INTRA-BLOCK ANALYSIS FOR FACTORIALS IN TWO-ASSOCIATE CLASS GROUP DIVISIBLE DESIGNS[1, 2]

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1. Introduction and summary. Group divisible incomplete block designs form an important class of incomplete block designs useful in a wide variety of experimental situations. Their properties, construction, and analysis have been thoroughly discussed in statistical literature, and we cite only several recent references [1], [2], and [3] to work of Bose and his co-workers dealing with partially balanced designs with two associate classes with which we shall be concerned.

The utility of incomplete block designs would be increased with means of incorporating factorial treatment combinations in them. The use of factorials is widespread and stimulated by the concepts of confounding, partial confounding, and fractional replication. A mathematical summary on factorials is given by Kempthorne [4]. Kramer and Bradley [5] considered factorials in near-balance incomplete block designs, and here we generalize to the wider class of group divisible designs with two associate classes. Harshbarger [6] used a 2^3 factorial in a Latinized rectangular lattice and this seems to be the first use of a factorial in a partially balanced incomplete block design.

We obtain the intra-block analysis of variance for two-associate class group divisible designs with the adjusted treatment sum of squares in a modified form that more clearly indicates the structure of that quantity. Factorial treatment combinations are then identified with basic treatments through the association scheme of a design. This identification is effected in such a way that the factors are divided into two groups. For example, the design for 18 treatments (see [2], Design S60), divisible into six groups of three, in blocks of six, treatments replicated five times, can be adapted to a 6×3 factorial scheme; by regarding the six groups as made up of a 2×3 classification, the same design can be used for a 2×3^2 factorial scheme. Single-degree-of-freedom comparisons are obtainable in much the usual way and use of fractional replication, essentially within the groups of factors, is possible. The analyses for factorials depend on the estimators of basic treatment effects.

We are not concerned with the construction of two-associate class group divisible designs and all known such designs for which $r \leqq 10$, $3 \leqq k \leqq 10$, where r is the number of replications and k is the number of plots per block, are given in [7].

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2. Definitions and notation. Bose, Clatworthy, and Shrikhande [7] list the following properties of group divisible designs with two associate classes:

(i) The experimental material is divided into b blocks of k units each; different treatments are applied to the different units in a block.

(ii) There are $v = mn$ treatments ($v > k$) and the treatments can be divided into m groups of n each such that any two treatments of the same group are first associates while two treatments from different groups are second associates. Each treatment occurs in the design r times, and $vr = bk$.

(iii) Each treatment has exactly $(n - 1)$ first associates and $n(m - 1)$ second associates.

(iv) Given any two treatments which are ith associates, the number of treatments common to the jth associate of the first and the kth associate of the second is p^i_{jk}, $(i, j, k = 1, 2)$, and is independent of the pair of treatments selected. In matrix notation, if P_i is the matrix with elements p^i_{jk},

$$P_1 = \begin{Vmatrix} (n-2) & 0 \\ 0 & n(m-1) \end{Vmatrix} \text{ and } P_2 = \begin{Vmatrix} 0 & (n-1) \\ (n-1) & n(m-2) \end{Vmatrix}.$$

(v) Two treatments which are ith associates occur together in exactly λ_i blocks, $i = 1, 2$.

(vi) The inequalities, $r \geqq \lambda_1$, $rk - \lambda_2 v \geqq 0$, hold.

(vii) The design parameters are related so that $(n - 1)\lambda_1 + n(m - 1)\lambda_2 = r(k - 1)$, or $rk - \lambda_2 v = r - \lambda_1 + n(\lambda_1 - \lambda_2)$. Group divisible designs have been divided into three subclasses, Singular, Semi-regular, and Regular, but we shall consider the class as a whole without subdivision.

We let V_{ij} denote the jth treatment of the ith group noted in (ii), $i = 1, \cdots, m$; $j = 1, \cdots, n$. Then the usual association scheme is given by the matrix V with elements V_{ij}. Two treatments with common first subscripts (in the same row of V) are first associates; otherwise they are second associates. The double subscript notation is introduced here for it will be convenient when we come to consider factorials. To use the design catalogue [7] it is only necessary to match our treatment designations with those in the association matrices where treatments are numbered serially.

The model that will be assumed for group divisible incomplete block designs is that

$$y_{ijs} = \mu + \tau_{ij} + \beta_s + \epsilon_{ijs}, \tag{2.1}$$

where y_{ijs} is the observation on V_{ij} in block s if that treatment occurs in block s, μ is the grand mean, τ_{ij} is the effect of V_{ij}, β_s is the effect of block s, and ϵ_{ijs} are independent normal variates with zero means and homogeneous variances, σ^2. Latin letters m, t_{ij} and b_s will be used for estimators of the parameters in (2.1). Restrictions on the parameters in (2.1) are

$$\textstyle\sum_i \sum_j \tau_{ij} = 0 \tag{2.2}$$

and

$$\sum_s \beta_s = 0. \tag{2.3}$$

The parameter β_s in (2.1) may sometimes be redefined when the blocks are arranged in replications or a Latin square [8], [9]. We shall not explicitly consider these situations since the modifications involved do not affect the estimation of the adjusted treatment sum of squares.

3. General regression theory. Let

$$y_\alpha = \mu + \sum_{i=1}^{k} \beta_i x_{i\alpha} + \epsilon_\alpha, \tag{3.1}$$

$\alpha = 1, \cdots, N$, represent a general regression model where the $x_{i\alpha}$ are constants and the ϵ_α are independent normal variates with zero means and homogeneous variances, σ^2. The β_i are regression parameters subject to r_1 linearly independent restrictions,

$$\sum_{i=1}^{k} \alpha_{ih}\beta_i = 0, \qquad h = 1, \cdots, r_1 < (k - 1), \tag{3.2}$$

defining a parameter space Ω. The α_{ih} in (3.2) are known constants. A null hypothesis introduces r_2 additional restraints through additional linearly independent equations like (3.2) for $h = r_1 + 1, \cdots, r_1 + r_2 < (k - 1)$ and thus defines a parameter space ω, a subspace of Ω.

The general theory of regression tests under the conditions set forth lets us state that $\text{Reg}\,(\beta \mid \Omega)/\sigma^2$, $\text{Reg}\,(\beta \mid \omega)/\sigma^2$, and $[\text{Reg}\,(\beta \mid \Omega) - \text{Reg}\,(\beta \mid \omega)]/\sigma^2$ have χ^2-distributions respectively with $(k - r_1)$, $(k - r_1 - r_2)$, and r_2 degrees of freedom independent of $\text{Res}\,(\beta \mid \Omega)/\sigma^2$, which also has a χ^2-distribution with $(N - k + r_1 - 1)$ degrees of freedom. $\text{Reg}\,(\beta \mid \Omega)$ is the sum of squares due to regression on the x-variables in (3.1) with the regression coefficients subject to the restraints (3.2); $\text{Res}\,(\beta \mid \Omega)$ is the resultant sum of squares of deviations about that regression line. $\text{Reg}\,(\beta \mid \omega)$ is the sum of squares due to regression on the x-variables in (3.1) with the regression coefficients subject to the totality of $(r_1 + r_2)$ restraints defining ω. We note that

$$\text{Reg}\,(\beta \mid \Omega) = \sum_{i=1}^{k} b_i g_i, \tag{3.3}$$

$$\text{Reg}\,(\beta \mid \omega) = \sum_{i=1}^{k} b_i' g_i, \tag{3.4}$$

and

$$\text{Res}\,(\beta \mid \Omega) = \sum_{\alpha=1}^{N} (y_\alpha - \bar{y})^2 - \sum_{i=1}^{k} b_i g_i, \tag{3.5}$$

where

$$g_i = \sum_{\alpha=1}^{N} (y_\alpha - \bar{y}) x_{i\alpha}, \tag{3.6}$$

b_i and b_i' are the least squares estimators of β_i under the restraints of Ω and ω respectively, and $\bar{y} = \sum_{\alpha=1}^{N} y_\alpha/N$. An F-test of the indicated hypothesis is possible based on

$$(3.7)\quad F = (N - k + r_1 - 1)[\text{Reg}\,(\beta \mid \Omega) - \text{Reg}\,(\beta \mid \omega)]/r_2\,\text{Res}\,(\beta \mid \Omega),$$

with r_2 and $(N - k + r_1 - 1)$ degrees of freedom.

To illustrate the application of this theory, we consider the model (2.1) corresponding to (3.1) and the restrictions (2.2) and (2.3) corresponding to (3.2) and defining Ω. Now N in the regression theory is replaced by vr, k by $(b + v)$, r_1 by 2, and r_2 by $(v - 1)$. The regression coefficients β_i become treatment and block effects, τ_{ij} and β_s. To test the hypothesis that $\tau_{ij} = 0$ for all i and j in (2.1), the hypothesis of "no treatment effects", it is only necessary to add $(v - 1)$ additional linearly independent restrictions on the τ_{ij} to insure that each $\tau_{ij} = 0$, thus defining ω. The adjusted treatment sum of squares with $(v - 1)$ degrees of freedom becomes

$$(3.8)\qquad \text{Reg}\,(\beta, \tau \mid \Omega) - \text{Reg}\,(\beta, \tau \mid \omega),$$

where

$$(3.9)\qquad \text{Reg}\,(\beta, \tau \mid \Omega) = \textstyle\sum_i \sum_j t_{ij} T_{ij} + \sum_s b_s B_s$$

and

$$(3.10)\qquad \text{Reg}\,(\beta, \tau \mid \omega) = \textstyle\sum_s b_s' B_s\,,$$

the latter sums of squares having respectively $(b + v - 2)$ and $(b - 1)$ degrees of freedom. T_{ij} is the total for treatment V_{ij} and B_s is the sth block total. b_s and b_s' are the estimators of β_s under Ω and ω respectively; t_{ij}', the estimator of τ_{ij} under ω, is necessarily zero. The error sum of squares for the intra-block analysis of variance is

$$(3.11)\qquad \text{Res}\,(\beta, \tau \mid \Omega) = \textstyle\sum_i \sum_j \sum_s (y_{ijs} - \bar{y})^2 - \text{Reg}\,(\beta, \tau \mid \Omega),$$

with $(vr - b - v + 1)$ degrees of freedom. In (3.11), note that the summation is restricted to values of i and j occurring with s through the properties of the designs considered; this will be the case throughout this paper. The unadjusted block sum of squares is given by (3.10) and has $(b - 1)$ degrees of freedom.

We shall use the theory summarized in this section in the subsequent discussions. A basis for this theory is given by Wilks ([10], Sections 8.3 and 8.43).

4. General analysis of variance modified. The basic intra-block analysis of variance for partially balanced incomplete block designs with two-associate classes is known ([7], Table 1.0). In our notation, the adjusted treatment sum of squares is

$$(4.1)\qquad \text{Adj. Treat. S.S.} = \textstyle\sum_i \sum_j t_{ij} Q_{ij}\,,$$

where

$$(4.2)\qquad Q_{ij} = T_{ij} - B_{ij\cdot}/k,$$

with $B_{ij.}$, the total of block totals for blocks containing V_{ij}. For the subclass of group divisible designs,

$$(4.3)\qquad v\lambda_2(\lambda_1 + rk - r)t_{ij} = k(\lambda_1 + \lambda_2 v - \lambda_2)Q_{ij} + k(\lambda_1 - \lambda_2)\sum_{\substack{p\\ p\neq j}} Q_{ip}$$

obtained from the reference ([7], Eqs. 1.11 to 1.19). If j in (4.3) is replaced by q and both sides of (4.3) summed over values of $q \neq j$, the resulting identity may be substituted back into (4.3) with simple algebraic reduction based on the relations (vii) of Section 2 to yield

$$(4.4)\qquad [(\lambda_2 + rk - r)t_{ij} + (\lambda_2 - \lambda_1)\sum_{\substack{p\\ p\neq j}} t_{ip}]/k = Q_{ij}.$$

The adjusted treatment sum of squares expressed in terms of the estimators of treatment effects alone is

$$(4.5)\qquad \text{Adj. Treat. S.S.} = \frac{(\lambda_1 + rk - r)}{k}\sum_i \sum_j t_{ij}^2 + \frac{(\lambda_2 - \lambda_1)}{k}\sum_i (\sum_j t_{ij})^2,$$

obtained by substituting Q_{ij} in (4.4) into (4.1).

The result of (4.5) is a form more suitable for the consideration of factorials than (4.1). Usually in analysis of variance, computing is based on (4.1). It is in fact simpler when using a desk calculator to substitute for the Q_{ij} in (4.3) to obtain

$$(4.6)\qquad \begin{aligned} t_{ij} = [kv\lambda_2 T_{ij} - k(\lambda_2 - \lambda_1)\textstyle\sum_j T_{ij} - v\lambda_2 B_{ij.} \\ + (\lambda_2 - \lambda_1)\textstyle\sum_j B_{ij.}]/v\lambda_2(\lambda_1 + rk - r) \end{aligned}$$

from two-way tables of values of T_{ij} and $B_{ij.}$. Substitution in (4.5) is then based on (4.6).

The analysis of variance is completed by the calculation of the unadjusted block sum of squares and the total sum of squares, for the error sum of squares is obtained by subtraction.

$$(4.7)\qquad \text{Unadj. Block S.S.} = \frac{1}{k}\sum_s B_s^2 - \frac{G^2}{rv}.$$

$$(4.8)\qquad \text{Total S.S.} = \sum_i \sum_j \sum_s y_{ijs}^2 - \frac{G^2}{rv}.$$

G is the grand total of all observations, $\sum_i \sum_j \sum_s y_{ijs}$. Degrees of freedom for Adj. Treat. S.S., Unadj. Block S.S., Total S.S., and Error S.S. are respectively $(v - 1)$, $(b - 1)$, $(rv - 1)$, and $[(r - 1)v - b + 1]$.

The variance of the difference between estimators of first-associate treatment effects is

$$(4.9)\qquad V(t_{ij} - t_{ij'}) = 2k\sigma^2/(\lambda_1 + rk - r),$$

$j \neq j'$; the variance of the difference between estimators of second-associate treatment effects is

$$V(t_{ij} - t_{i'j'}) = 2k\sigma^2(\lambda_1 + \lambda_2 v - \lambda_2)/v\lambda_2(\lambda_1 + rk - r), \tag{4.10}$$

$i \neq i'$. These variances are estimated by substituting the error mean square from the analysis of variance for σ^2.

The efficiencies of first and second associate treatment comparisons have been given by Bose and his associates [7]. These efficiencies are obtained by taking the ratio of the variance of the treatment contrast for a randomized block design to the corresponding variance for the incomplete block design given equal values of r and on the assumption that both designs yield the same experimental error. The efficiency for the comparison of two treatments that are first associates is

$$E_1 = (\lambda_1 + rk - r)/rk \tag{4.11}$$

and, for two treatments that are second associates, the efficiency is

$$E_2 = v\lambda_2(\lambda_1 + rk - r)/rk(\lambda_1 + \lambda_2 v - \lambda_2). \tag{4.12}$$

E_1 and E_2 are in more explicit forms than given previously and are derivable from (4.9) and (4.10) and the fact that the corresponding variance for the randomized block design is $2\sigma^2/r$.

5. The basic two-factor factorial. To introduce factorials into two-associate class group divisible designs, we first consider a basic two-factor factorial. Then it will be possible to show how multi-factor factorials may be used.

Consider A and C factors with m and n levels respectively providing $v = mn$ treatment combinations associated with the V_{ij} so that

$$\tau_{ij} = \alpha_i + \gamma_j + \delta_{ij} \tag{5.1}$$

with restrictions,

$$\textstyle\sum_i \alpha_i = 0, \tag{5.2}$$

$$\textstyle\sum_j \gamma_j = 0, \tag{5.3}$$

$$\textstyle\sum_i \delta_{ij} = 0, \tag{5.4}$$

and

$$\textstyle\sum_i \delta_{ij} = 0. \tag{5.5}$$

Equations (5.2) to (5.5) represent $(m + n + 1)$ linearly independent restrictions on the $(mn + m + n)$ new parameters. α_i, γ_j, and δ_{ij} are parameters representing respectively the effects of the ith level of the A-factor, the jth level of the C-factor and the interaction of the ith level of the A-factor and the jth level of the C-factor. Corresponding Latin letters will be used for estimators of these effects.

The change to factorial parameters may be regarded simply as a one-to-one transformation in the parameter space. It follows that

$$t_{ij} = a_i + c_j + d_{ij} \tag{5.6}$$

and substitution in (4.5) yields

$$\begin{aligned}\text{Adj. Treat. S.S.} = \frac{n\lambda_2 v}{k}\sum_i a_i^2 + \frac{m(\lambda_1 + rk - r)}{k}\sum_j c_j^2 \\ + \frac{(\lambda_1 + rk - r)}{k}\sum_i\sum_j d_{ij}^2\end{aligned} \tag{5.7}$$

after reduction based on (vii) of Section 2. Use of the general regression theory is sufficient to obtain

$$\text{Adj. S.S. } (A) = \frac{n\lambda_2 v}{k}\sum_i a_i^2, \tag{5.8}$$

$$\text{Adj. S.S. } (C) = \frac{m(\lambda_1 + rk - r)}{k}\sum_j c_j^2, \tag{5.9}$$

and

$$\text{Adj. S.S. } (AC) = \frac{(\lambda_1 + rk - r)}{k}\sum_i\sum_j d_{ij}^2, \tag{5.10}$$

with $(m - 1)$, $(n - 1)$, and $(m - 1)(n - 1)$ degrees of freedom respectively. The complete analysis of variance is given in Table 1. Definition of (5.8), (5.9), and (5.10) is complete when we note that

$$a_i = \textstyle\sum_j t_{ij}/n = \bar{t}_{i\cdot}\,, \tag{5.11}$$

$$c_j = \textstyle\sum_i t_{ij}/m = \bar{t}_{\cdot j}\,, \tag{5.12}$$

and

$$d_{ij} = t_{ij} - \bar{t}_{i\cdot} - \bar{t}_{\cdot j}\,, \tag{5.13}$$

computed most easily from the two-way table of values of t_{ij}. Independence of the sums of squares in (5.8), (5.9), and (5.10) follows from Cochran's theorem [11].

We sketch the use of the general regression theory of Section 3 and the application of it to our problem by considering Adj. S.S. (A).

To effect the regression with the complete model obtained by substituting for τ_{ij} of (5.1) in (2.1), it is necessary to minimize

$$\textstyle\sum_i\sum_j\sum_s (y_{ijs} - \mu - \alpha_i - \gamma_j - \delta_{ij} - \beta_s)^2, \tag{5.14}$$

subject to the restraint (2.3) and to the $(m + n + 1)$ linearly independent restraints of (5.2) to (5.5) through use of Lagrange multipliers. The resulting

TABLE 1
Intra-block analysis of variance for the basic two-factor factorial

Source of Variation	Degrees of Freedom	Sum of Squares*
Treatments (adjusted)	$(v - 1) = (mn - 1)$	$K_1 \sum_i \sum_j t_{ij}^2 + K_2 \sum_i (\sum_j t_{ij})^2$
A-factor (adjusted)	$(m - 1)$	$(nK_1 + n^2K_2) \sum_i \bar{t}_i^2$
C-factor (adjusted)	$(n - 1)$	$mK_1 \sum_j \bar{t}_{.j}^2$
AC-interaction (adjusted)	$(m - 1)(n - 1)$	$K_1 \sum_i \sum_j (t_{ij} - \bar{t}_{i.} - \bar{t}_{.j})^2$
Blocks (unadjusted)	$(b - 1)$	$\frac{1}{k} \sum_s B_s^2 - \frac{G^2}{rv}$
Error	$[mn(r - 1) - b + 1]$	By subtraction
Total.	$(mnr - 1)$	$\sum_i \sum_j \sum_s y_{ijs}^2 - \frac{G^2}{rv}$

* $K_1 = (\lambda_1 + rk - r)/k$, $K_2 = (\lambda_2 - \lambda_1)/k$, and $nK_1 + n^2K_2 = n\lambda_2 v/k$.

estimators are those given in (5.11) to (5.13) for α_i, γ_j, and δ_{ij}, and the estimator of μ is G/vr. It follows that

$$\text{Reg}(\alpha, \gamma, \delta, \beta \mid \Omega) = \textstyle\sum_i a_i A_i + \sum_j c_j C_j + \sum_i \sum_j d_{ij} D_{ij} + \sum_s b_s B_s, \tag{5.15}$$

where $A_i = \sum_j T_{ij}$, $C_j = \sum_i T_{ij}$, $D_{ij} = T_{ij}$, and B_s is defined after (3.9). Ω is the parameter space defined by the indicated restrictions.

The null hypothesis of no A-effects implies $(m - 1)$ additional linearly independent restrictions sufficient to make each $\alpha_i = 0$, and they reduce consideration to a parameter space ω_A, a subspace of Ω. Under these conditions it is necessary to minimize

$$\textstyle\sum_i \sum_j \sum_s (y_{ijs} - \mu - \gamma_j - \delta_{ij} - \beta_s)^2 \tag{5.16}$$

with use of Lagrange multipliers and the restraints (2.3) and (5.3) to (5.5). Estimators of μ, γ_j, and δ_{ij} are unchanged; a new estimator b_s' of β_s is obtained. Now

$$\text{Reg}(\gamma, \delta, \beta \mid \omega_A) = \textstyle\sum_j c_j C_j + \sum_i \sum_j d_{ij} D_{ij} + \sum_s b_s' B_s. \tag{5.17}$$

The estimators b_s and b_s' are fairly complex, but we need only note that

$$b_s' = b_s + \frac{1}{k} \sum_{\substack{i \\ \text{in } s}} n_s(i) a_i, \tag{5.18}$$

where $n_s(i)$ is the number of times a treatment combination with the ith level of the A-factor occurs in block s.

Adj. S.S. (A) is the difference, Reg $(\alpha, \gamma, \delta, \beta \mid \Omega)$ − Reg $(\gamma, \delta, \beta \mid \omega_A)$; and, using (5.15), (5.17), and (5.18), we have

$$\text{Adj. S.S. } (A) = \sum_i a_i A_i - \frac{1}{k} \sum_s \sum_{\substack{i \\ \text{in } s}} n_s(i) a_i B_s. \tag{5.19}$$

But

$$\sum_s \sum_{\substack{i \\ \text{in } s}} n_s(i) a_i B_s = \sum_i a_i \sum_{\substack{s \\ \text{with } i}} n_s(i) B_s = \sum_i a_i \sum_j B_{ij\cdot}\,,$$

and, from the definition of A_i, $A_i - \sum_j B_{ij\cdot} = \sum_j Q_{ij}$. It follows that

$$\text{Adj. S.S. } (A) = \textstyle\sum_i a_i \sum_j Q_{ij} \tag{5.20}$$

and

$$\textstyle\sum_j Q_{ij} = n\lambda_2 v a_i / k, \tag{5.21}$$

the latter result obtained from (4.4), (5.2) to (5.6), and algebraic reduction based on (vii) of Section 2. The final form for Adj. S.S. (A) given in (5.8) is now evident and the degrees of freedom are $(m - 1)$, since $(m - 1)$ additional restrictions were required to reduce Ω to ω_A. Adj. S.S. (C) and Adj. S.S. (AC) are obtained in much the same way.

It is of interest in some applications to have the variances of $(a_i - a_{i'})$, $i \neq i'$, and of $(c_j - c_{j'})$, $j \neq j'$. These variances are most easily obtained from the forms of the multipliers of $\sum_i \bar{t}_{i\cdot}^2 = \sum_i a_i^2$ and of $\sum_j \bar{t}_{\cdot j}^2 = \sum_j c_j^2$ in the analysis of variance of Table 1. It follows that

$$V(a_i - a_{i'}) = 2k\sigma^2 / n\lambda_2 v, \qquad i \neq i', \tag{5.22}$$

and

$$V(c_j - c_{j'}) = 2k\sigma^2 / m(\lambda_1 + rk - r), \qquad j \neq j'. \tag{5.23}$$

The error mean square of the analysis of variance is used to estimate σ^2 and consequently the variances of (5.22) and (5.23). Alternate derivations of (5.22) and (5.23) may be obtained through the forms (5.11) and (5.12) given the variances and covariances of the t_{ij}. Considerable algebra is involved in the derivation of these variances and covariances, and we do not include it here. It may, however, be useful to have these results and we now state without proof that

$$V(t_{ij}) = k\sigma^2 \left[\frac{(n-1)}{n(\lambda_1 + rk - r)} + \frac{(m-1)}{mn\lambda_2 v} \right], \tag{5.24}$$

$$\text{Cov } (t_{ij} t_{ij'}) = k\sigma^2 \left[\frac{m-1}{mn\lambda_2 v} - \frac{1}{n(\lambda_1 + rk - r)} \right], \qquad j \neq j', \tag{5.25}$$

and

$$\text{Cov } (t_{ij} t_{i'j'}) = -k\sigma^2 / mn\lambda_2 v, \qquad i \neq i'. \tag{5.26}$$

Efficiencies of factorial contrasts may be obtained in the same way as E_1 and E_2 in (4.11) and (4.12). The variances corresponding to (5.22) and (5.23) respectively for a randomized block design are $2\sigma^2/rn$ and $2\sigma^2/rm$, on the assumption again of equal experimental errors for the complete and incomplete block designs. The efficiency for contrasts among A-factor effects is

$$E_A = \lambda_2 v/rk \tag{5.27}$$

and the efficiency for contrasts among C-factor effects is

$$E_C = (\lambda_1 + rk - r)/rk. \tag{5.28}$$

The variance for an interaction contrast in the group divisible design is

$$(\lambda_1 + rk - r)\sigma^2/k$$

from Table 1 and is σ^2/r for the randomized block design. Consequently, the efficiency for an AC-interaction contrast is also

$$E_{AC} = (\lambda_1 + rk - r)/rk. \tag{5.29}$$

Note that $E_C = E_{AC} = E_1$. The two-associate class group divisible designs have three subclasses as noted earlier. For the singular subclass, $\lambda_1 = r$ and $E_C = E_{AC} = 1$; for the semi-regular subclass, $\lambda_2 v = rk$ and $E_A = 1$. In the next section we discuss individual comparisons and multifactor factorials. We now note, somewhat in advance, that all individual comparisons and subfactor effects of the A-factor have the efficiency E_A, those of the C-factor have efficiency E_C, and those of the AC-interaction have efficiency E_{AC}.

6. Individual comparisons and multi-factor factorials. Individual or single-degree-of-freedom comparisons are possible in much the usual way.

Let ξ be an $(m - 1)$ by m orthogonal matrix and η, an $(n - 1)$ by n orthogonal matrix used to transform the α's and γ's respectively. Contrasts on A-factor effects would be

$$\xi_u = \sum_i \xi_{iu}\alpha_i, \qquad u = 1, \cdots, (m - 1), \tag{6.1}$$

and on C-factor effects,

$$\eta_v = \sum_j \eta_{vj}\gamma_j, \qquad v = 1, \cdots, (n - 1). \tag{6.2}$$

To test the hypothesis that $\xi_u = 0$, we form the contrast

$$I_u = \sum_i \xi_{iu}\bar{t}_{i\cdot} = \sum_i \sum_j \xi_{iu}t_{ij}/n \tag{6.3}$$

and

$$\begin{aligned} \text{Adj. S.S. } (I_u) &= n\lambda_2 v\left(\textstyle\sum_i \xi_{iu}\bar{t}_{i\cdot}\right)^2/k \textstyle\sum_i \xi_{iu}^2 \\ &= \lambda_2 v\left(\textstyle\sum_i \sum_j \xi_{iu}t_{ij}\right)^2/k \textstyle\sum_i \sum_j \xi_{iu}^2 . \end{aligned} \tag{6.4}$$

Similarly, to test the hypothesis that $\eta_v = 0$, we form the contrast

$$J_v = \sum_j \eta_{vj}\bar{t}_{\cdot j} = \sum_i \sum_j \eta_{vj}t_{ij}/m \tag{6.5}$$

and

$$\text{Adj. S.S. } (J_v) = m(\lambda_1 + rk - r)\left(\textstyle\sum_j \eta_{vj}\bar{t}_{\cdot j}\right)^2 / k \textstyle\sum_j \eta_{vj}^2 \tag{6.6}$$
$$= (\lambda_1 + rk - r)\left(\textstyle\sum_i \sum_j \eta_{vj} t_{ij}\right)^2 / k \textstyle\sum_i \sum_j \eta_{vj}^2 .$$

The contrast for interaction of ξ_u and η_v is

$$(\xi\eta)_{uv} = \textstyle\sum_i \sum_j \xi_{iu}\eta_{vj}\delta_{ij} . \tag{6.7}$$

The hypothesis, $(\xi\eta)_{uv} = 0$, is tested through use of the contrast

$$(IJ)_{uv} = \textstyle\sum_i \sum_j \xi_{iu}\eta_{vj}t_{ij} \tag{6.8}$$

and

$$\text{Adj. S.S. } (IJ)_{uv} = (\lambda_1 + rk - r) \tag{6.9}$$
$$\cdot \left(\textstyle\sum_i \sum_j \xi_{iu}\eta_{vj}t_{ij}\right)^2 / k \textstyle\sum_i \sum_j (\xi_{iu}\eta_{vj})^2 .$$

Cochran's theorem [11] is sufficient to demonstrate the independence of all adjusted sums of squares, Adj. S.S. (I_u), $u = 1, \cdots, (m - 1)$, Adj. S.S. (J_v), $v = 1, \cdots, (n - 1)$, and Adj. S.S. $(IJ)_{uv}$, $u = 1, \cdots, (m - 1)$, $v = 1, \cdots, (n - 1)$, and that they are appropriate for use in analysis of variance. Each has one degree of freedom and F-tests are effected using the error mean square of Table 1.

Special definition of the matrices ξ and η permits the use of special contrasts. For example, rows of ξ and η may be defined such that contrasts on A-factor and C-factor effects measure trends (linear, quadratic, cubic, $\cdots$) over the factor levels.

Suppose the A-factor has levels which themselves are factorial combinations of other factors. Let there be p such factors, $A_1, \cdots, A_p$, with levels $m_1, \cdots, m_p$. It is only required that $m = \prod_{i=1}^{p} m_i$. Then ξ may be chosen in the obvious way so that the contrasts defined may be grouped to obtain main-effect and interaction comparisons for the subfactors of A. The corresponding adjusted sums of squares, each with one degree of freedom, may be grouped if desired to obtain Adj. S.S. (A_1) with $(m_1 - 1)$ degrees of freedom, Adj. S.S. (A_2) with $(m_2 - 1)$ degrees of freedom, Adj. S.S. for interaction of A_1 and A_2 with $(m_1 - 1)(m_2 - 1)$ degrees of freedom, etc. Alternately, these sums of squares may be computed by forming the usual two-way, three-way, etc., tables of values of $\bar{t}_{i\cdot}$ and effecting the computation as though they were observations in a single replication on factorial treatment combinations only finally multiplying the resulting sums of squares by the coefficient $n\lambda_2 v/k$ of (6.4). Similarly the C-factor may consist of factorial combinations of q factors, $C_1, \cdots, C_q$, with levels $n_1, \cdots, n_q$ such that $\prod_{j=1}^{q} n_j = n$ and appropriate contrasts and adjusted sums of squares may be obtained with proper selection of the rows of η. When ξ and η have been defined, the corresponding contrasts for interaction of A-factor and C-factor contrasts follow immediately. These in turn yield adjusted sums of squares that may be grouped to yield sums of squares for interaction of A_1 and C_1, A_1, A_2, and C_1, etc.

Now we have shown how multi-factor factorials may be used in two-associate class group divisible designs. It is also evident that fractional factorials may be used. The levels of the A-factor may be designated to be m treatment combinations of a fractional factorial which is a fraction of a full factorial with, say, hm treatment combinations, h an integer. The levels of the A-factor would then form a $(1/h)$-th fraction of the full factorial. Similarly, the n levels of the C-factor might be a fraction of a second set of factorial treatment combinations. Analysis of the resulting fractional factorial experiment would again depend only on proper specification of ξ and η. We would have r replications of a fractional factorial in the experiment. This may be a very useful system when it is necessary to use small incomplete blocks in a study.

7. Remarks. We have shown how factorials may be incorporated in group divisible partially balanced incomplete block designs with two associate classes. The factorial treatment combinations were so matched with treatments in the rectangular association schemes for these designs as to yield quite simple analyses. Other correspondences between factorial treatments and the treatments of the basic designs may be possible, but we would expect that they would result in considerably more complex analyses and in lack of orthogonality among the factorial comparisons. The problem of the recovery of inter-block information is being considered.

The group divisible subclass of the two-associate class of partially balanced incomplete block designs is only one of five subclasses given in [7]. The others listed are Simple, Triangular, Latin Square Type, and Cyclic, and comprise only a minor percentage of the designs listed in the reference. In particular, many designs of the Simple and Cyclic subclasses have values of v which are prime numbers and are not therefore suitable for factorials. Factorials have been developed in Simple and Triangular designs for special cases, but a general development has not been found.

In the view of the authors, important applications of these factorial incomplete block designs should be forthcoming. They should be useful in large animal experimentation where litter sizes sharply limit the amounts of homogeneous experimental material available. In taste testing, fatigue and other factors limit the number of samples that can be considered at a session, and these designs have applications there. In industrial experimentation, it may not be possible to make many observations while normal production is interrupted, and again use of incomplete blocks may be desirable. Some numerical examples on the uses of these designs are being prepared for an applied paper [12].

Marvin Zelen [13] did some preliminary work on the use of factorials in incomplete block designs (and subsequently has obtained additional results independent of us.) While the formulation and presentation given here are our own, we wish to acknowledge his cooperation through helpful discussions when this research was initiated.

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