THE UNIQUENESS OF THE TRIANGULAR ASSOCIATION SCHEME

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1. Summary. Parameters for a class of partially balanced incomplete block designs with two associate classes are immediately implied by the triangular association scheme. This paper deals with the more difficult question of whether or not these parameters imply the triangular association scheme.

2. Introduction. A partially balanced incomplete block design with two associate classes [1] is said to be triangular [2], [3] if the number of treatments $v = n(n - 1)/2$ and the association scheme is an array of n rows and n columns with the following properties:

(a) The positions in the principal diagonal are blank.

(b) The $n(n - 1)/2$ positions above the principal diagonal are filled by the numbers $1, 2, \cdots, n(n - 1)/2$ corresponding to the treatments.

(c) The positions below the principal diagonal are filled so that the array is symmetrical about the principal diagonal.

(d) For any treatment i the first associates are exactly those treatments which lie in the same row and the same column as i.

The following relations clearly hold:

(1) The number of first associates of any treatment is $n_1 = 2n - 4$.

(2) With respect to any two treatments θ_1 and θ_2 which are first associates, the number of treatments which are first associates of both θ_1 and θ_2 is

$$p_{11}^1(\theta_1, \theta_2) = n - 2.$$

(3) With respect to any two treatments θ_3 and θ_4 which are second associates, the number of treatments which are first associates of both θ_3 and θ_4 is $p_{11}^2(\theta_3, \theta_4) = 4$.

We wish to examine the converse, i.e., whether or not relations (1), (2) and (3) imply (a), (b), (c), and (d). We shall give a proof for $n \geqq 9$ which shows that the converse is true. The cases with $n < 9$ will not be considered, although the author has found that it is true for several small values of n, and conjectures that it is true for the rest.

As background for this problem, it is interesting to recall what has been found for some other classes of partially balanced designs. In the analogous problem for the group divisible designs it is easy to show that the converse is true [4]. For the latin square designs the converse is true for a sufficiently large number of treatments, but is not always true, as has been shown by example [5].

The present problem is closely related to problems considered in [6] and [7].

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The arguments used here could be substituted for some of the arguments in those papers.

3. A characterization of the triangular association scheme. The proof will consist of showing that there exist sets of treatments which satisfy the following theorem.

THEOREM. *The triangular association scheme for* $n(n-1)/2$ *treatments exists if and only if there exist sets of treatments* S_j, $j = 1, \cdots, n$, *such that*:

(i) *Each* S_j *consists of* $n - 1$ *treatments.*

(ii) *Any treatment is in precisely two sets* S_j.

(iii) *Any two distinct sets* S_i, S_j *have exactly one treatment in common.*

Proof. Necessity follows from the observation that the n rows of treatments in the triangular association scheme are the n sets S_j.

Sufficiency follows from noting a correspondence between the rows and columns of the association scheme and the sets S_j. To display the correspondence, we denote the unique element common to sets S_i and S_j by $\alpha(i, j) = \alpha(j, i)$. Then the correspondence is as follows: We let set S_i correspond to the ith row and column, and put element $\alpha(i, j)$ in the ith row and jth column of the association scheme. Because $\alpha(i_1, j_1) = \alpha(i_2, j_2)$ implies that $i_1 = i_2$ and $j_1 = j_2$, the element $\alpha(i, j)$ occurs only in the ith row (column) and jth column (row). This fills up the association scheme as described in (a), (b) and (c). Further, if we let "belonging to the same set S_j" correspond to "being first associates", then (d) is satisfied.

4. The existence of sets S_j which satisfy the Theorem. In this section we shall show for $n \geqq 9$ that there exist sets S_j which satisfy the Theorem. The proof makes conspicuous use of the condition (3) that $p_{11}^2 = 4$. In fact, in constructing the proof, the author was attracted to the singular fact that this parameter does not depend on n.

Throughout the proof, we shall employ certain conventions. In citing a reason why something is or is not true, we often shall write "$p_{11}^1(\theta_1, \theta_2)$" or "$p_{11}^2(\theta_3, \theta_4)$," whereby we mean to refer to particular treatments θ_1, θ_2, θ_3, and θ_4. Also, we shall write "$(\theta_1, \theta_2) = 1$ (or 2)," meaning that treatments θ_1 and θ_2 are first (or second) associates.

In developing the proof, the author used a matrix in which the ith row and column correspond to the ith treatment, and the entry in the intersection of the ith row and jth column is 1 or 2, depending on whether treatments i and j are first or second associates. Though this matrix is not explicitly used below, it is implicit, and it is believed that the reader will find the use of this matrix helpful in following the proof.

We begin by proving a lemma which will be used repeatedly in the sequel.

LEMMA 1. *With respect to any two initial treatments* θ_1 *and* θ_2 *which are first associates, the* $n - 3$, $(n \geqq 9)$ *treatments which are first associates of* θ_1 *and second associates of* θ_2 *pairwise are first associates.*

Proof. For simplicity we shall replace θ_1 by 1 and θ_2 by 2. From (1) and (2) it follows that there are $n - 2$ treatments which are first associates of both treatments 1 and 2, and $n - 3$ treatments which are first associates of treatment 1 and second associates of treatment 2. We shall refer to the treatments of the first set as treatments $3, \cdots, n$; and to those of the second set as treatments $n + 1, \cdots, 2n - 3$. These sets will be denoted respectively by

$$T_1 = T_1(3, \cdots, n)$$

and $T_2 = T_2(n + 1, \cdots, 2n - 3)$.

We first show that any treatment α in T_2 cannot have more than one second associate in T_2. We observe that $p_{11}^2(2, \alpha) = 4$, of which one such treatment is treatment 1. Thus, treatment α has at most three first associates in T_1. Because $p_{11}^1(1, \alpha) = n - 2$, treatment α has at least $n - 5$ first associates in T_2, and hence at most one second associate in T_2.

We now shall show that even this one second associate is impossible. Consider any two treatments α and β in T_2, and assume that $(\alpha, \beta) = 2$. We have established that treatment 1 and the $n - 5$ treatments other than α and β in T_2 are first associates of both α and β. But for $n \geqq 9$ the condition that $p_{11}^2(\alpha, \beta) = 4$ is violated, which shows that $(\alpha, \beta) = 1$. This completes the proof of Lemma 1.

Our next lemma shows the existence of sets S_j which satisfy (i) and (ii) of the theorem.

LEMMA 2. *For $n \geqq 9$, any initial treatment θ is an element of exactly two sets of treatments S_1 and S_2 which are such that a set contains $n - 1$ treatments, the treatments in a set pairwise are first associates, and θ is the unique element common to S_1 and S_2.*

Proof. We begin by showing that Lemma 1 implies that there are $n - 4$ treatments in T_1 which pairwise are first associates. For this purpose, it is convenient to define sets $T_1' = T_1'(3, \cdots, n - 2)$ and

$$T_2' = T_2'(n + 2, \cdots, 2n - 3).$$

From Lemma 1 and the condition that $p_{11}^1(1, \alpha) = n - 2$ for every treatment α in T_2, it follows that every treatment in T_2 has two first associates and $n - 4$ second associates in T_1. Without essential loss of generality, let treatment $n + 1$ be a second associate of every treatment in T_1', and let $(n - 1, n + 1) = (n, n + 1) = 1$. Then by Lemma 1, letting $\theta_1 = 1$ and $\theta_2 = n + 1$, the treatments in T_1' pairwise are first associates.

We still have to determine how treatments $n - 1$ and n intersect the treatments in T_1', T_2' and each other. We shall show that $(n - 1, n) = 2$ and either we have Case 1: $(n - 1, \alpha) = 1$, $(n, \alpha) = 2$ for all treatments α in T_1' and $(n - 1, \beta) = 2$, $(n, \beta) = 1$ for all treatments β in T_2'; or we have Case 2: $(n - 1, \alpha) = 2$, $(n, \alpha) = 1$ for all α in T_1' and $(n - 1, \beta) = 1$, $(n, \beta) = 2$ for all β in T_2'.

Suppose that treatment $n - 1$ is a second associate of some treatment in T_2', say treatment β. We shall show that we have Case 1. From Lemma 1 and $p_{11}^1(1, \beta)$ it follows that treatment β has two first associates and $n - 4$ second associates among the treatments of T_1' and treatments $n - 1$ and n. Therefore, treatment β has at least $n - 6$ second associates in T_1'. Without essential loss of generality, let these be treatments $3, \cdots, n - 4$. Applying Lemma 1 with $\theta_1 = 1$ and $\theta_2 = \beta$, it follows that these treatments are first associates of treatment $n - 1$.

Suppose that $(n - 3, n - 1) = 2$. Then it would be necessary that $p_{11}^2(n - 3, n - 1) = 4$. However, treatments $1, \cdots, n - 4$ are first associates of both treatments $n - 3$ and $n - 1$, violating $p_{11}^2(n - 3, n - 1) = 4$ for $n \geqq 9$. Similarly, treatment $n - 2$ cannot be a second associate of treatment $n - 1$.

We have shown that if treatment $n - 1$ has a second associate in T_2', then it is a first associate of every treatment in T_1'. Further, the treatments in T_1' and treatments 2 and $n + 1$ satisfy the condition that $p_{11}^1(1, n - 1) = n - 2$, implying that treatment $n - 1$ is a second associate of treatment n and the treatments in T_2'.

By applying Lemma 1 with $\theta_1 = 1$ and $\theta_2 = n - 1$, it follows that $(n, \beta) = 1$ for all β in T_2'. Because treatment 2 and the treatments in T_2 satisfy $p_{11}^1(1, n)$, it follows that $(n, \alpha) = 2$ for all α in T_1'. This demonstrates Case 1.

If $(n - 1, \beta) \neq 2$ for any β in T_2', then $(n - 1, \beta) = 1$ for all β in T_2'. But treatment 2 and the treatments in T_2 satisfy $p_{11}^1(1, n - 1)$, and it follows that $(n - 1, \alpha) = 2$ for all α in T_1' and that $(n - 1, n) = 2$. We now apply Lemma 1 with $\theta_1 = 1$ and $\theta_2 = n - 1$ to show that $(n, \alpha) = 1$ for all α in T_1'. Because treatments $2, \cdots, n - 2, n + 1$ satisfy $p_{11}^1(1, n) = n - 2$, it follows that $(n, \beta) = 2$ for all β in T_2'. This establishes Case 2.

We now observe a set S_1 which contains treatments 1, 2, the treatments in T_1' and either treatment $(n - 1)$ or n. Also, a set S_2 which contains treatments 1, the treatments in T_2, and the one of treatments $n - 1$ and n which is not in S_1. These sets are such that their elements pairwise are first associates. They are the sets of Lemma 2.

To show that there are no other such sets, we shall consider the way in which the treatments in T_1' are associated with the treatments in T_2'. Consider any treatment α in T_1', and the condition $p_{11}^1(1, \alpha) = n - 2$. Treatment 2, the remaining $(n - 5)$ treatments in T_1', and either treatment $n - 1$ or n are $n - 3$ treatments which satisfy this condition. Hence there is exactly one more such treatment in T_2'. Similarly, any treatment β in T_2' has exactly one first associate in T_1'. It follows that no other set of $n - 1$ treatments exists such that its treatments pairwise are first associates. This completes the proof of Lemma 2.

The sets found in Lemma 2 obey (i) and (ii) of the Theorem. To find the number s of sets and to prove (iii), we observe that each of s sets contains $n - 1$ elements, so that there are $s(n - 1)$ (not necessarily distinct) elements in the s sets. But every treatment occurs in exactly two sets, so that $s(n - 1) = 2v = n(n - 1)$ or $s = n$. Thus the number of pairs of sets is $n(n - 1)/2 = v$, and because every treatment occurs in exactly two sets, we have (iii).

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