# STATISTICAL PROPERTIES OF INVERSE GAUSSIAN DISTRIBUTIONS. II.

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**0. Summary.** Given a fixed number $n$ of observations on a variate $x$ which has the Inverse Gaussian probability density function

$$\exp\left\{-\frac{\phi^2 x}{2\lambda} + \phi - \frac{\lambda}{2x}\right\}\sqrt{\frac{\lambda}{2\pi x^3}}, \qquad 0 < x < \infty,$$

for which $E(x) = \lambda/\phi = \mu$, it is shown how to find functions of the sample mean $m$ whose expectations can be expressed suitably in terms of the parameter $\phi$ (or $\mu$). In particular, it is shown that the conditional expectation of any unbiased estimator $\tilde{\kappa}_r$ of the $r$th cumulant $\kappa_r$ is

$$E(\tilde{\kappa}_r \mid m) = 2m(\tfrac{1}{2}\lambda n^2)^{r-1} e^{\frac{1}{2}g} \int_1^\infty (u - 1)^{2r-3} e^{-\frac{1}{2}g u^2}\, du/(r-2)!$$

where $g = \lambda n/m$. This expectation may be evaluated either by series given in the paper or by using published tables of numerical values of certain functions to which it can be related. The conditional variance of the usual mean square estimator $s^2$ of $\kappa_2$ is also found. These results give an asymptotic series for the conditional variance of a generalization $\chi_s^2 = (n-1)s^2/E(s^2 \mid m)$ of a statistic discussed by Cochran. Exact formulae for the expectation of the statistic $s^2/m^3$ and its mean square error as an estimator of $\lambda^{-1}$ are given or described. This statistic is a consistent estimator of $\lambda^{-1}$ and has asymptotically an efficiency of $\phi/(\phi + 3)$.

**1. Introduction.** An earlier paper [1], which will be called "Paper I," was mainly devoted to the characteristics of an Inverse Gaussian variate and its reciprocal and to the maximum likelihood estimation of the parameters. The work reported in that paper was started some considerable number of years ago because of some unusual features of some experimental data which had been obtained in the physics research laboratories of the University of Reading, England. At a casual examination these data showed a strong tendency for the dispersions to increase when samples were considered with increasing means. This tendency was confirmed by calculating arithmetic means and the sums of the squared deviations from the means for a large number of samples, plotting the two against one another and fitting regression curves, using some rather arbitrary assumptions about weighting. A comparable technique was used by Fisher, Thornton, and Mackenzie [2] in analysing some bacterial counts. Our case led to the question of the effect of Brownian motion in the experiments, and to some unsolved problems in theoretical statistics. Although the values

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of the parameters in the Reading data were such as to make theoretically precise solutions to these problems unnecessary, some exact and asymptotic results have recently been obtained on the conditional distributions of certain statistics at fixed values of the sample mean, and on the overall distributions of some statistics which can be based on them, on the assumption that the random variation was of the Inverse Gaussian nature which would be caused by Brownian motion. These results are of some theoretical interest even if the physical origin of the basic Inverse Gaussian family of distributions is ignored. It may also be added that in the originating physical experiments there were some relatively minor disturbing factors [3] which made the Brownian motion theory an incomplete statistical model.

**2. Some general formulae.** In this paper the standard form adopted for the probability density function of an Inverse Gaussian variate $x$ will be

$$f(x;\phi,\lambda) = \exp\left\{-\frac{\phi^2 x}{2\lambda} + \phi - \frac{\lambda}{2x}\right\}\sqrt{\frac{\lambda}{2\pi x^3}}, \tag{1}$$

with $0 < x < \infty$ (cf. (1d) of Paper I). The population mean of $x$ is $\mu = \phi/\lambda$. The integral over $(0, \infty)$ of (1), with complex values for $\phi$ and $\lambda$, is equal to unity if the real parts of $\phi^2/\lambda$ and $\lambda$ are positive. The distribution (1) will rarely appear explicitly; for we shall be mainly concerned with the sample mean and conditional distributions referred to fixed values of the sample mean.

As has previously been shown [4], a Laplacian form for a probability density function facilitates the derivation of the regression mean, against the sample mean, of any statistic which is algebraically independent of the population mean $\mu$ but whose overall expectation is a suitable known function of $\mu$. The Inverse Gaussian family is of this form and this property has already been used in Paper I. We now proceed to give some general formulae which will be used subsequently to derive some further statistical results.

We shall use $m$ to stand for the arithmetic mean of a fixed number $n$ of independent observations on (1). As was shown in Paper I, the distribution of $m$ is of the same form as (1), with $\phi$ replaced by $\phi n$. In the problems considered in this paper, many of the mathematical operations are most conveniently expressed in terms of the variables defined by

$$g = \lambda n/m, \qquad \theta = \phi n. \tag{2}$$

The probability density function of $g$ is

$$\exp\left(-\tfrac{1}{2}\theta^2 g^{-1} + \theta - \tfrac{1}{2}g\right)/\sqrt{2\pi g} = gf(g;\theta,1). \tag{3}$$

This is of the same form as the density function of $y$, the reciprocal of the Inverse Gaussian variate $x$, as given in (30) in Paper I, with $\lambda = 1$ and $\theta = 1/\mu$.

The moments of $g$ are directly obtainable from (12) and (33) of Paper I. In particular, Var $(g) = \theta + 2$ (cf. (35) of that paper). For convenience of reference, we give

$$
\begin{aligned}
&\theta E(g^{-1}) = 1\\
&\theta^3 E(g^{-2}) = E(g) = \theta + 1\\
&\theta^5 E(g^{-3}) = E(g^2) = \theta^2 + 3\theta + 3\\
&\theta^7 E(g^{-4}) = E(g^3) = \theta^3 + 6\theta^2 + 15\theta + 15\\
(4)\quad &\theta^9 E(g^{-5}) = E(g^4) = \theta^4 + 10\theta^3 + 45\theta^2 + 105\theta + 105\\
&\theta^{11} E(g^{-6}) = E(g^5) = \theta^5 + 15\theta^4 + 105\theta^3 + 420\theta^2 + 945\theta + 945\\
&\theta^{13} E(g^{-7}) = E(g^6) = \theta^6 + 21\theta^5 + 210\theta^4 + 1260\theta^3 + 4725\theta^2\\
&\qquad\qquad + 10395\theta + 10395
\end{aligned}
$$

By simple algebraic operations on these results one can easily find polynomials in $g$ whose expectations are the positive integral powers of $\theta$ from the first to the sixth.

Under fairly general conditions on the arbitrary functions $h(u)$ and $l(g)$ which appear in the following equation, it is true that

$$(5)\qquad E\left\{l(g)e^{\frac{1}{2}g}\int_1^\infty h(u)e^{\frac{1}{2}gu^2}\,du \mid \theta\right\} = e^\theta\int_1^\infty e^{-\theta u}u^{-1}h(u)E\{l(Gu^{-2}) \mid \theta u\}\,du,$$

where $G$ is a random variable with $Gf(G; Ou, 1)$ as its probability density function. It is both necessary and sufficient that the integrals and expectations in (5) exist for all the required values of the variables on which they depend. A proof may be established by expressing the expectation on the left as an integral and reversing the order of the integrations with respect to $g$ and $u$.

To develop the first application of (5), take $l(g) = g^{-1}$. Then, since

$$E\{G^{-1}u^2 \mid \theta u\} = \theta^{-1}u,$$

therefore

$$(6)\qquad E\left\{g^{-1}e^{\frac{1}{2}g}\int_1^\infty h(u)e^{-\frac{1}{2}gu^2}\,du \mid \theta\right\} = \theta^{-1}\int_0^\infty e^{-\theta v}h(v+1)\,dv.$$

From the effective uniqueness of the inverse of the Laplace transform, the variate whose overall expectation is taken on the left of (6) must be equal to the conditional expectation, with a fixed value of $g$ or $m$, of a variate whose overall expectation can be equated to the expression on the right of (6). The problem of finding the conditional expectation, or regression function, thus becomes the relatively simple one of expressing the overall expectation in terms of $\theta$ and then inverting the Laplace transform of $h(v + 1)$ which appears on the right of (6). In this procedure for inversion it is permissible to take $\theta$ to be complex, if necessary, so long as its real part is positive.

If $\tilde{T}$ is some unbiased estimator of a function $T(\theta)$ of $\theta$ of known form, it therefore follows that the conditional expectation of this estimator is

$$
\begin{aligned}
E(\tilde{T} \mid g) &= g^{-1}e^{\frac{1}{2}g}\int_1^\infty h(u)e^{\frac{1}{2}-gu^2}\,du = g^{-1}\int_0^\infty h(v+1)e^{-g(v+\frac{1}{2}v^2)}\,du\\
(7)\qquad &= g^{-1}\sum_{r=0}^\infty \frac{(-\frac{1}{2}g)^r}{r!}\frac{\partial^{2r}}{\partial g^{2r}}[gT(g)].
\end{aligned}
$$

However, this symbolic result (7) is not always the most convenient one to use, as it sometimes leads to infinite series which can be avoided by introducing functions for which tables of numerical values have been published. When

$$T(\theta) = C\theta^{-s} \tag{8}$$

$C$ and $s$ being independent of $\theta$, the inversion of the transform gives, in an obvious notation,

$$E_g^{-1}(C\theta^{-s}) = E(\tilde{T} \mid g) = Cg^{-1}e^{\frac{1}{2}g} \int_1^{\infty} (u - 1)^{s-2} e^{-\frac{1}{2}gu^2} \, du/(s - 2)! \tag{9}$$

$$= Cg^{-\frac{1}{2}(s+1)} e^{\frac{1}{2}g} Hh_{s-2}(g^{1/2}), \tag{10}$$

where $Hh_{s-2}(\cdot)$ represents the Hermite polynomial function as given by Jeffreys and Jeffreys ([5], 23.081, who give also convergent and asymptotic series expansions of it. When $s$ is an integer greater than 2, these expansions of the Hermite polynomials are infinite series. However, the right-hand side of (9) can be expressed in closed form in terms of more extensively tabulated functions by integrating it by parts. A convenient general formula, to avoid some rather repetitious work if the requisite Hermite polynomials are not known in this form, is

$$\int_a^{\infty} f(u) e^{-u^2/2b} \, du = (1 - bDP)^{-1} f(u) \mid_{u=0} \int_a^{\infty} e^{-u^2/2b} \, du + bP(1 - bDP)^{-1} f(u) \mid_{u=a} e^{-a^2/2b}. \tag{11}$$

Here $a$ and $b$ are positive constants, $f(u)$ is a polynomial function of $u$, $P$ is an operator with the general property typified by $PF(u) = [F(u) - F(0)]/u$, $F(u)$ being a polynomial in $u$, and $D$ is the operator of differentiation with respect to $u$. The compound symbol $(1 - bDP)^{-1}$ is an abbreviation for the series $1 + bDP(1 + bDP(1 + bDP(1 + \cdots)))$, which $= 1 + bDP + b^2DPDP + b^3DPDPDP + \cdots$.

In (9) the general formula (11) would take $f(u) = (u - 1)^{s-2}$, $a = 1$, $b = g^{-1}$. As an example, suppose that $T(\theta) = \theta^{-6}$. Then we shall have

$$\begin{aligned}
f(u) &= u^4 - 4u^3 + 6u^2 - 4u + 1, &&\text{which} = \phantom{-}1 \text{ at } u = 0\\
Pf(u) &= u^3 - 4u^2 + 6u - 4 &&, \text{which} = -1 \text{ at } u = 1\\
DPf(u) &= 3u^2 - 8u + 6 &&, \text{which} = \phantom{-}6 \text{ at } u = 0\\
PDPf(u) &= 3u - 8 &&, \text{which} = -5 \text{ at } u = 1\\
DPDPf(u) &= 3 &&, \text{which} = \phantom{-}3 \text{ at } u = 0
\end{aligned}$$

Thus

$$\begin{aligned}
E_g^{-1}(\theta^{-6}) &= g^{-1}e^{\frac{1}{2}g} \int_1^{\infty} (u - 1)^4 e^{-\frac{1}{2}gu^2} \, du/4! \\
&= g^{-1}\{(1 + 6g^{-1} + 3g^{-2})I - g^{-1} - 5g^{-2}\}/24,
\end{aligned} \tag{12}$$

where

$$I = e^{\frac{1}{2}g} \int_1^\infty e^{-\frac{1}{2}gu^2}\, du = e^{-\frac{1}{2}g} \sqrt{2\pi/g} \int_{\sqrt{g}}^\infty e^{-\frac{1}{2}v^2}\, dv/\sqrt{2\pi}, \tag{13}$$

for which Laplace found a continued fraction expansion (cf. [6], p. 263; [7], p. v; [8], Eq. (92.11)). It is of interest that the expression within the braces in (12) is the difference between the numerator of one of the convergents of this continued fraction and $I$ times the denominator of the same convergent, so that a rather large number of significant figures is needed in $I$ if (12) is to be evaluated accurately. A comparable situation occurs with other formulae in this field of study.

The following special formulae, which are obtainable by combining (4) and (5) with $h(u) = 1$, are also useful:

$$E(I) = e^\theta \int_\theta^\infty x^{-1} e^{-x}\, dx, \tag{14}$$

which has a well-known asymptotic series expansion, obtainable by integrating by parts, and may be expressed as a continued fraction (cf. [8], Eq. (92.16)); and

$$\begin{aligned}
2\cdot 1!E(gI) &= -\theta^2 E(I) + \theta + 1\\
2^2\cdot 2!E(g^2I) &= \theta^4 E(I) - \theta^3 + \theta^2 + 6\theta + 6\\
2^3\cdot 3!E(g^3I) &= -\theta^6 E(I) + \theta^5 - \theta^4 + 2\theta^3 + 42\theta^2 + 120\theta + 120\\
2^4\cdot 4!E(g^4I) &= \theta^8 E(I) - \theta^7 + \theta^6 - 2\theta^5 + 6\theta^4 + 360\theta^3 + 2040\theta^2\\
&\quad + 5040\theta + 5040\\
2^5\cdot 5!E(g^5I) &= -\theta^{10}E(I) + 0!\theta^9 - 1!\theta^8 + 2!\theta^7 - 3!\theta^6 + 4!\theta^5\\
&\quad + 372\theta^4 + 3528\theta^3 + 15624\theta^2 + 9!\theta + 9!
\end{aligned} \tag{15}$$

In deriving these last formulae (15), it was found convenient to use, on the right side of (5), the identity

$$\begin{aligned}
e^\theta \int_\theta^\infty \sum_{i=0}^{r} (i!c_i x^{-i}) x^{-1} e^{-x}\, dx &= e^\theta \int_\theta^\infty x^{-1}e^{-x}\, dx(c_0 - c_1 + c_2 - c_3 + \cdots)\\
&\quad + 0!\theta^{-1}(c_1 - c_2 + c_3 - \cdots)\\
&\quad + 1!\theta^{-2}(c_2 - c_3 + \cdots)\\
&\quad + 2!\theta^{-3}(c_3 - \cdots)\\
&\quad + \cdots + (r-1)!\theta^{-r}c_r.
\end{aligned} \tag{16}$$

For completeness, the following results are also recorded:

$$\begin{aligned}
E(g^{-1}I) &= \theta^{-2}\\
E(g^{-2}I) &= \theta^{-3} + 2\theta^{-4}\\
E(g^{-3}I) &= \theta^{-4} + 5\theta^{-5} + 8\theta^{-6}\\
E(g^{-4}I) &= \theta^{-5} + 9\theta^{-6} + 33\theta^{-7} + 48\theta^{-8}.
\end{aligned} \tag{17}$$

In conjunction with (4), these results enable polynomials in $g^{-1}$ to be found which, when multiplied by $I$, give expressions whose expectations are the negative integral powers of $\theta$ from the first to the seventh. The results for the first, third, fifth, sixth and seventh negative powers are given essentially in (12) and (23).

It is also convenient to introduce a further new variable to simplify the presentation of the results in the following sections. We write

$$\text{(18a)} \quad J = 1 - gI$$

$$\text{(18b)} \quad = 1 - ge^{\frac{1}{2}g} \int_1^\infty e^{-\frac{1}{2}gu^2}\, du$$

$$\text{(19)} \quad \begin{aligned} &= g^{-1} - 3g^{-2} + \cdots + (-1)^{r+1} 3\cdot 5 \cdots (2r-1) g^{-r} \\ &\quad + (-1)^r 3\cdot 5 \cdots (2r-1)(2r+1) g^{-r} e^{\frac{1}{2}g} \int_1^\infty u^{-2r-2} e^{-\frac{1}{2}gu^2}\, du. \end{aligned}$$

It can be shown that $J$ decreases monotonely from 1 to 0, and that $gJ$ increases monotonely from 0 to 1, when $g$ increases from 0 to infinity.

The form (19) gives an asymptotic series which is satisfactory when $g$ is sufficiently large. A useful expansion for moderately large values of $g$ is the continued fraction

$$\text{(20)} \quad J = \cfrac{g^{-1}}{1 + \cfrac{3g^{-1}}{1 + \cfrac{2g^{-1}}{1 + \cfrac{5g^{-1}}{1 + \cfrac{4g^{-1}}{1 + \text{etc.}}}}}} = \cfrac{1}{g + \cfrac{3}{1 + \cfrac{2}{g + \cfrac{5}{1 + \cfrac{4}{g + \text{etc.}}}}}}$$

The constants in the successive partial numerators are the positive integers, following the sequence with alternating reversals 1; 3, 2; 5, 4; 7, 6; 9, 8; etc. Alternatively Laplace's continued fraction could be used to find $I$. When $g$ approaches zero,

$$\text{(21)} \quad J = 1 - (\tfrac{1}{2}\pi g)^{1/2} + 0(g).$$

In the notation adopted in the National Bureau of Standards' Tables of Probability Functions ([9], p. xix),

$$I = g^{-1/2} F(g^{1/2}), \qquad J = 1 - L(2^{-1/2} g^{1/2}).$$

Hence a published table of either $F(x)$ or $L(x)$ may be used to shorten the computations, provided it gives enough significant digits. Table I gives $I$, $J$ and $gJ$ for certain values of $g$, chosen because Sheppard's 1939 tables [7] could be used for them without interpolation. Burgess's Table 3 [6], which is unfortunately 'seriously infested with error' according to page x of Sheppard's tables [7], would similarly lead directly to values for which $\frac{1}{2}g$ is an exact square.

TABLE I

| $g$ | $I$ | $J$ | $gJ$ |
|---|---|---|---|
| 1 | .65567 95424 | .34432 04576 | .34432 04576 |
| 4 | .21068 46146 | .15726 15414 | .62904 61657 |
| 9 | .10153 00996 | .08622 91039 | .77606 19348 |
| 16 | .05916 30957 | .05339 04683 | .85424 74935+ |
| 25 | .03856 16209 | .03595 94764 | .89898 69106 |
| 36 | .02706 29435− | .02573 40346 | .92642 52463 |
| 49 | .02001 48834 | .01927 07158 | .94426 50756 |
| 64 | .01539 14954 | .01494 42939 | .95643 48119 |
| 81 | .01219 85870 | .01191 44568 | .96507 10004 |
| 100 | .00990 28596 | .00971 40353 | .97140 35283 |

**3. Conditional means and variances of certain statistics.** The cumulants of the general Inverse Gaussian variate were given in (9) of Paper I. Since they are of the form of (8), we can use (9) of the present paper and get

$$E(\tilde{\kappa}_r \mid m; \lambda, n) = 2m(\tfrac{1}{2}\lambda n^2)^{r-1} e^{\frac{1}{2}g} \int_1^{\infty} (u-1)^{2r-3} e^{-\frac{1}{2}gu^2}\, du/(r-2)!, \tag{22}$$

from which

$$\begin{aligned} E_m^{-1}(\kappa_2) &= E(\tilde{\kappa}_2 \mid m; \lambda, n) = nm^2 J = \lambda^{-1} m^3 g J = \lambda^2 n^3 g^{-2} J \\ E_m^{-1}(\kappa_3) &= E(\tilde{\kappa}_3 \mid m; \lambda, n) = \tfrac{1}{2} n^2 m^3 \{(g+3)J - 1\} \\ E_m^{-1}(\kappa_4) &= E(\tilde{\kappa}_4 \mid m; \lambda, n) = \tfrac{1}{8} n^3 m^4 \{(g^2 + 10g + 15)J - g - 7\}. \end{aligned} \tag{23}$$

These formulae may be verified by using (4) and (17). The terms within the braces in (23) are the same as occur in the successive convergents of the continued fraction expansion (20), so that, in a similar way as the computation of (12), high precision is needed in $J$ to give accurate numerical results. For example, when $g = 25$, if we took $I = 0.0385616$ or $J = 0.035959$ or $L = 0.964041$, which are correct according to the usual rounding-off rules to the last digit shown, the rounding error (which is 0.21 of the last place shown for $I$, and 0.48 of the last place shown for $J$ and $L$) would give a value for $E_m^{-1}(\kappa_4)$ which would be 12 per cent too high if I were used, or 11 per cent too low if $J$ or $L$ were used.

When $g$ is large, the asymptotic expansion of the Hermite polynomial given by Jeffreys and Jeffreys ([5], 23.082) gives

$$E(\tilde{\kappa}_r \mid m; \lambda, n) \sim \frac{4\lambda}{m}\left(\frac{m^2}{2\lambda}\right)^r \sum_{i=0}^{\infty} \frac{(2i + 2r - 3)!}{i!(r-2)!(-2g)^i} \tag{24}$$

From this, or by using the asymptotic series expansion (19) for $J$,

$$E(\tilde{\kappa}_2 \mid m; \lambda, n) \sim \lambda^{-1} m^3 (1 - 3g^{-1} + 15g^{-2} - 105g^{-3} + 945g^{-4} - \cdots)$$

$$E(\tilde{\kappa}_3 \mid m; \lambda, n) \sim 3\lambda^{-2} m^5 (1 - 10g^{-1} + 105g^{-2} - 1260g^{-3}$$

$$+ 17325g^{-4} - \cdots) \tag{25}$$

$$E(\tilde{\kappa}_4 \mid m; \lambda, n) \sim 15\lambda^{-3}m^7(1 - 21g^{-1} + 378g^{-2} - 6930g^{-3} + 135135g^{-4} - \cdots).$$

**4. Conditional variance of Cochran's $\chi_s^2$ statistic.** By the same technique of inverting a Laplace transform, as was shown previously ([4], p. 48), the conditional variance of the usual unbiased mean square estimator—viz, $s^2 = \sum(x - m)^2/(n - 1)$—of $\kappa_2$ can be found. We first find

$$E(s^4 \mid m; \lambda, n) = \frac{n+1}{n-1} E_m^{-1}(\kappa_2^2) + \frac{1}{n} E_m^{-1}(\kappa_4) \tag{26}$$

for which an exact expression in terms of $I$ or $J$ can be found by using (12) and (23), since $\kappa_2^2 = \theta^{-6}\lambda^4 n^6$. The required conditional variance is obtainable by subtracting the square of $E(s^2 \mid m; \lambda, n)$ from (26). The asymptotic form is

$$\operatorname{Var}(s^2 \mid m; \lambda, n) \sim \frac{2m^6}{(n-1)\lambda^2}\left\{1 + \frac{3(n-6)m}{\lambda n} - \frac{6(12n-47)m^2}{\lambda^2 n^2} + \frac{30(47n-152)m^3}{\lambda^3 n^3} - \cdots\right\}. \tag{27}$$

For comparison with some of the results already given [4] for the Inverse Poisson (or chi-square type) and the Poisson types of distribution, and also to provide a more direct comparison with the exact chi-square distribution which occurs in the distribution of the maximum likelihood estimator of $\lambda$, we may consider a measure of dispersion equivalent to that studied by Cochran [10] for other distributions:

$$\chi_s^2 = (n - 1)s^2/E(s^2 \mid m; \lambda, n). \tag{28}$$

Obviously

$$E(\chi_s^2 \mid m; \lambda, n) = n - 1.$$

By using (25) and (27), we get, when $g$ is large,

$$\operatorname{Var}(\chi_s^2 \mid m; \lambda, n) \sim 2(n-1)\left\{1 + \frac{3m}{\lambda}\left(1 - \frac{4}{n}\right) + \frac{54m^2}{\lambda^2 n}\left(1 - \frac{59}{18n}\right) + \right\} \cdots. \tag{29}$$

It will be noticed that the absolute values of the leading terms in the series in both (27) and (29) are minimized, as functions of the sample size $n$, and simultaneously become very insensitive to the precise value of $\lambda$, when $n$ is quite small—between 3 and 6 approximately. With the Inverse Poisson distribution the second term in the series corresponding to (29) was found to vanish when $n = 5$. It may therefore be surmised that the standard chi-square distribution will be a good approximation to that of $\chi_s^2$ with samples of about this size, with either the Inverse Gaussian or the Inverse Poisson distribution, assuming that the correct regression formula $E_m^{-1}(\kappa_2)$ is used in (28). An approximate confidence or fiducial interval for $\lambda$ may be derived from this result.

**5. An approximate estimator of $\lambda^{-1}$.** The statistic $s^2m^{-3}$ might be used as an estimator of $\lambda^{-1}$, as an alternative to the estimator of maximum likelihood discussed in Paper I. The conditional mean and variance of $s^2m^{-3}$ are obtainable from the above results, to provide a partial check on its suitability for this purpose. Thus from (23),

$$E(s^2m^{-3} \mid m; \lambda, n) = \lambda^{-1}gJ = \lambda^{-1}(g - g^2I). \tag{30}$$

The series expansions of this formula and of the formula for the conditional variance are obtainable immediately from (25) and (27). Both the expectation and the variance depend to some extent on $m$, in contrast to the corresponding results which follow from (22) in Paper I, viz,

$$E\left\{\sum_{i=1}^{n} (x_i^{-1} - m^{-1})/(n - 1) \mid m; \lambda, n\right\} = 1/\lambda, \tag{31}$$

$$\operatorname{Var}\left\{\sum_{i=1}^{n} (x_i^{-1} - m^{-1})/(n - 1) \mid m; \lambda, n\right\} = 2/\lambda^2(n - 1). \tag{32}$$

The overall expectation of $s^2m^{-3}$ is, from (4), (15) and (30),

$$E\{s^2m^{-3} \mid \phi, \lambda, n\} = [2 + 2\phi n - (\phi n)^2 + (\phi n)^3 - (\phi n)^4 e^{\phi n} \int_{\phi n}^{\infty} x^{-1}e^{-x}\, dx]/8\lambda \tag{33}$$

$$= \frac{1}{\lambda}\left\{1 - \frac{1}{2^2 \cdot 2!}\left[\frac{4!}{\phi n} - \frac{5!}{(\phi n)^2} + \cdots + \frac{(-1)^{r-1}(r - 1)!}{(\phi n)^{r-4}} + (\phi n)^4 e^{\phi n} \int_{\phi n}^{\infty} \frac{(-1)^r r!}{x^{r+1}} e^{-x}\, dx\right]\right\}. \tag{34}$$

The formula (34) might be used as an asymptotic series for computations, or a continued fraction might be found.

The conditional mean square error, $E(s^4m^{-6} - 2\lambda^{-1}s^2m^{-3} + \lambda^{-2} \mid m; \lambda, n)$, may be evaluated by using the results of (23) and (26). The exact overall mean square error can be found from it by using (4) and (15), but the full expression is too lengthy to be given here. Its asymptotic series expansion is

$$E\{(s^2m^{-3} - \lambda^{-1})^2 \mid \phi, \lambda, n\} \sim \frac{2}{\lambda^2(n - 1)}\left\{1 + \frac{3(n - 6)}{\phi n} + \frac{15(5n - 33)}{(\phi n)^2} + \frac{105(27n - 33)}{(\phi n)^3} + \cdots\right\}. \tag{35}$$

This formula is sufficient to show that $s^2m^{-3}$ is a "consistent" estimator of $\lambda^{-1}$. By comparison with (31) and (32), its efficiency, on the basis of the mean square error or the variance, is seen to be asymptotically $\phi/(\phi + 3)$.

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