TRANSIENT ATOMIC MARKOV CHAINS WITH A DENUMERABLE NUMBER OF STATES[1]

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1. Introduction. Many of the more interesting transient Markov chains have the property that for any set of states A and any initial distribution, the probability of entering A infinitely often (i.o.) is either zero always or one always. This type of chain has been termed *atomic* by D. Blackwell [1] and is exemplified by the three-dimensional random walk or by the successive sums of independent, identically distributed random variables.

In this paper we investigate the "fine structure" of an atomic chain, that is, we try to characterize the class of all sets A such that $P(x_n \varepsilon A \text{ i.o.}) = 0$. The study is restricted to atomic chains with a countable set of states which, for convenience of notation, we identify with the integers, and with stationary transition probabilities $p_{ij}^{(n)}$.

The martingale convergence theorem is used in [1] to show that a necessary and sufficient condition for atomicity is that every bounded solution ϕ of

$$\phi(i) = \sum_j p_{ij}\phi(j)$$

be constant. We use as our main tool the semi-martingale convergence theorem and the corresponding equation $\phi(i) \geqq \sum_j p_{ij}\phi(j)$ and obtain a complete, but not simple, characterization of the fine structure of transient atomic chains.

To illustrate the use of the above characterization we prove two theorems regarding the return to equilibrium times $x_0, x_1, \cdots$ in the coin-tossing game. The latter of these is then used to prove that there exists no set of numbers $\{\lambda_m\}$ such that[2] $P(x_n \varepsilon A \text{ i.o.}) = 0 \Leftrightarrow \sum_{m \varepsilon A} \lambda_m < \infty$.

This last result shows that, in general, there is no simple resolution to the question of defining the fine structure. There are, however, a number of interesting transient atomic chains which have the property that every infinite set of states is entered infinitely often with probability one. These chains are the subject of papers by Chung and Derman [2], and Breiman [3].

2. Use of the semi-martingale theorem.

THEOREM 1. *Let $x_0, x_1, \cdots$ be an atomic chain. Then for ϕ any nonnegative solution of*

$$\phi(i) \geqq \sum_j p_{ij}\phi(j) \tag{a}$$

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[1] This paper was prepared with the support of the Office of Ordnance Research, U.S. Army, under Contract DA-04-200-ORD-171.

[2] The referee has informed us that a similar theorem for the three-dimensional random walk has been proved by P. Erdös and B. J. Murdoch (unpublished).

which is finite for at least one value of i, $\phi(x_n)$ *converges almost surely (a.s.) to a constant independent of the initial distribution.*

PROOF. Let ϕ be a nonnegative solution of (a) with $\phi(0) < \infty$, and let R be the set of all states i such that P (entering $i \mid x_0 = 0) > 0$. From the atomicity and $P(x_n \varepsilon \tilde{R} \text{ i.o.} \mid x_0) = 0$, where $\tilde{R}$ is the complement of R, follows $P(x_n \varepsilon \tilde{R} \text{ i.o.}) = 0$. Therefore, it is sufficient to prove the theorem for initial distributions concentrated on R noting that ϕ is finite on R. Pick any such distribution $\{p_j\}$ such that $\sum_j p_j \phi(j) < \infty$ and $p_j > 0$, all $j \varepsilon R$. The random variables $\phi(x_n)$ form a semi-martingale with respect to the fields generated by $x_0, x_1, \cdots$, since

$$E(\phi(x_n) \mid x_{n-1}, \cdots, x_0) = E(\phi(x_n) \mid x_{n-1}) = \sum p_{x_{n-1}j}\phi(j) \leqq \phi(x_{n-1}),$$

$$E|\phi(x_n)| = E\phi(x_n) \leqq E\phi(x_0) < \infty.$$

By the semi-martingale theorem [4], $\phi(x_n)$ converges a.s. Suppose this limit is nonconstant, then there will be a number $a > 0$ such that if A is the set of states defined by $\{j; \phi(j) \geqq a\}$, then $0 < P(x_n \varepsilon A \text{ i.o.}) < 1$. Hence the limit is constant, and since $\phi(x_n)$ must converge to this same constant with the initial distribution concentrated at any single state in R, the theorem is valid.

We note that the same result is true for ϕ any bounded solution of (a) because for any sufficiently large constant α, $\phi + \alpha$ is a positive solution.

A simple but informative corollary of the above theorem demonstrates the special applicability of (a) to the transient case.

COROLLARY. *All the states of an atomic chain are recurrent if and only if all bounded solutions of* (a) *are constant. For a transient atomic chain there is at least one nonconstant bounded solution of* (a)

PROOF. Let $\phi(i) = P$ (entering $i_0 \mid x_0 = i$), so that $\phi(i_0) = 1$, and

$$\phi(i) = E(P(\text{entering } i_0 \mid x_1, x_0) \mid x_0 = i) \geqq E(\phi(x_1) \mid x_0 = i) = \textstyle\sum_j p_{ij}\phi(j).$$

If every solution of (a) is constant, then for every i_0 we have $P(\text{entering } i_0 \mid x_0 = i) = 1$. This implies that return to every state is certain. Now assume that every state is recurrent and let ϕ be any bounded solution of (a). If $\phi(i) \neq \phi(j)$, $\phi(x_n)$ cannot possibly converge to a constant since both i and j are entered i.o. with probability one. If there are transient states present the function ϕ defined above cannot be constant for all i_0.

3. Characterization of the fine structure.

We use the notation

$$u_{ik} = E(\text{number of visits to } k \mid x_0 = i),$$

$$u_{ik}^{(n)} = \delta_{ik} + p_{ik} + \cdots + p_{ik}^{(n-1)} \qquad \delta_{ik} = \begin{cases} 1, & i = k, \\ 0, & i \neq k, \end{cases}$$

and recall that $u_{ik} = \lim_n u_{ik}^{(n)}$.

THEOREM 2. *If* $x_0, x_1, \cdots$ *is an atomic chain, then for every nonnegative sequence of numbers* $\{\alpha_k\}$ *with* $\sum_k u_{0k}\alpha_k < \infty$ *and every* $\epsilon > 0$, *the set of states* $A_\alpha = \{i; \sum_k u_{ik}\alpha_k \geqq \epsilon\}$ *has the property* $P(x_n \epsilon A_\alpha \text{ i.o.}) = 0$. *Conversely, every* *set of states A such that* $P(x_n \varepsilon A$ i.o.$) = 0$ *is included in at least one of the sets* A_α *as defined above.*

PROOF. Let $\{\alpha_k\}$ be a sequence fulfilling the conditions of the theorem and let $\phi(i) = \sum_k u_{ik}\alpha_k$. The identity $\sum_j p_{ij}u_{jk} = u_{ik} - \delta_{ik}$ leads to the equation $\Sigma_j p_{ij}\phi(j) = \phi(i) - \alpha_i$. Thus, theorem 1 applies and $\phi(x_n)$ converges a.s. to a constant. Since the properties in which we are interested do not, in an atomic chain, depend on initial conditions, it is sufficient to take $x_0 = 0$. Then, iterating the equation which ϕ satisfies,

$$E(\phi(x_n) \mid x_0 = 0) = \sum_k (u_{0k} - u_{0k}^{(n)})\alpha_k \rightarrow 0$$

and by a semi-martingale inequality ([4], p. 325) which states that

$$E(\text{a.s. limit}) \leqq E\phi(x_n)$$

we are able to conclude that the a.s. limit of $\phi(x_n)$ is identically zero. This implies that $P(\phi(x_n) \geqq \epsilon$ i.o.$) = 0$ and proves one part of the theorem.

To get the second part, let A be any set of states with $P(x_n \varepsilon A$ i.o.$) = 0$. Form the function $\phi(i) = P(\text{entering } A \mid x_0 = i)$, so that $\phi(i) = 1$, all $i \varepsilon A$. It is easy to verify that ϕ satisfies (a), and thus $\phi(x_n)$ converges a.s. to some constant. We deduce that this constant is zero by noting that $P(\text{entering } A$ after $n - 1$ steps$) = E\phi(x_n)$. Since $P(x_n \varepsilon A$ i.o.$) = 0$ we conclude that $E\phi(x_n) \rightarrow 0$ and apply the bounded convergence theorem to get the result. Let the nonnegative sequence $\{\alpha_i\}$ be defined by $\phi(i) = \alpha_i + \sum_j p_{ij}\phi(j)$. Iterating this equation

$$\phi(i) = \sum_j p_{ij}^{(n)}\phi(j) + \sum_j u_{ij}^{(n)}\alpha_j .$$

By the boundedness of ϕ the second sum converges to $\sum_j u_{ij}\alpha_j$. The first sum must also converge to some bounded limit sequence $\{\lambda(i)\}$. Since

$$\lambda(i) = \sum_j p_{ij}\lambda(j),$$

by Blackwell's theorem as quoted above this sequence is constant, and by the convergence of $\phi(x_n)$ to zero, $\lambda(i) \equiv 0$. The set A is contained in the set $A_\alpha = \{i;\ \sum_k u_{ik}\alpha_k \geqq 1\}$ which proves the theorem.

4. Two theorems concerning the coin-tossing game. We apply theorem 2 to the Markov chain $x_0, x_1, \cdots$ whose values are the successive times of return to equilibrium in the fair coin-tossing game. The set of states is the set of all nonnegative even integers and we use the fact that this chain, being the sum of independent and identically distributed random variables, is atomic. It is well known that

$$\begin{aligned} u_{ik} &= 0, && k < i, \\ &\sim \frac{c}{\sqrt{(k - i)}}, && k \gg i. \end{aligned}$$

As it is evident that the characterization given in theorem 2 is invariant under asymptotic equivalence, we use $1/\sqrt{k - i}$ throughout this section in place of u_{ik} with the convention $\sqrt{0} = 1$ and $1/\sqrt{-} = 0$.

The first theorem we prove is similar to a theorem stated by Chung and Erdös [5].

THEOREM 3. *Let the sequence of even positive integers* $\{m_i\}$ *be such that the sequence* $\{\Delta_i\}$ *defined by* $\Delta_i = m_i - m_{i-1}$ *is nondecreasing. Then*

$$P(x_n \varepsilon \{m_i\} \text{ i.o.}) = 0 \Leftrightarrow \sum_i \frac{1}{\sqrt{m_i}} < \infty .$$

PROOF. If $\sum_i 1/\sqrt{m_i} < \infty$, the assertion follows immediately from the Borel-Cantelli lemma. Now assume that $P(x_n \varepsilon \{m_i\} \text{ i.o.}) = 0$, but that $\sum_i 1/\sqrt{m_i} = \infty$. By theorem 2, there is a nonnegative sequence $\{\alpha_i\}$ such that $\sum_k \alpha_k/\sqrt{k} < \infty$ and $\{m_i\} \subset \{i;\ \sum_k \alpha_k/\sqrt{k - i} \geqq \epsilon\}$. From this we have for all m_i

$$\sum_{k \geqq m_i} \frac{\alpha_k}{\sqrt{k - m_i}} \geqq \epsilon.$$

Define $\lambda_k^{(N)}$ by

$$\lambda_k^{(N)} = \frac{\sum_{i=0}^{N} \dfrac{1}{\sqrt{m_i}\,\sqrt{k - m_i}}}{\sum_{i=0}^{N} \dfrac{1}{\sqrt{m_i}}}.$$

It is evident that $\sum_k \lambda_k^{(N)} \alpha_k \geqq \epsilon$, all N, and that $\lim_N \lambda_k^{(N)} = 0$ for k fixed. We will show that $\lambda_k^{(N)} < c/\sqrt{k}$, all k, N, and conclude from the bounded convergence theorem the contradiction that $\lim_N \sum_k \lambda_k^{(N)} \alpha_k = 0$. To begin with, assume that $k \geqq m_N$, then

$$\sqrt{k}\,\lambda_k^{(N)} \leqq \frac{\sqrt{m_N} \sum_{i=0}^{N} \dfrac{1}{\sqrt{m_i}\,\sqrt{m_N - m_i}}}{\sum_{i=0}^{N} \dfrac{1}{\sqrt{m_i}}}.$$

By splitting the top sum into the two parts $m_i \leqq m_N/2$, $m_i > m_N/2$ and using our assumption concerning Δ_i, we get $\sqrt{k}\lambda_k^{(N)} \leqq 4$. Now if $k \leqq m_N$, let m_n be the largest of the m_i which is $\leqq k$. With this

$$\sqrt{k}\,\lambda_k^{(N)} \leqq \frac{\sqrt{m_n} \sum_{i=0}^{N} \dfrac{1}{\sqrt{m_i}\,\sqrt{m_n - m_i}}}{\sum_{i=0}^{N} \dfrac{1}{\sqrt{m_i}}}$$

and repeating the above argument results again in $\sqrt{k}\lambda_k^{(N)} \leqq 4$.

It is clear that in the above context, a little more attention to the appropriate inequalities would result in a considerable weakening of the growth condition on the sequence $\{m_i\}$.

We can get a result in another direction by combining our characterization with different inequalities. Let all the states between and including n_1 and n_2, $n_2 \geqq n_1$, be called an interval and denoted by $[n_1, n_2]$.

THEOREM 4. *If the sequence of disjoint finite intervals* $\{I_j\}$, $I_j = [m_j, M_j]$ *is such that for some* $\delta > 0$, $m_{j+1} \geqq (1 + \delta)M_j$, *then, denoting*

$$l_j = M_j - m_j + 2,$$

$$P(x_n \,\varepsilon \bigcup_j I_j \text{ i.o.}) = 0 \Leftrightarrow \sum_j \sqrt{l_j/M_j} < \infty .$$

PROOF. Define a sequence of intervals $I'_j = [m'_j, M'_j]$ by $m'_j = M_j$, $M'_j = M_j + \sqrt{l_j}$, where $\sqrt{l_j}$ is here to be interpreted as the greatest even integer less than $\sqrt{l_j}$. Let $\sum_j \sqrt{l_j/M_j} < \infty$ and define α_k by

$$\alpha_k = \begin{cases} 1 & \text{if } k \,\varepsilon \bigcup_j I'_j , \\ 0 & \text{otherwise.} \end{cases}$$

By these definitions

$$\sum_k \frac{\alpha_k}{\sqrt{k}} \leqq \frac{1}{2} \sum_j \sqrt{l_j/M_j} < \infty .$$

Thus the set $A = \{i; \sum_k \alpha_k/\sqrt{k-i} \geqq \frac{1}{2}\}$ has the property that $P(x_n \,\varepsilon\, A \ i.\ o.) = 0$. The set A includes, in particular, the integers i such that $i \leqq M_j$ and

$$\tfrac{1}{2} \leqq \sum_{k \varepsilon I'_j} \frac{1}{\sqrt{k - i}} \leqq \tfrac{1}{2}(\sqrt{M'_j - i} - \sqrt{m'_j - i}).$$

This inequality can be easily shown to be satisfied by all $i \geqq m_j$, which proves the theorem going one way.

To go the other way, assume that $P(x_n \,\epsilon \bigcup_j I_j \ i.\ o.) = 0$. Then there is a non-negative sequence α_k such that $\sum_k \alpha_k/\sqrt{k} < \infty$ and $\bigcup_j I_j \subset \{i; \sum_k \alpha_k/\sqrt{k-i} \geqq \epsilon\}$, from which, if $i \,\varepsilon\, I_j$, then $\sum_{k \geqq m_j} \alpha_k/\sqrt{k-i} \geqq \epsilon$. We wish to conclude that part of this sum is negligible and argue that if $i \,\varepsilon\, I_j$,and if j is sufficiently large

$$\sum_{k \geqq m_{j+1}} \sqrt{\frac{k}{k - i}} \frac{\alpha_k}{\sqrt{k}} \leqq \sqrt{\frac{m_{j+1}}{m_{j+1} - M_j}} \sum_{k \geqq m_{j+1}} \frac{\alpha_k}{\sqrt{k}} \leqq \frac{\epsilon}{2}$$

so that if $i \,\varepsilon\, I_j$,

$$\sum_{m_{j+1} \geqq k \geqq m_j} \frac{\alpha_k}{\sqrt{k - i}} \geqq \frac{\epsilon}{2} .$$

We sum this last inequality over $i \,\varepsilon\, I_j$ to get

$$\sum_{m_{j+1} > k \geqq m_j} \alpha_k \left(\sum_{i \varepsilon I_j} \frac{1}{\sqrt{k - i}} \right) \geqq \frac{\epsilon}{4} l_j .$$

It can be easily shown that

$$\sum_{i \varepsilon I_j} \frac{1}{\sqrt{k - i}} \leqq 4 \sqrt{\frac{M_j l_j}{k}} ,$$

and using this we conclude that

$$\sum_k \frac{\alpha_k}{\sqrt{k}} \geqq \frac{\epsilon}{16} \sum_j \sqrt{l_j/M_j} .$$

5. The nonsimplicity of the fine structure of the coin-tossing game. The purpose of this section is to prove the following theorem.

THEOREM 5. *Let* $x_0, x_1, \cdots$ *be the successive times of return to equilibrium in the fair coin-tossing game. Then there exists no weighting* $\{\lambda_m\}$, $\lambda_m \geqq 0$ *of the positive even integers such that*

$$P(x_n \varepsilon A \text{ i.o.}) = 0 \Leftrightarrow \sum_{m \varepsilon A} \lambda_m < \infty.$$

PROOF. Consider any set $\cup_j I_j$ where the I_j are disjoint finite intervals which we can represent as $[m_j, m_j(1 + \alpha_j)]$, $0 < \alpha_j \leqq 1$, where $m_{j+1} \geqq 3m_j$. By theorem 4

$$P(x_n \varepsilon \bigcup_j I_j \text{ i.o.}) = 0 \Leftrightarrow \sum_j \sqrt{\alpha_j} < \infty.$$

Let now $\{\lambda_m\}$ be any weighting of the positive even integers having the property stated in the theorem. By this property, $\lim_m \lambda_m = 0$ since otherwise we could find an indefinitely sparse set A which would be entered i.o. with probability one. We define a function $\phi(\alpha)$, $0 \leqq \alpha < \infty$ by

$$\phi(\alpha) = \liminf_n \sum_{m=n}^{ne^\alpha} \lambda_m,$$

where in writing the upper limit of summation as ne^α it is immaterial whether we take the next greater integer, or the previous integer.

PROPOSITION. $\phi(\alpha)$ *is monotone nondecreasing,* $\phi(\alpha + \beta) \geqq \phi(\alpha) + \phi(\beta)$ *and there is a neighborhood of the origin in which* $\phi(\alpha) < \infty$.

PROOF. The first assertion is immediate. As to the second, we write:

$$\liminf_n \left(\sum_{m=n}^{ne^\alpha e^\beta} \lambda_m \right) = \liminf_n \left(\sum_{m=n}^{ne^\alpha} \lambda_m + \sum_{m=ne^\alpha}^{ne^\alpha e^\beta} \lambda_m \right)$$

$$\geqq \liminf_n \left(\sum_{m=n}^{ne^\alpha} \lambda_m \right) + \liminf_n \left(\sum_{m=n}^{ne^\beta} \lambda_m \right).$$

Finally, suppose that $\phi(\alpha) = \infty$ for all $\alpha > 0$, and consider any sequence $\{\alpha_j\}$, $0 < \alpha_j \leqq 1$, such that $\sum_j \sqrt{\alpha_j} < \infty$. Since $\lim_n \sum_{m=n}^{n(1+\alpha_j)} \lambda_m = \infty$ for all j, we find a sequence of intervals $I_j = [m_j, m_j(1 + \alpha_j)]$ as far apart as desired having the undesirable property $\sum_j \sum_{m \varepsilon I_j} \lambda_m = \infty$.

To complete the proof of the theorem, we note that as a well-known consequence of the proposition there is a neighborhood N of the origin and a constant $q < \infty$ such that $\phi(\alpha) \leqq q\alpha$, $\alpha \varepsilon N$. Take $\{\alpha_j\}$, $\alpha_j > 0$, such that $\sum_j \alpha_j < \infty$ but $\sum_j \sqrt{\alpha_j} = \infty$, and $\{\alpha_j\} \subset N$. Then we may find a sequence $\{m_j\}$ increasing as rapidly as desired such that

$$\sum_{m=m_j}^{m_j(1+\alpha_j)} \lambda_m \leqq 2q\alpha_j.$$

Hence, taking $I_j = [m_j, m_j(1 + \alpha_j)]$ we have

$$\sum_j \sqrt{\alpha_j} = \infty \qquad \text{but} \qquad \sum_j \sum_{m \varepsilon I_j} \lambda_m < \infty.$$

It is a pleasure to acknowledge my debt to David Blackwell who brought my attention to the problems treated above.

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