THEOREM 8. *If the arrival and departure epochs of a single-counter queue are both Poisson, then the service time distribution is exponential, or a step function at* 0.

The author has had valuable discussions with A. W. Marshall and T. E. Harris in connection with this work.

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ON THE POWER OF OPTIMUM TOLERANCE REGIONS WHEN SAMPLING FROM NORMAL DISTRIBUTIONS[1]

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1. Introduction and Summary. In [1], optimum β-expectation tolerance regions were found by reducing the problem to that of solving an equivalent hypothesis testing problem. The regions produced when sampling from a k-variate normal distribution were found to be of similar β-expectation and optimum in the sense of minimax and most stringency. It is the purpose of this paper to discuss the "Power" or "Merit" of such regions, when sampling from the k-variate normal distribution.

Let $X = (X_1, \cdots, X_n)$ be a random sample point in n dimensions, where each X_i is an independent observation, distributed by $N(\mu, \sigma^2)$. It is often desirable to estimate on the basis of such a sample point a region which contains a given fraction β of the parent distribution. We usually seek to estimate the center $100\beta\%$ of the parent distribution and/or the $100\beta\%$ left-hand tail of the parent distribution.

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2. Formulation of the Power Function. Suppose sampling from $N(\mu, \sigma^2)$, where *Case* I: μ, σ^2 *unknown.* For this case the solution of the equivalent hypothesis testing problem (as formulated on p. 171 of [1]) is given by

$$\phi_y(x_1, \cdots, x_n) = 1 \quad \text{if} \quad |W| \leqq a_\beta, \qquad = 0 \quad \text{if} \quad |W| > a_\beta,$$

where

$$W = \frac{y - \bar{x}}{s_x}, \qquad \bar{x} = n^{-1} \sum x_i, \qquad s_x^2 = (n-1)^{-1} \sum_{i=1}^{n} (x_i - \bar{x})^2,$$

and $\phi_y(x_1, \cdots, x_n)$ is the characteristic function of the minimax most stringent tolerance region $S(X_1, \cdots, X_n)$. The a_β are constants chosen to give

$$S(x_1, \cdots, x_n) = [\bar{x} - a_\beta s_x, \bar{x} + a_\beta s_x]$$

size β, and are tabulated in Table I of [1].

The power of ϕ (as defined on p. 170 of [1]) and hence of S is determined by the distribution of W under the alternative of the equivalent hypothesis testing problem. That is, we have

$$\text{Power} = P_{\text{Alt.}}(|W| \leqq a_\beta), \qquad = P_{\text{Alt.}}\left(\frac{|W|}{(\alpha^2 + n^{-1})} \leqq \frac{a_\beta}{(\alpha^2 + n^{-1})}\right). \tag{2.1}$$

Now, under the alternative, $y - \bar{x}$ has variance $(\alpha^2 + n^{-1})\sigma^2$. Thus, $W / (\alpha^2 + n^{-1})^{\frac{1}{2}}$ under the alternative, is the Student's "T" variable with $(n - 1)$ degrees of freedom. Hence

$$\text{Power} = P\left(|T| \leqq \frac{a_\beta}{(\alpha^2 + n^{-1})^{1/2}}\right). \tag{2.2}$$

The power measures the "degree of confidence" we have in $S(X_1, \cdots, X_n)$ of covering the centre $100\beta\%$ of $N(\mu, \sigma^2)$, when the "desirability" of covering the centre $100\beta\%$ set is given by

$$Q_{\mu,\sigma^2}(S) = \int_S dN(\mu, \alpha^2\sigma^2), \qquad 0 < \alpha < 1.$$

For example, if it is 99.1% desirable to cover the 95% center part of $N(\mu, \sigma^2)$, then $\alpha = \frac{3}{4}$, and the power is found by (2.2) using $\alpha = \frac{3}{4}$. Values of the power for the regions S, where the desirability of the $100\beta\%$ sets are .99, are given for $\beta = .75$, .90, .95 and .975, in Table I.

As an example, consider forming S on the basis of a sample of 7. Then from the tables in [1],

$$S = [\bar{x} - 2.616\, s_x, \bar{x} + 2.616\, s_x].$$

Now, suppose we wish to have 99% confidence that $S(x_1, \cdots, x_7)$ contains 95% of $N(\mu, \sigma^2)$. Then, the "confidence" that S covers 95% of the parent dis-

TABLE I

Power of β-expectation tolerance regions,
$[\bar{x} - a_\beta s_x, \bar{x} + a_\beta s_x]$

	Measure of Desirability = .99			
α	.870167	.760906	.638572	.446594
β / n	.975	.95	.90	.75
2	.9759	.9545	.9141	.7931
3	.9776	.9618	.9361	.8564
4	.9792	.9682	.9516	.9011
5	.9806	.9733	.9576	.9154
7	.9826	.9772	.9669	.9381
11	.9848	.9809	.9763	.9586
21	.9871	.9847	.9818	.9756
31	.9880	.9863	.9842	.9795
41	.9885	.9872	.9855	.9817
61	.9890	.9881	.9869	.9842
121	.9895	.9890	.9884	.9869

tribution, that is the power of S, is found by entering Table I for $n = 7$ in the $\beta = .95$ column. The power is found to be .9772. That is, if $X_1, \cdots, X_7, Y$ are independent normally distributed chance variables, $X_1, \cdots, X_7$ having identical distributions with mean μ and standard deviation σ, Y having mean μ and standard deviation $\alpha\sigma$, then

$$\Pr(\bar{X} - 2.616\, s_x < Y < \bar{X} + 2.616\, s_x) = \begin{cases} .95 \text{ if } \alpha = 1 \\ .9772 \text{ if } \alpha = .760906. \end{cases}$$

Case II. *Mean unknown, variance known.* The minimax and most stringent tolerance region $S(X_1, \cdots, X_n)$ of similar β-expectation is given by

$$S(x_1, \cdots, x_n) = [\bar{x} - b_\beta\sigma, \bar{x} + b_\beta\sigma],$$

where σ^2 is the known value of the variance, and b_β are constants chosen to give S size β. Using the same procedure as for Case I, we have

$$\text{Power} = P\left(|Z| \leqq \frac{b_\beta}{(\alpha^2 + n^{-1})}\right), \tag{2.3}$$

where Z is the standard normal variate. Values of the power for this case are given in Table II.

Case III. *Mean known, variance unknown.* The minimax and most stringent tolerance region $S(X_1, \cdots, X_n)$ of similar β-expectation is

$$S(x_1, \cdots, x_n) = [\mu - t_{(1-\beta)/2}\, s'_x, \quad \mu + t_{(1-\beta)/2}\, s'_x],$$

TABLE II

Power of β-expectation tolerance regions,
$[\bar{x} - b_\beta \sigma, \bar{x} + b_\beta \sigma]$

	Measure of Desirability = .99			
α	.870167	.760906	.638572	.446594
n \ β	.975	.95	.90	.75
2	.9856	.9792	.9655	.9079
3	.9868	.9822	.9726	.9312
4	.9875	.9839	.9766	.9449
5	.9879	.9850	.9792	.9538
7	.9885	.9863	.9822	.9644
11	.9890	.9876	.9850	.9742
21	.9894	.9887	.9874	.9822
31	.9896	.9891	.9882	.9848
41	.9897	.9893	.9887	.9861
61	.9898	.9896	.9891	.9874
121	.9899	.9898	.9896	.9887

TABLE III

Power of β-expectation tolerance regions,
$[\mu - t_{(1-\beta)/2} s'_x, \mu + t_{(1-\beta)/2} s'_x]$, where $s'_x = \sqrt{n^{-1} \sum (x_i - \mu)^2}$

	Measure of Desirability = .99			
α	.870167	.760906	.638572	.446594
n \ β	.975	.95	.90	.75
1	.9765	.9578	.9280	.8651
2	.9787	.9678	.9535	.9243
3	.9806	.9751	.9626	.9501
4	.9821	.9771	.9695	.9581
5	.9832	.9787	.9747	.9644
7	.9847	.9811	.9779	.9733
11	.9863	.9838	.9814	.9787
21	.9879	.9865	.9851	.9835
40	.9889	.9881	.9873	.9864
60	.9892	.9887	.9882	.9875
120	.9896	.9893	.9891	.9887

where μ is the known value of the mean, t_α is the point exceeded with probability α by the Student's "T" variable with n degrees of freedom. Using a similar procedure as above, the power of these regions are clearly given by

$$\text{Power} = P(|T| \leqq t_{(1-\beta)/2}/\alpha),$$

where T is the Student's variable with n degrees of freedom. Values of this power are given in Table III. s'_x is defined as $\sqrt{n^{-1}\sum_1^n (x_i - \mu)^2}$.

3. Sampling from a k-variate normal distribution. Consider the case of sampling from a multivariate normal distribution

$$c \exp[-\tfrac{1}{2}(\omega - \mu)\Lambda(\omega - \mu)'],$$

where $\mu = (\mu_1, \cdots, \mu_k)$ and Λ^{-1} is the variance covariance matrix of $\omega = (X_1, \cdots, X_k)$. Suppose μ and Λ^{-1} are unknown, and suppose it is desired to form regions S which cover the centre part of the parent distribution. The choice of the measure of desirability is

$$Q_{\mu,\lambda^{-1}} = N(\mu, \alpha^2\Lambda^{-1}), \qquad 0 < \alpha < 1. \tag{3.1}$$

It was shown in [1] that the solution of forming these regions, that is, of an equivalent hypothesis testing problem (see p. 176 of [1]) is

$$\begin{aligned} \phi_y(\omega_1, \cdots, \omega_n) &= 1 \quad \text{when} \quad (\xi - \bar{\omega})A^{-1}(\xi - \omega)' \leqq c_\beta, \\ &= 0 \quad \text{when} \quad (\xi - \bar{\omega})A^{-1}(\xi - \bar{\omega})' > c_\beta, \end{aligned} \tag{3.2}$$

where $\bar{\omega} = n^{-1}\sum_{\alpha=1}^n \omega_\alpha$,

$$A = (n-1)^{-1}\sum_{\alpha=1}^n (\omega_\alpha - \bar{\omega})'(\omega_\alpha - \bar{\omega}),$$

and c_β are constants chosen to give the ellipsoidal region

$$S(\omega_1, \cdots, \omega_n) = \{\xi \mid (\xi - \bar{\omega})A^{-1}(\xi - \bar{\omega})' \leqq c_\beta\} \tag{3.3}$$

size β, that is, β-expectation. These regions were found to be minimax and most stringent. Letting $\gamma^2 = (\xi - \bar{\omega})A^{-1}(\xi - \bar{\omega})'$, the power clearly takes the form

$$\begin{aligned} \text{Power} &= P_{\text{Alt.}}\{\gamma^2 \leqq c_\beta\} = P_{\text{Alt.}}\left\{\frac{\gamma^2}{\alpha^2 + n^{-1}} \leqq \frac{c_\beta}{\alpha^2 + n^{-1}}\right\} \\ &= P\left\{T^2 \leqq \frac{c_\beta}{\alpha^2 + n^{-1}}\right\}, \end{aligned} \tag{3.4}$$

where T^2 is Hotelling's T^2 variable with $(n - 1)$ degrees of freedom, and Alt. refers to the Alternative hypothesis in the formulation of p. 176 of [1].

By making the transformation

$$T^2 = (n-1)\frac{k}{n-k}F,$$

it is well known that Hotelling's T^2 distribution goes into Fisher's F distribution with k, $n - k$ degrees of freedom. That is, the power of S is given by

$$\text{Power} = P\left(F \leqq \frac{n-k}{k(n-1)}\frac{c_\beta}{(\alpha^2 + n^{-1})}\right).$$

TABLE IV

Power of β-expectation tolerance regions,
$(\xi - \bar{\omega})A^{-1}(\xi - \bar{\omega})' \leqq c_\beta$

	Measure of Desirability = .99			
α	.88927	.79697	.69432	.53403
$n \backslash \beta$	.925	.95	.90	.75
3	.9755	.9531	.9105	.7810
4	.9770	.9598	.9318	.8502
5	.9784	.9661	.9522	.9022
7	.9809	.9752	.9606	.9291
11	.9838	.9794	.9751	.9578
21	.9869	.9845	.9818	.9770
30	.9880	.9865	.9847	.9812
31	.9881	.9866	.9849	.9815
32	.9882	.9868	.9851	.9818

In [1], it was shown that $c_\beta = (1 + n^{-1}) \cdot (n - 1) \cdot (k / n - k) \cdot F_{1-\beta}$, where $F_{1-\beta}$ is the point exceeded with probability $1 - \beta$ using the F distribution with k, $n - k$ degrees of freedom. Hence the regions (3.3) have power given by

$$\text{Power} = P\left(F \leqq \frac{1 + n^{-1}}{\alpha^2 + n^{-1}} F_{1-\beta}\right). \tag{3.5}$$

Values of the power function (3.5) are given for the case of sampling from the bi-variate normal distribution ($k = 2$), when the correlation coefficient ρ is zero, and desirability of the centre $100\beta\,\%$ sets is .99, in Table IV.

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THE CONVERGENCE OF CERTAIN FUNCTIONS OF SAMPLE SPACINGS[1]

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1. Introduction and summary. Suppose $g(u_1, \cdots, u_k)$ is a continuous function of its arguments, homogeneous of order r, monotonic nondecreasing in each of its

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