beta_l \theta_l''(t_l), \quad l = 1, 2, \cdots, n. \tag{5.4}$$

Next multiplying both sides of the equation (5.4) by β_l and adding for all $l = 1, 2, \cdots, n$, we get

$$\textstyle\sum_{l=1}^n \sum_{j=1}^p \beta_l a_{lj}(a_{0j} - \sum_{k=1}^n \beta_k a_{kj}) \Theta_j''(\sum_{k=1}^n a_{kj} t_k) = \sum_{l=1}^n \beta_l^2 \theta_l''(t_l). \tag{5.5}$$

Now using the equation (5.3), we get a simplification of the following expression

$$(5.6)\quad \sum_{j=1}^{p} (a_{0j} - \sum_{k=1}^{n} \beta_k a_{kj})^2 \Theta_j''(\sum_{k=1}^{n} a_{kj} t_k) + \sum_{k=1}^{n} \beta_k^2 \theta_k''(t_k)$$
$$= \sum_{j=1}^{p} \{a_{0j}^2 - 2a_{0j}(\sum_{k=1}^{n} \beta_k a_{kj}) + (\sum_{k=1}^{n} \beta_k a_{kj})^2\} \Theta_j''(\sum_{k=1}^{n} a_{kj} t_k) + \sum_{k=1}^{n} \beta_k^2 \theta_k''(t_k)$$
$$= -(\sigma_0^2 - \delta_0^2) - 2\sum_{j=1}^{p} \{(\sum_{k=1}^{n} \beta_k a_{kj})(a_{0j} - \sum_{k=1}^{n} \beta_k a_{kj})\} \Theta_j''(\sum_{k=1}^{n} a_{kj} t_k) + 2\sum_{k=1}^{n} \beta_k^2 \theta_k''(t_k).$$

Finally using the equation (5.5) to the right-hand side of (5.6), we obtain

$$(5.7)\quad \sum_{j=1}^{p} (a_{0j} - \sum_{k=1}^{n} \beta_k a_{kj})^2 \Theta_j''(\sum_{k=1}^{n} a_{kj} t_k) + \sum_{k=1}^{n} \beta_k^2 \theta_k''(t_k) = -(\sigma_0^2 - \delta_0^2).$$

Since it is given that the matrix $(A\Sigma A' + \Delta)$ is non-singular, it can be easily shown under the condition $E(x_0|x_1, x_2, \cdots, x_n) = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n$, β_k is given by the kth element of the vector $\alpha_0 \Sigma A'(A\Sigma A' + \Delta)^{-1}$ for $k = 1, 2, \cdots, n$. Similarly $a_{0j} - \sum_{k=1}^{n} \beta_k a_{kj}$ is given by the jth element of the vector

$$\alpha_0[I - \Sigma A'(A\Sigma A' + \Delta)^{-1}A] \quad \text{for } j = 1, 2, \cdots, p.$$

Thus under the given restrictions on a_{ij}'s, it follows that $a_{0j} - \sum_{k=1}^{n} \beta_k a_{kj} \neq 0$ for all $j = 1, 2, \cdots, p$ and similarly $\beta_k \neq 0$ for all $k = 1, 2, \cdots, n$. Then the proof of the necessity part follows at once, using Linnik's result (Lemma 3.3) to the equation (5.7).

The proof that the condition is sufficient follows easily from Cramér ([2], pp. 314–315).

REFERENCES

[1] H. V. Allen, "A theorem concerning the linearity of regression," *Stat. Res. Memoirs.*, Vol. 2 (1938), pp. 60–68.
[2] H. Cramér, *Mathematical Methods of Statistics*, Princeton University Press, 1946.
[3] T. Ferguson, "On the existence of linear regression in linear structural relations," *University of California Publications in Statistics*, Vol. 2 (1955), pp. 143–166.
[4] E. Fix, "Distributions which lead to linear regressions," *Proceedings of the First Berkeley Symposium on Mathematical Statistics and Probability*, University of California Press, 1949, pp. 79–91.
[5] R. G. Laha, "On Characterizations of Probability Distributions and Statistics," Ph.D. thesis submitted to Calcutta University, 1955.
[6] R. G. Laha, "On a characterization of stable law with finite expectation," *Ann. Math. Stat.*, Vol. 27 (1956), pp. 187–195.
[7] R. G. Laha, "On a characterization of the normal distribution from properties of suitable linear statistics," *Ann. Math. Stat.*, Vol. 28 (1957), pp. 126–139.
[8] Yu, V. Linnik, "On an analytical extension of Cramer's theorem on the normal law," *Reports of the Leningrad and Moscow University Probability Seminars*, 1954.
[9] C. R. Rao, M.A. thesis submitted to Calcutta University, 1943.
[10] C. R. Rao, "Note on a problem of Ragnar Frisch," *Econometrica*, Vol. 15 (1947), pp. 245–249.
[11] C. R. Rao, "A correction to 'Note on a problem of Ragnar Frisch,' " *Econometrica*, Vol. 17 (1949), p. 212.
[12] C. Rothschild and E. Mourier, "Sur les lois de probabilité à regression linéaire et écart type lié constant," *C. R. Acad Sci, Paris*, Vol. 225 (1947), pp. 1117–1119.
[13] A. A. Zinger and Yu. U. Linnik, "On an analytical extension of a theorem of Cramér and its applications (Russian)", ***Vestnik Leningrad Univ.***, Vol. 10 (1955), pp. 51–56.