ON QUEUES WITH POISSON ARRIVALS

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1. Summary. The system to be studied consists of a service unit and a queue of customers waiting to be served. Service-times of customers are independent, nonnegative variates with the common distribution $B(v)$ having a finite first moment b_1. Customers arrive in a Poisson process (see Feller [4], p. 364) of intensity λ; they form a queue and are served in order of arrival, with no defections from the queue. For previous work on this queueing system see for instance Pollaczek [11], Khintchine [9], Lindley [10], Kendall [7], [8], Smith [12], Bailey [1], and Takács [14].

Let $W(t)$ be the time a customer would have to wait if he arrived at t. The forward Kolmogorov equation for the distribution of $W(t)$ is solved in principle by the use of Laplace integrals, and $E\{\exp\{-sW(t)\}\}$ is determined in terms of $W(0)$ and the root of a possibly transcendental equation. It is shown that any analytic function of the root can be expanded in Lagrange's series, which provides a way of actually computing the transition probabilities of the process. Let z be the first zero of $W(t)$. A series for $E\{\exp\{-\tau z\}\}$ is obtained, and it is proved that $\text{pr}\{z < \infty\} = 1$ if and only if $\lambda b_1 \leqq 1$. From a functional relation between $E\{W(t)\}$ and $\text{pr}\{W(t) = 0\}$ the covariance function R of $W(t)$ is determined. If the service-time distribution $B(v)$ has a finite third moment, then R is absolutely integrable, and the spectral distribution of $W(t)$ is absolutely continuous.

2. The distributions of waiting-time and busy-time. Let $W(t)$ be the instantaneous waiting-time. That is, let $W(t)$ be the time that a customer arriving at t would have to wait before beginning his service. Evidently $W(t)$ jumps upward discontinuously every time someone arrives who has a nonzero service time. Otherwise $W(t)$ approaches 0 with slope -1 until it jumps again or reaches 0. At 0 it stays $=0$ until another jump occurs. The magnitudes of the jumps are the (independent) service-times of the customers arriving at the jumps. $W(t)$ is a continuous parameter Markov process of the mixed type considered in Feller [5]. Let $P(w, t) = \text{pr}\{W(t) \leqq w\}$. As shown in Takács [14], the forward Kolmogorov equation of the process is

$$\frac{\partial P(w, t)}{\partial t} = \frac{\partial P(w, t)}{\partial w} - \lambda P(w, t) + \lambda \int_{0-}^{w} P(w - v, t)\, dB(v), \tag{2.1}$$

and if $\varphi(s, t) = E\{\exp\{-sW(t)\}\}$ for $\text{Re}(s) \geqq 0$, we obtain

$$\frac{\partial \varphi(s, t)}{\partial t} = \varphi(s, t)[s - \lambda(1 - B^*)] - sP(0, t), \tag{2.2}$$

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where $B^*(s)$ is the Laplace-Stieltjes transform of $B(v)$. Let $\varphi^*(s, \tau)$ be the Laplace transform of φ with respect to t, and let $P^*(\tau)$ be that of $P(0, t)$.

Then the Kolmogorov equation implies

$$\varphi^* = \frac{\varphi(s, 0) - sP^*}{\tau - s + \lambda[1 - B^*]}. \tag{2.3}$$

By the busy-time, we mean the epoch of the first zero of $W(t)$ subsequent to 0, when $W(0)$ is any admissible starting point. The busy-time can be investigated in terms of a modified process which is like $W(t)$ except that it stays at 0 once it arrives there. Let z be the first zero in $t \geqq 0$, and define

$$Z(t) = W(t), \qquad \text{for } t \leqq z,$$
$$Z(t) = 0, \qquad \text{for } t > z.$$

Let $F(u, t) = \text{pr}\{Z(t) \leqq u\}$; the forward Kolmogorov equation for the process is

$$\begin{aligned}\frac{\partial F(u, t)}{\partial t} = \frac{\partial F(u, t)}{\partial u} &- \lambda[F(u, t) - F(0, t)] \\ &+ \lambda \int_{0-}^{u} F(u - v, t)\, dB(v) - \lambda F(0, t)B(u).\end{aligned} \tag{2.4}$$

Let $\psi(s, t) = E\{\exp\{-sZ(t)\}\}$; let $\psi^*(s, \tau)$ be the Laplace transform (with respect to t) of ψ, and also let $F^*(\tau) = E\{e^{-\tau z}\}$, $f^* = \tau F^*$, for $\text{Re}(\tau) > 0$.

Then the Kolmogorov equation 2.4 yields

$$\psi^* = \frac{\psi(s, 0) - f^*[s - \lambda + \lambda B^*]}{\tau - s + \lambda[1 - B^*]}. \tag{2.5}$$

To solve for the unknown functions P^* and F^* we argue that the transforms φ^* and ψ^* must converge (cf. Bailey [1]) for $\text{Re}(s) > 0$ whenever $\text{Re}(\tau) > 0$, and that in this region zeros of $\tau - s + \lambda[1 - B^*]$ must coincide with zeros of the respective numerators. We show that there is an unique zero $\eta(\tau)$ of

$$\tau - s + \lambda[1 - B^*]$$

in $\text{Re}(s) > 0$ for $\text{Re}(\tau) > 0$. Choose a real δ such that $0 < \delta < \text{Re}(\tau)$, and a real $\epsilon > \text{Re}(\tau)$. Consider the line $\text{Re}(s) = \delta$, and the circle, with center at $\tau + \lambda$, defined by $|\tau - s + \lambda| = \lambda + \epsilon$. Define a contour C to be the circle when $\text{Re}(s) > \delta$, and to be the line when $\text{Re}(s) = \delta$. On the circle, we have the inequality

$$|\tau - s + \lambda| = \lambda + \epsilon > \lambda \geqq |\lambda B^*(s)|.$$

And on the line $\text{Re}(s) = \delta$:

$$|\tau - s + \lambda| \geqq \text{Re}(\tau) - \delta + \lambda > \lambda \geqq |\lambda B^*(s)|,$$

so that the inequality

$$|\tau - s + \lambda| > |\lambda B^*(s)|$$

holds over the whole contour C. Now $\tau - s + \lambda$ has no zeros on $\mathrm{Re}(s) = \delta$, nor any on the circle $|\tau - s + \lambda| = \lambda + \epsilon$; and $B^*(s)$ is single-valued and analytic in $\mathrm{Re}(s) > 0$. So by Rouché's theorem we conclude that $\tau - s + \lambda$ and $\tau - s + \lambda - \lambda B^*(s)$ have the same number of zeros in $\mathrm{Re}(s) > 0$, namely one, because δ can be made arbitrarily small, and ϵ arbitrarily large.

It follows that, with $\eta = \eta(\tau)$,

$$P^*(\tau) = \frac{\varphi(\eta, 0)}{\eta} = \frac{E\{e^{-\eta W(0)}\}}{\eta}, \tag{2.6}$$

$$F^*(\tau) = \psi(\eta, 0) = E\{e^{-\eta Z(0)}\}. \tag{2.7}$$

In the proof above we saw that $|\tau - s + \lambda| > |\lambda B^*(s)|$, if s is on the contour C. So by Lagrange's expansion (p. 133 of [16]), for any function Γ analytic on and inside C, we have

$$\Gamma(\eta) = \Gamma(\tau + \lambda) + \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \frac{d^{n-1}}{ds^{n-1}} \left[\frac{d\Gamma}{ds} (B^*(s))^n\right]_{s=\tau+\lambda}; \tag{2.8}$$

this expansion is valid for $\mathrm{Re}(\tau) > 0$, and provides a way of actually evaluating P^* and F^*. Except for the matter of inverting transforms, the solutions for the distributions of $W(t)$, $Z(t)$, and z are complete. It is easy to see that both φ^* and ψ^* can be inverted explicitly as Laplace transforms with respect to τ and give rise to exponential functions which, together with P^* and F^*, determine φ and ψ in terms of initial conditions.

The results of this section may be collected in the following statements.

THEOREM 1. *The function* $\varphi(s, t) = E\{\exp\{-sW(t)\}\}$ *is determined as the solution to Eq.* (2.2) *by the conditions*

(i)
$$\varphi(s, t) = \varphi(s, 0) \exp\{st - \lambda t[1 - B^*]\} - s\int_0^t P(0, t - y) \exp\{sy - \lambda y[1 - B^*]\}\, dy,$$

(ii)
$$P^*(\tau) = \int_0^\infty e^{-\tau t} P(0, t)\, dt = \eta^{-1}\varphi(\eta, 0) = \eta^{-1} E\{e^{-\eta W(0)}\},$$

where $\eta = \eta(\tau)$ *is the unique root of* $\tau - \eta + \lambda = \lambda B^*(\eta)$ *in the right half-plane.*

THEOREM 2. *The function* $\psi(s, t) = E\{\exp\{-sZ(t)\}\}$ *is determined by the conditions*

(i)
$$\psi(s, t) = \psi(s, 0) \exp\{st - \lambda t[1 - B^*]\} - [s - \lambda + \lambda B^*]\int_0^t F(0, t - y) \exp\{sy - \lambda y[1 - B^*]\}\, dy,$$

(ii) $$\int_0^\infty e^{-\tau t}F(0, t)\, dt = \tau\psi(\eta, 0) = \tau E\{e^{-\eta Z(0)}\},$$

where η *is as in Theorem* 1.

THEOREM 3. *If* Γ *is analytic in the open right half-plane and* η *is as in Theorem* 1, *then* $\Gamma(\eta)$ *may be expanded by Lagrange's series.*

3. The probability that z **is finite.** Since $F^* = E\{e^{-\tau z}\} = \psi(\eta, 0)$ where η is the root of $\tau - \eta + \lambda[1 - B^*(\eta)] = 0$, it seems natural to consider Tauberian and Abelian theorems in an effort to find the probability that z is finite, and to ascertain the existence of moments. We therefore turn attention to the behavior of η as $\tau \to 0$ along the real axis. There is an advantage to considering, instead of η, the linear function ξ of it defined by $\eta = \lambda(1 - \xi)$. Set $K(s) = B^*(\lambda - \lambda s)$, so that K generates a discrete probability distribution with mean λb_1; the equation for η may now be rewritten

$$\frac{\tau}{\lambda} + \xi = K(\xi),$$

and this fact suggests that as $\tau \to 0$ along the real axis, ξ approaches a root of the familiar equation from branching-process theory, $\xi = K(\xi)$. Let ζ be the least nonnegative real root of $\xi = K(\xi)$; for properties of ζ see Feller [4] or Harris [6]. We now show that $\xi \to \zeta$ as $\tau \to 0$ along the real axis.

If τ is real then so is ξ; for if not, then η is conjugate and not unique. Also, if $\tau > 0$, then $\xi < \zeta$, because $\tau > 0$ implies $K(\xi) > \xi$, and in $\xi < 1$ this is possible only if $\xi < \zeta$, since $K(0) > 0$ and $\xi = K(\xi)$ has at most two roots in $(0, 1)$, one of them being at 1. To show that $0 < \tau < \tau'$ implies $\xi(\tau) > \xi(\tau')$, write $\xi = \xi(\tau)$, $\xi' = \xi(\tau')$. Then the hypothesis and $\xi < \zeta$, $\xi' < \zeta$ imply

$$K(\xi) - K(\xi') < \xi - \xi'.$$

Now $K'(y)$ is steadily increasing in $0 < y < 1$, so if for some u we have both $u < \zeta$ and $K'(u) > 1$, then $K(u) > u$ and $K(1) > 1$; so $K'(y) \leqq 1$ for $y < \zeta$. Now if $\xi \leqq \xi'$, this would imply

$$K(\xi') - K(\xi) \leqq \xi' - \xi,$$

which is impossible. It remains to show that given $u < \zeta$, there exists a $\tau > 0$ such that $\xi(\tau) > u$. The equation $x = \lambda[K(u) - u]$ uniquely determines an $x > 0$, and for this x we must have $\xi(x) = u$, or else $\tau - \eta + \lambda[1 - B^*(\eta)] = 0$ does not have a unique root η. If now $0 < \tau < x$, then $\xi(\tau) > \xi(x) = u$, as was to be proved.

It follows that as $\tau \to 0$ along the real axis, $\eta \to 0$ or $\lambda(1 - \zeta)$ according as $\lambda b_1 \leqq 1$ or $\lambda b_1 > 1$. We are now in a position to prove that

$$\text{pr}\,\{z < \infty\} = \lim_{t\to\infty} F(0, t) = E\{\exp\,\{\lambda(\zeta - 1)Z(0)\}\}.$$

As $\tau \to 0$ along the real axis, the continuity of ψ yields

$$\begin{aligned} F^*(\tau) &\to \psi(\lambda(1 - \zeta), 0) \\ &\to E\{\exp\{\lambda(\zeta - 1)Z(0)\}\}. \end{aligned}$$

Since $F(0, t)$ is nondecreasing, so is

$$\int_0^t xF(0, dx);$$

thus $F(0, t)$ and F^* satisfy the hypothesis of Theorem 4.5 of Widder [15]. This proves

THEOREM 4. *The probability that the first zero z of $W(t)$ is finite is*

$$\text{pr}\,\{z < \infty\} = \lim_{\tau \to 0} F^*(\tau) = E\{\exp\,\{\lambda(\zeta - 1)Z(0)\}\}.$$

This limit is 1 if $\lambda b_1 \leqq 1$, and is <1 if $\lambda b_1 > 1$.

A discussion similar to the above has been given by Takács [14] for the case $\varphi(s, 0) = B^*(s)$, and this case is also treated by Kendall [7]. We mention in addition the following results, provable by simple Abelian arguments: If $\lambda b_1 < 1$ and $E\{Z(0)\} < \infty$, then

$$E\{z\} = \frac{E\{Z(0)\}}{1 - \lambda b_1};$$

if $\lambda b_1 \geqq 1$, or $E\{Z(0)\} = \infty$, then $E\{z\} = \infty$.

4. The expectation of $W(t)$.

THEOREM 5. *If $E\{W(0)\} < \infty$, then $E\{W(t)\}$ exists for $t > 0$ and is given by*

$$E\{W(t)\} = E\{W(0)\} + \int_0^t [P(0, u) - 1 + \lambda b_1]\, du. \tag{4.1}$$

A result similar to this appears in Clarke [2]. To prove (4.1), we differentiate (2.2) with respect to s, and let $s \to 0$. From (4.1) we see that if $\lambda b_1 > 1$, then $d/dt\, E\{W(t)\}$ is positive and bounded away from 0, so that $E\{W(t)\}$ increases indefinitely. Let $M^*(\tau)$ be the Laplace transform with respect to t of $E\{W(t)\}$; then (4.1) implies

$$M^*(\tau) = \frac{E\{W(0)\} + P^*}{\tau} - \frac{1 - \lambda b_1}{\tau^2}$$
$$= \frac{E\{W(0)\} + \eta^{-1}E\{\exp\,\{-\eta W(0)\}\}}{\tau} - \frac{1 - \lambda b_1}{\tau^2}.$$

From this it can be shown that if $B(v)$ has a finite second moment b_2, and $\lambda b_1 < 1$, then

$$\lim_{t \to \infty} E\{W(t)\} = \lim_{\tau \to 0} \tau M^* = \frac{\lambda b_2}{2(1 - \lambda b_1)}.$$

5. The stationary distribution. We call an initial distribution $P(w, 0)$ of $W(0)$ stationary if it is invariant under the transition probabilities for $W(t)$, that is, when $P(w, t) = P(w, 0)$ for all t and w. Let $A(w)$ be the distribution whose Laplace-Stieltjes transform is given by the Pollaczek-Khintchine formula

$$A^*(s) = \frac{s(1 - \lambda b_1)}{s - \lambda[1 - B^*]}, \qquad \lambda b_1 < 1.$$

We show that $A(w)$ is the unique stationary distribution. From (2.3) we see that a given $P(w, 0)$ is stationary if and only if the corresponding $\varphi(s, 0)$ satisfies

$$\frac{\varphi(s, 0)}{\tau} = \frac{\varphi(s, 0) - [s\varphi(\eta, 0)]/\eta}{\tau - s + \lambda[1 - B^*(s)]},$$

$$P^* = \frac{\varphi(\eta, 0)}{\eta} = \frac{P(0, 0)}{\tau}.$$

These imply

$$\varphi(s, 0) = \frac{sP(0, 0)}{s - \lambda[1 - B^*]},$$

and a simple Abelian argument proves $P(0, 0) = 1 - \lambda b_1$. This shows that $A(w)$ is unique.

To invert the transform $A^*(s)$ explicitly, we write it as

$$\frac{1 - \lambda b_1}{1 - \lambda b_1(1 - B^*)/b_1 s}$$

and notice that since $B(v)$ is the distribution of a nonnegative random variable with mean $0 < b_1 < \infty$, therefore

$$\frac{1 - B^*}{b_1 s}$$

is the Laplace transform of the density function

$$h(v) = \frac{U(v) - B(v)}{b_1},$$

where $U(v)$ is the unit step at 0. Define

$$H_0(w) = U(w),$$

$$H_{n+1}(w) = \int_0^w H_n(w - v)h(v)\,dv.$$

Then $A^*(s)$ may be inverted, and we have proved

THEOREM 6. *$A(w)$ is the unique stationary distribution of $W(0)$.* It may be written as

$$A(w) = (1 - \lambda b_1) \sum_{n=0}^{\infty} (\lambda b_1)^n H_n(w),$$

which shows that $A(w)$ is decomposable into a single step of magnitude $1 - \lambda b_1$ at 0 and an absolutely continuous portion, and that the equilibrium solution of Pollaczek and Khintchine has the form of a compound geometric distribution, i.e.,

$$W(\infty) = \sum_{i=0}^{k} x_i,$$

where the x's are mutually independent with the common density $h(v)$, and $\text{pr}\{k = n\} = (1 - \lambda b_1)(\lambda b_1)^n$.

6. The covariance function and the spectral distribution. In this last section we assume that $\lambda b_1 < 1$, and that $W(0)$ has the stationary distribution $A(w)$. Let $E\{W^n(0)\} = a_n$, when this exists. The covariance function $R(t)$ of the process is

$$R(t) = \int_{0-}^{\infty} wE\{W(t) \mid W(0) = w\}\, dA(w) - a_1^2, \tag{6.1}$$

and the Laplace transform of $R(t)$ is

$$R^*(\tau) = \int_{0-}^{\infty} \left[\frac{w^2 - wa_1}{\tau} + \frac{we^{-\eta w}}{\eta\tau} - \frac{w(1 - \lambda b_1)}{\tau^2}\right] dA(w). \tag{6.2}$$

Intuitively one might expect that in view of the Poisson arrival process and the independent service-times, the $W(t)$ process would have no periodic components, and thus a smooth spectral distribution. That this is so under weak conditions is a consequence of the following:

THEOREM 7. *If* $\lambda b_1 < 1$, *and* $B(v)$ *has a finite moment* b_3 *of the third order, then*

$$\int_0^{\infty} | R(t) |\, dt < \infty. \tag{6.3}$$

To prove that $R(t)$ is $L_1(0, \infty)$, we show that $R^*(\tau)$ is the Laplace-Stieltjes transform of an absolutely continuous (AC) function of bounded total variation (BTV). We make use of the following result: Let u be a nonnegative random variable such that $E\{u\}$ and $E\{u^2\}$ both exist; then

$$E\{e^{-\tau y}\} = \frac{1 - E\{e^{-\tau u}\}}{\tau E\{u\}}$$

defines an unique variate $y \geqq 0$ such that distr $\{y\}$ is AC and $E\{y\} = E\{u^2\} / 2E\{u\}$.

By differentiating $A^*(s)$ successively, it can be verified that if b_3 exists, so do a_1 and a_2, and $a_1 = \lambda b_2 / 2(1 - \lambda b_1)$. Let $N^* = \lambda B^*(\eta) / (\tau + \lambda)$, and let

$$T_1^* = \frac{1 - [(\lambda e^{-\eta w})/(\tau + \lambda)]}{[\tau w/(1 - \lambda b_1)] + \tau\lambda^{-1}},$$

$$T_2^* = \frac{1 - [\lambda(1 - \lambda b_1)(1 - N^*)]/\tau}{[\tau a_1/(1 - \lambda b_1)] + \tau\lambda^{-1}},$$

By use of $\eta = \tau + \lambda - \lambda B^*(\eta)$ and algebra we can write the integrand of (6.2) as

$$w\lambda^{-1}[w(1 - \lambda b_1)^{-1} + \lambda^{-1}][\lambda(1 - \lambda b_1)(1 - N^*)\tau^{-1} - T_1^*](1 - N^*)^{-1}$$
$$-w\lambda^{-1}[a_1(1 - \lambda b_1)^{-1} + \lambda^{-1}][\lambda(1 - \lambda b_1)(1 - N^*)\tau^{-1} - T_2^*](1 - N^*)^{-1}.$$

By Taylor series arguments and repeated use of the result stated earlier, it can be shown that if b_3 exists, then each of T_1^*, T_2^*, and N^* is $E\{\exp\{-\tau y\}\}$ for some suitable $y \geqq 0$, such that $E\{y\} < \infty$ and distr $\{y\}$ is AC. It follows from Lemma 5 of Smith [13] that for each w, the integrand of (6.2) is the Laplace-Stieltjes transform of an AC function of BTV. Therefore $R^*(\tau)$ is also.

From (6.3), and from the remarks on p. 522 of Doob [3], it follows that if $\lambda b_1 < 1$, and $B(v)$ has a finite moment b_3 of third order, then $W(t)$ has an absolutely continuous spectral distribution. The associated spectral density $g(x)$ is continuous and is given by

$$g(x) = 4 \int_0^\infty R(t) \cos 2\pi xt \, dt = 4 \, Re\{R^*(2\pi ix)\},$$

since R^* is well defined along the imaginary axis.

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