Attention should be drawn to a paper of Korolyuk [2] wherein the author gives different versions of the probabilities we have presented for the case $x = y$.

REFERENCES

[1] J. BLACKMAN, "An extension of the Kolmogorov distribution," *Ann. Math. Stat.*, Vol. 27 (1956), pp. 513–520.
[2] V. S. KOROLYUK, "On the difference of the empirical distribution of two independent samples," *Izv. Akad. Nauk. SSSR*, Vol. 19 (1955), pp. 81–96.

APPENDIX

BY J. H. B. KEMPERMAN

By a *path* of length n we shall mean an ordered sequence of $n + 1$ integers $(z_0, \cdots, z_n)$, such that

$$z_i - z_{i-1} \geqq -1 \qquad (i = 1, \cdots, n).$$

For each path $\pi_n = (z_0, \cdots, z_n)$, let

$$P(\pi_n) = \prod_{i=1}^{n} p(z_i - z_{i-1}),$$

(the weight or "probability" of π_n). Here, the $p_i = p(i)$, $(i = -1, 0, +1, \cdots)$, denote given (real or complex) numbers, $p(-1) \neq 0$. Finally, let

$$e_z(n) = \sum_{\pi_n}{}' p(\pi_n),$$

the summation being extended over all the paths $\pi_n = (z_0, \cdots, z_n)$ with $z_0 = 0$, $z_n = z$, $z_i \neq z (i = 0, 1, \cdots, n - 1)$.

THEOREM. *For* $n = 1, 2, \cdots$,

$$\begin{aligned} e_z(n) = -zr_z(n)/n + \sum_{j=1}^{\infty} j(j+1)p_j \\ \sum_{0<m\leqq+z} r_z(-m)r_{-j}(m+n-1)/(m+n-1). \end{aligned} \tag{8}$$

Here, for arbitrary integers h *and* s, $r_h(s)$ *is defined as the coefficient of* w^{h+s} *in the formal development*

$$(p_{-1} + p_0 w + p_1 w^2 + \cdots)^s = \sum_h r_h(s) w^{h+s};$$

especially, $r_h(s) = 0$ *if* $h + s < 0$.

PROOF. Let n and z be given integers, $n \geqq 1$. For any path $(z_0, \cdots, z_n)$ with $z_0 = 0$, $z_n = z$, we have

$$z_i - z_{i-1} = z - \sum_{\substack{\nu=1 \\ \nu \neq i}}^{n} (z_\nu - z_{\nu-1}) \leqq z + n - 1,$$

$(i = 1, \cdots, n - 1)$, thus, $e_z(n)$ does not depend on the p_i with $i \geqq n + z$. Further, $r_h(s)$ does not depend on the p_i with $i \geqq h + s$, hence, the inner sum in (8) does not depend on the p_i with $i \geqq n + z$; moreover, the jth inner sum equals 0 when $j \geqq n + z$. Consequently, it suffices to prove the theorem for the special case that $p_i = 0$ for i sufficiently large.

In this case,

$$f(w) = \sum_{i=-1}^{\infty} p_i w^i$$

is analytic at each point $w \neq 0$. Further, for $|w|$ sufficiently small

$$f(w)^s = \sum_{-\infty}^{\infty} r_h(s) w^h, \tag{9}$$

hence, for $s \geqq 0$

$$r_h(s) = \sum_{\pi_s}{}' P(\pi_s),$$

summing over all the paths $\pi_s = (z_0, \cdots, z_s)$ with $z_0 = 0$, $z_s = h$. Observing that to each path $(z_0, \cdots, z_n)$ with $z_n = z$ there corresponds a unique integer m with $0 \leqq m \leqq n$, $z_i \neq z (i = 0, 1, \cdots, m - 1)$, $z_m = z$, it follows that

$$r_z(n) = \sum_{m=0}^{n} e_z(m) r_0(n - m) \qquad (n = 0, 1, \cdots),$$

hence,

$$E_z = R_z / R_0, \tag{10}$$

where

$$R_h = \sum_{n=0}^{\infty} r_h(n) t^n, \qquad E_z = \sum_{n=0}^{\infty} e_z(n) t^n, \tag{11}$$

t denoting a sufficiently small parameter, $t \neq 0$.

Further, from (9), for each integer h,

$$R_h + \sum_{-h \leqq n < 0} r_h(n) t^n = \sum_{n=-h}^{\infty} r_h(n) t^n$$
$$= \sum_{n=-h}^{\infty} \frac{t^n}{2\pi\sqrt{-1}} \int_{|w|=R} f(w)^n w^{-h-1}\, dw = \frac{1}{2\pi\sqrt{-1}} \int_{|w|=R} \frac{(wf(w)t)^{-h}}{w - twf(w)}\, dw,$$

where R denotes a fixed positive number with $f(w) \neq 0$ for $0 < |w| \leqq R$. Here, from $p(-1) \neq 0$, the integrand is regular at $w = 0$. Moreover, for $t \neq 0$, $|t|$ sufficiently small, the equation $f(\xi) = t^{-1}$ has a unique solution satisfying $0 < |\xi| < R$. Thus,

$$R_h = (-t\xi f'(\xi))^{-1} \xi^{-h} - \sum_{0 < m \leqq h} r_h(-m) t^{-m}.$$

Finally, (10) and

$$\xi f'(\xi) = \xi f'(\xi) + f(\xi) - t^{-1} = -t^{-1} + \sum_{j=0}^{\infty} (j+1)p_j\xi^j$$

imply

$$E_z = \xi^{-z} + (-1 + t\sum_{j=0}^{\infty} (j+1)p_j\xi^j) \sum_{0<m\leqq z} r_z(-m)t^{-m}.$$

In view of (11), it suffices to prove that, for each integer h and $|t|$ sufficiently small, $t \neq 0$,

$$\xi^{-h} = -h \sum_{\substack{m=-h \\ m\neq 0}}^{\infty} r_h(m)t^m/m + c_h,$$

where c_h denotes a constant. Now, for $|t|, |\xi|$ small, the mapping $t \to \xi$ defined by $f(\xi) = t^{-1}$ is a 1:1 analytic transformation. Hence, integrating along a small positively oriented circle about 0, we have, for $m \neq 0$,

$$\int \xi^{-h}t^{-m-1}\,dt = -\int \xi^{-h}\,d(f(\xi)^m/m) = -\frac{h}{m}\int f(\xi)^m\xi^{-h-1}\,d\xi = -2\pi\sqrt{-1}\,\frac{h}{m}\,r_h(m).$$

REMARK. Results and methods analogous to the above may be found in the paper "The passage problem for a stationary Markov chain" by J. H. B. Kemperman, to appear in these Annals.

Let k be a fixed positive integer and choose $p(-1) = p(k) = 1$, $p(i) = 0$ for $i \neq -1, k$. Then $e_n(z)$ is equal to the number of sequences $(z_0, \cdots, z_n)$ with $z_i - z_{i-1} = -1$ or $+k$

$$(i = 1, \cdots, n), \quad z_0 = 0, \quad z_n = z, \quad z_i \neq z \qquad (i = 0, 1, \cdots, n-1).$$

Further, $H(i)$ is equal to the number of sequences $(z_n, z_{n-1}, \cdots, z_0)$ with

$$n = -\alpha + i(k+1) \geqq 1, \qquad z_i - z_{i-1} = -1 \text{ or } k$$

$(i = 1, \cdots, n)$, $z_n = \alpha$, $z_0 = 0$, $z_i \neq \alpha (i = 0, \cdots, n-1)$. Hence,

$$H_\alpha(i) = e_\alpha(-\alpha + i(k+1))$$

and the above Theorem yields

$$H_\alpha(i) = -\alpha r_\alpha(n)/n + k(k+1)\sum_{0<m\leqq\alpha} r_\alpha(-m)r_{-k}(m+n-1)/(m+n-1),$$

where $n = -\alpha + i(k+1)$. Noting that $r_h(s)$ is equal to the coefficient of w^{k+s} in the expansion of $(1 + w^{k+1})^s$ about 0, formula (1) easily follows.