PÓLYA TYPE DISTRIBUTIONS IV. SOME PRINCIPLES OF SELECTING A SINGLE PROCEDURE FROM A COMPLETE CLASS[1]

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0. Introduction. In previous publications [1], [2], and [3], various aspects of decision theory in which the underlying distributions are Pólya type have been studied. For example, complete classes of decision procedures were determined, all Bayes procedures were characterized, and the problem of admissibility was investigated as related to various kinds of loss functions.

Usually the minimal complete class of decision procedures, to which the statistician would obviously restrict himself in practical application, is still quite large. Consequently, without any additional knowledge or further conditions, it is a hopeless task to justify preferring any given admissible procedure to another. It is therefore of importance to introduce new criteria which will single out a procedure for use. It is the object of this paper to discuss some further principles which select a single statistical procedure from the class of all "monotone" procedures.

In the $n = 2$ action problem (essentially the testing problem) some of the classical principles used to determine a single admissible procedure for use are related to the concepts of unbiasedness, maximum likelihood, invariance, minimax, etc. These principles have received much attention and their justification and relevance are well understood for the parametric testing problem. For a detailed analysis of these classical concepts in the case of two action problems when the underlying distributions are Pólya type, the reader is referred to [1]. Our present discussion deals with the extension and analysis of some of these principles to the n-action problem. In the sense that the estimation problem may be obtained as a limit of finite action problems, the ideas here shed further light on the estimation problem.

The language and notation we use is that of the introduction of the previous paper [3]. However, a knowledge of the results of [3] is not necessary for an understanding of the present discussion although a reading of the introduction would more than provide sufficient familiarity with the terminology to be used here as well as a general background for Pólya type distributions. Henceforth, we assume that the notation of this manuscript is that of [3]. Nevertheless, for clarity of exposition, we review briefly some of the main quantities to be used.

Let the distribution of the observed real random variable X (usually a sufficient

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statistic), depending on the unknown parameter ω ($\omega \varepsilon \Omega$, an interval of the real line), have the form

$$P(x, \omega) = \int_{-\infty}^{x} p(\xi, \omega)\, d\mu(\xi), \tag{1}$$

where the density $p(\xi, \omega)$ possesses a monotone likelihood ratio (Pólya type 2) and μ is a countably additive measure defined at least for the Borel field of sets containing the open subsets of the real line. Occasionally, we shall assume the stronger condition that the density is Pólya type 3.

The main transformation property of Pólya type 2 densities used in our analysis is as follows: If $g(x)$ changes sign at most once (say from negative to positive values), then

$$h(\omega) = \int g(x)p(x, \omega)\, d\mu(x)$$

changes sign at most once. Moreover, if $h(\omega)$ does indeed change signs, then it must change in the same direction as g, i.e., from negative to positive. For a thorough discussion of these properties the reader is referred to [2].

There are n possible actions, and $L_i(\omega)$ ($i = 1, \cdots, n$) represents the measure of the loss when taking action i and ω is the state of nature. We require that the set

$$S_i = \{\omega \mid L_i(\omega) < L_j(\omega), j \neq i\} = (\omega_{i-1}^0, \omega_i^0)$$

where the ω_i^0 satisfy

$$-\infty = \omega_0^0 < \omega_1^0 < \omega_2^0 < \cdots < \omega_n^0 = \infty.$$

The set S_i represents the set of ω values where action i is favored if the state of nature were known. Also, we assume that $L_i(\omega) - L_{i+1}(\omega)$ has exactly one sign change which must occur at ω_i^0.

We shall assume throughout what follows that the loss functions $L_i(\omega)$ and the density $p(x, \omega)$ satisfy sufficient smoothness conditions to guarantee the existence of all integrals involving these quantities and to justify all differentiation operations. In most particular examples these smoothness requirements can be readily verified.

A statistical procedure is an n-tuple

$$\varphi(x) = (\varphi_1(x), \cdots, \varphi_n(x)),$$

where $\varphi_i(x)$ is interpreted as the probability of taking action i when observing x. A "monotone" procedure is characterized by a tuple

$$(x_1, x_2, \cdots, x_{n-1}; \lambda_1, \cdots, \lambda_{n-1})$$

where $x_1 \leqq x_2 \leqq \cdots \leqq x_{n-1}$, $0 \leqq \lambda_i \leqq 1$. Explicitly, when the x_i are distinct, then

$$\varphi_i(x) = \begin{cases} 1 & \text{if } x_{i-1} < x < x_i, \\ 0 & \text{if } x < x_{i-1},\ x > x_i, \\ \lambda_i & \text{if } x = x_i, \\ 1 - \lambda_{i-1} & \text{if } x = x_{i-1}, \end{cases} \qquad i = 1, \cdots, n,$$

and by definition $x_0 = -\infty, \lambda_0 = 0, x_n = +\infty, \lambda_n = 1$. In the case where some of the x_i coincide then appropriate changes in the form of the definition of $\varphi_i(x)$ at the values x_i must be made. If the measure μ of (1) has no atoms (jumps), then a monotone procedure is fully specified (up to equivalence almost everywhere with respect to μ) by the critical values $(x_1, x_2, \cdots, x_{n-1})$. For the sake of simplicity of exposition, we restrict ourselves henceforth to the case of a continuous distribution. However, we remark in passing that all of the results of this paper may be extended, subject to suitable modifications, to the general case where we allow the measure μ to possess atoms. The risk corresponding to any given strategy $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_n)$ is given by the expression

$$\rho(\omega, \varphi) = \int p(x, \omega) \left\{ \sum_{i=1}^{n} L_i(\omega)\varphi_i(x) \right\} d\mu(x). \tag{2}$$

The collection of all monotone procedures constitutes a complete class [4]. When the loss functions satisfy additional assumptions, then all non-degenerate monotone procedures are also admissible [3].

The set of all monotone strategies $\mathfrak{M}$ form an $n - 1$ dimensional family in the sense that they depend on the $n - 1$ critical values which determine the procedures. Our problem, in choosing a specific strategy from $\mathfrak{M}$, is in essence finding $n - 1$ conditions which will cut the class $\mathfrak{M}$ down to a unique member. Alternatively, we could impose some global restrictions which also single out a monotone procedure. For instance, if an a priori distribution of nature $F(\omega)$ is known to be meaningful, then the Bayes procedure with respect to F determines a specific monotone procedure. [See [3], [5].] The assumption of the existence of F is often hard to justify and appears contrived.

Another global condition frequently followed is to choose a monotone minimax procedure. However, minimax procedures are often very unreasonable on the basis of statistical intuition and there exists feeling that minimax philosophy is in general too conservative and unrealistic. Of course, modifications of the minimax principle lead to the so-called regret principles. Various complications appear also for the case of the criteria of minimax regret [6].

A third method for choosing a monotone procedure is inherent in the construction of complete classes as introduced in [4]. Suppose that for a given problem there has been in use a common or accepted mode of action which is not a monotone procedure. Then, there exists at least one monotone procedure which improves everywhere on it for the decision problem of more than two actions. If the original procedure is described by an n-tuple of functions $\varphi = (\varphi_1, \cdots, \varphi_n)$, then any monotone procedure $\varphi^0 = (\varphi_1^0, \cdots, \varphi_n^0)$ (and there is at least one) which satisfies

$$\int_{-\infty}^{\infty} p(x \mid \omega) \left[\sum_{j=1}^{i} \varphi_j^0(x) - \sum_{j=1}^{i} \varphi_j(x) \right] d\mu(x) \begin{cases} \geqq 0 & \text{for} \quad \omega \leqq \omega_i^0 \\ \leqq 0 & \text{for} \quad \omega \geqq \omega_i^0 \end{cases}$$

improves on φ.

This method is constructive. That is, for any non-monotone procedure in use we can explicitly exhibit a monotone procedure which yields a smaller risk uniformly for any choice of the state of nature ω. The apparent disadvantage to this idea is that it involves only an improvement relative to a given non-monotone procedure and sheds no light on the intrinsic question of selecting a specific monotone procedure from the class $\mathfrak{M}$.

In this study we will analyze three principles of selecting a single monotone procedure from $\mathfrak{M}$. The first represents an extension of the maximum likelihood estimate to the circumstance of the n-action problem. The monotone test obtained in this case has a lot of intuitive appeal and will be referred to as the maximum likelihood procedure.

The following section examines another approach called the principle of maximum probabilities (abbreviated M.P.). This principle, as well as the maximum likelihood procedure, does not depend on the specific values of the loss functions but rather on the preference regions $S_i = (\omega_{i-1}^0 \, \omega_i^0)$. Any other loss function satisfying the properties of a monotone preference pattern and giving rise to the same preference sets S_i will possess the same class of monotone procedures obeying the principle of M.P.

The precise description of this principle is as follows: A decision procedure which is defined by an n-tuple of functions $\varphi = (\varphi_1, \cdots, \varphi_n)$ is said to have the property of maximum probabilities (φ has M.P.) if for every i

$$h_i(\omega') \geqq h_i(\omega'') \qquad \text{for any } \omega' \text{ in } S_i, \omega'' \not\in S_i, \tag{3}$$

where

$$h_i(\omega) = \int \varphi_i(x) p(x, \omega) \, d\mu(x). \tag{4}$$

For the case of two actions a procedure φ has the property of M.P. if and only if φ is unbiased in the classical sense. Therefore, this principle may be considered to be a generalization to the case of n actions of the concept of unbiasedness. The quantity $h_i(\omega)$ may be interpreted as the unconditional probability for the procedure φ of taking action i when the state of nature is ω. The condition (3) states that $h_i(\omega)$ is larger when ω is in S_i than when ω is outside S_i. This last property is the reason for the name, principle of maximum probabilities.

It will be shown that there always exist monotone procedures having the property of M.P. for the case of $n \leqq 5$ actions. In fact, we shall exhibit a one parameter family of such procedures. When $n > 5$, in general there ceases to exist such monotone procedures.

The final principle investigated is the principle of unbiasedness (in the sense of Lehmann [7]). A decision procedure φ is said to be risk unbiased with respect to the loss functions L_i if $E_\theta[L(\omega, \varphi(x))] \geqq E_\theta[L(\theta, \varphi(x))]$ for all ω and θ, where $E_\theta(\cdot)$ denotes the expected value given that the state of nature is θ, and

$$L(\omega, \varphi(x)) = \sum L_i(\omega)\varphi_i(x).$$

For the case of two actions, this definition reduces to the usual concept of unbiasedness. This principle of unbiasedness differs from the principle of M.P. in that the former depends in a very crucial way on the magnitudes of the loss functions while the latter depends only on the preference regions. We shall prove that if $L_j(\omega) = L_{ij}$ for ω in S_i and the L_{ij} satisfy suitable assumptions, then there exists a unique admissible monotone procedure unbiased in the sense of Lehmann. The method of proof of the existence will in effect be constructive. In general, risk unbiased procedures need not exist.

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1. Maximum likelihood principle. We assume throughout this section that the density $p(x, \omega)$ of (1) has a strict monotone likelihood ratio and further that $p(x, \omega)$ possesses continuous second order partial derivatives. The fact that p is of Pólya type 2 implies (see [2]) that

$$\begin{vmatrix} p(x, \omega) & \frac{\partial}{\partial \omega} p(x, \omega) \\ \frac{\partial}{\partial x} p(x, \omega) & \frac{\partial^2}{\partial x \partial \omega} p(x, \omega) \end{vmatrix} \geqq 0 \tag{5}$$

for all x and ω. An additional assumption is imposed to the effect that the inequality of (5) is strict for all x and ω. Finally, we assume that for each x in X the equation

$$\frac{\partial}{\partial \omega} p(x, \omega) = 0 \tag{6}$$

has a unique solution, $\omega = \omega(x)$, which is a differentiable function of x. These assumptions are not as stringent as may appear offhand. A wide class of distributions, including the exponential family ($p(x, \omega) = e^{\omega x}\beta(\omega)$), the noncentral t, the noncentral χ^2, etc., fulfills these requirements. For the exponential family, $\omega(x)$ is the solution of the equation $-\beta'(\omega)/\beta(\omega) = x$.

LEMMA 1. *$\omega(x)$ is a strictly increasing function of x.*

Proof. Differentiating Eq. (6) with respect to x leads to

$$\frac{\partial^2 p(x, \omega(x))}{\partial x \partial \omega} + \frac{\partial^2 p(x, \omega(x))}{\partial \omega^2} \omega'(x) = 0. \tag{7}$$

By assumption,

$$\begin{vmatrix} p(x, \omega(x)) & \frac{\partial}{\partial \omega} p(x, \omega(x)) \\ \frac{\partial}{\partial x} p(x, \omega(x)) & \frac{\partial^2}{\partial x \partial \omega} p(x, \omega(x)) \end{vmatrix} > 0,$$

which implies $\partial^2 p(x, \omega(x))/\partial x \partial \omega > 0$ because of (6). Since $p(x, \omega)$ assumes a maximum at $\omega = \omega(x)$, $\partial^2 p(x, \omega(x))/\partial \omega^2 \leqq 0$. Thus from (7), $\omega'(x) > 0$.

As x varies over the sample space X, $\omega(x)$ varies over the whole Ω interval. Suppose not; then there exists an ω_0 such that ω_0 is not the upper endpoint of

Ω and for $\omega > \omega_0$, $\partial p(x, \omega)/\partial\omega < 0$ for all x. (Or similarly for the lower end of Ω.) But this contradicts the fact that for all $\omega \varepsilon \Omega$,

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial\omega} p(x, \omega)\, d\mu(x) = \frac{\partial}{\partial\omega} \int_{-\infty}^{\infty} p(x, \omega)\, d\mu(x) = 0.$$

Since $\omega(x)$ is a $1 - 1$ strictly monotonic mapping of X onto Ω, the inverse function ω^{-1} is well-defined. Set $x_i^0 = \omega^{-1}(\omega_i^0)$, $i = 1, \cdots, n - 1$. The maximum likelihood principle dictates that the monotone procedure which should be used is the one defined by the critical numbers $(x_1^0, \cdots, x_{n-1}^0)$. For $x \varepsilon (x_{i-1}^0, x_i^0)$, take action i, $i = 1, \cdots, n$, $x_0^0 = -\infty$ and $x_n^0 = +\infty$. This principle has the feature that for any observed x the proper action i is taken whose corresponding interval $(\omega_{i-1}^0, \omega_i^0)$ includes the maximum likelihood estimate of ω. In less precise language, that action is taken which is most likely.

2. Principle of maximum probabilities (M.P.). The principle of maximum probabilities is one type of extension of the concept of unbiasedness in hypothesis testing. Consider the n-action problem defined by the points $-\infty = \omega_0^0 < \omega_1^0 < \cdots < \omega_n^0 = +\infty$ in which action i is preferred in the interval $S_i = (\omega_{i-1}^0, \omega_i^0)$. A decision procedure which is defined by an n-tuple of functions $\varphi = (\varphi_1, \cdots, \varphi_n)$ is said to have the property M.P. if for every i, $h_i(\omega') \geqq h_i(\omega'')$ for any $\omega' \varepsilon S_i$, $\omega'' \not\varepsilon S_i$, where

$$h_i(\omega) = \int_{-\infty}^{\infty} \varphi_i(x) p(x, \omega)\, d\mu(x).$$

Our object is to try to establish the existence of monotone procedures possessing the property of M.P

It is necessary in studying this concept to assume that the density $p(x, \omega)$ is strictly Pólya type 3, and that the equation $\partial p(x, \omega)/\partial\omega = 0$ is well-defined and has a unique solution $\omega = \omega(x)$ for each value of x. For any constants $a < b$ it is tacitly assumed that differentiation with respect to ω is valid inside the integral sign of

$$\int_a^b p(x, \omega)\, d\mu(x).$$

Also, assume that μ is a continuous measure without discrete mass points whose spectrum is an interval. This last assumption is not essential but without it additional care must be taken in handling randomizations and the lack of uniqueness of various quantities caused by gaps in the spectrum.

For the purpose of exposition our analysis is divided into a series of lemmas.

A randomized strategy is now defined by $n - 1$ points $(x_1, \cdots, x_{n-1})$. Let $i(i = 1, \cdots, n - 2)$ be fixed for the moment and define $(x_i(\alpha), x_{i+1}(\alpha))$ by the equations

$$h_{i+1}(\omega_i^0) = \int_{x_i}^{x_{i+1}} p(x, \omega_i^0)\, d\mu(x) = \alpha,$$

(8)

$$h_{i+1}(\omega_{i+1}^0) = \int_{x_i}^{x_{i+1}} p(x, \omega_{i+1}^0)\, d\mu(x) = \alpha.$$

$(x_i(\alpha), x_{i+1}(\alpha))$ are uniquely defined since by Theorem 3 of [1] there is a unique monotone strategy which improves on the non-monotone strategy $\varphi(x) \equiv \alpha$. Moreover, it is clear that $h_{i+1}(\omega') \geqq h_{i+1}(\omega'')$ for any $\omega' \varepsilon S_{i+1}$ and $\omega'' \not\varepsilon S_{i+1}$ when (8) is satisfied.

LEMMA 2. *$x_i(\alpha)$ is a monotone decreasing and $x_{i+1}(\alpha)$ is a monotone increasing function of α.*

Proof. From (8),

$$\int_{x_i(\alpha)}^{x_{i+1}(\alpha)} [p(x, \omega_i^0) - p(x, \omega_{i+1}^0)]\, d\mu(x) = 0 \tag{9}$$

for all α. Since $p(x, \omega)$ is strictly Pólya type 3, $p(x, \omega_i^0) - p(x, \omega_{i+1}^0)$ has at most one zero; by (9) it has at least one. In order that the relation (9) be preserved for all α, either $x_i(\alpha)$ increases and $x_{i+1}(\alpha)$ decreases, or $x_i(\alpha)$ decreases and $x_{i+1}(\alpha)$ increases, as α increases. It is clear from (8) that the latter must hold.

It also follows from the variation diminishing properties of the density $p(x, \omega)$ [2] that

$$h_1(\omega) = \int_{-\infty}^{x_1} p(x, \omega)\, d\mu(x)$$

is a monotone decreasing function of ω, and

$$h_n(\omega) = \int_{x_{n-1}}^{\infty} p(x, \omega)\, d\mu(x)$$

is a monotone increasing function of ω for any x_1 and x_{n-1} respectively.

Consider $x_i(\alpha)$ and $x_{i+1}(\alpha)$, which are defined by the equations $h_{i+1}(\omega_i^0) = \alpha = h_{i+1}(\omega_{i+1}^0)$, and $x'_{i+1}(\alpha)$ and $x'_{i+2}(\alpha)$, which are defined by $h_{i+2}(\omega_{i+1}^0) = \alpha = h_{i+2}(\omega_{i+2}^0)$. Then

LEMMA 3. *For all α, $x_i(\alpha) < x'_{i+1}(\alpha)$ and $x_{i+1}(\alpha) < x'_{i+2}(\alpha)$, $i = 1, \cdots, n-3$.*

Proof. Let

$$I_{[a,b]} = \begin{cases} 1, & a \leqq x \leqq b, \\ 0, & \text{otherwise.} \end{cases}$$

Suppose $x_i(\alpha) \geqq x'_{i+1}(\alpha)$. Then $I_{[x_i, x_{i+1}]} - I_{[x'_{i+1}, x'_{i+2}]}$ is always of one sign or at worse changes sign from $-$ to $+$. But

$$\int_{-\infty}^{\infty} [I_{[x_i, x_{i+1}]} - I_{[x'_{i+1}, x'_{i+2}]}] p(x, \omega)\, d\mu(x) \begin{cases} > 0 & \text{for} \quad \omega < \omega_{i+1}^0 \\ = 0 & \text{for} \quad \omega = \omega_{i+1}^0 \\ < 0 & \text{for} \quad \omega > \omega_{i+1}^0 \end{cases} \tag{10}$$

which is an impossibility in that it changes sign in the wrong direction [2] so $x_i(\alpha) < x'_{i+1}(\alpha)$.

Suppose $x_{i+1}(\alpha) \geqq x'_{i+2}(\alpha)$. Then $I_{[x_i, x_{i+1}]} - I_{[x'_{i+1}, x'_{i+2}]}$ is always of one sign which contradicts (10).

As $\alpha \to 1$, $x_i(\alpha)$, $x'_{i+1}(\alpha) \to -\infty$ and $x_{i+1}(\alpha)$, $x'_{i+2}(\alpha) \to +\infty$ (or the ends of the spectrum of μ), and as $\alpha \to 0$, $x_i(\alpha) \to x_i^*$, $x_{i+1}(\alpha) \to x_i^*$, and $x'_{i+1}(\alpha) \to$

x_{i+1}^*, $x'_{i+2}(\alpha) \to x_{i+1}^*$. Lemma 5 below asserts that $x_i^* < x_{i+1}^*$ but first it is necessary to prove Lemma 4.

LEMMA 4. *$\partial p(x_i^*, \omega)/\partial\omega$ does not vanish at ω_i^0 or ω_{i+1}^0 but does vanish for some ω_i^* where $\omega_i^0 < \omega_i^* < \omega_{i+1}^0$, $i = 1, \cdots, n-2$.*

Proof. By the mean value theorem for some $\omega_i^*(\alpha) \varepsilon [\omega_i^0, \omega_{i+1}^0]$,

$$\frac{\partial}{\partial\omega}\int_{x_i(\alpha)}^{x_{i+1}(\alpha)} p(x, \omega)\, d\mu(x)\Bigg|_{\omega=\omega_i^*(\alpha)} = \int_{x_i(\alpha)}^{x_{i+1}(\alpha)} \frac{\partial}{\partial\omega} p(x, \omega_i^*(\alpha))\, d\mu(x) = 0$$

for every $\alpha \varepsilon [0, 1]$. As $\alpha \to 0$, $\omega_i^*(\alpha) \to \omega_i^*$; $\partial p(x_i^*, \omega_i^*)/\partial\omega = 0$. Suppose $\omega_i^* = \omega_i^0$. Then, $\partial p(x_i^*, \omega)/\partial\omega > 0$ for $\omega > \omega_i^0$ which implies that $p(x_i^*, \omega_i^0) < p(x_i^*, \omega_{i+1}^0)$. Since $p(x, \omega)$ is continuous in each variable, there exists $\epsilon > 0$ such that $p(x, \omega_i^0) < p(x, \omega_{i+1}^0)$ for all x satisfying $|x - x_i^*| < \epsilon$. But this implies that for sufficiently small α,

$$\int_{x_i(\alpha)}^{x_{i+1}(\alpha)} p(x, \omega_i^0)\, d\mu(x) < \int_{x_i(\alpha)}^{x_{i+1}(\alpha)} p(x, \omega_{i+1}^0)\, d\mu(x),$$

a contradiction of the definition of $x_i(\alpha)$ and $x_{i+1}(\alpha)$. Similarly, $\omega_i^* \neq \omega_{i+1}^0$. Thus, $\omega_i^* \varepsilon (\omega_i^0, \omega_{i+1}^0)$.

LEMMA 5. *$x_i^* < x_{i+1}^*$, $i = 1, \cdots, n-2$.*

Proof. By Lemma 3, $x_i^* \leqq x_{i+1}^*$. Suppose $x_i^* = x_{i+1}^*$. Then, $\partial p(x_i^*, \omega_i^*)/\partial\omega = \partial p(x_i^*, \omega_{i+1}^*)/\partial\omega = 0$, where $\omega_i^* \varepsilon (\omega_i^0, \omega_{i+1}^0)$, $\omega_{i+1}^* \varepsilon (\omega_{i+1}^0, \omega_{i+2}^0)$, which is impossible by assumption.

This lemma can now be utilized to construct decision procedures possessing the property of M.P. For the 2-action problem any monotone procedure (defined by a single number x_1) is unbiased. In the 3-action problem each monotone procedure (x_1, x_2) which satisfies $h_2(\omega_1^0) = \alpha = h_2(\omega_2^0)$ for some $\alpha \varepsilon [0, 1]$ is unbiased. This means the monotone M.P. procedures are a one parameter family since once x_1 is specified as possible, x_2 and α are determined. For $n = 4$ consider $x_1(\alpha_1)$, $x_2(\alpha_1)$ defined by $h_2(\omega_1^0) = \alpha_1 = h_2(\omega_2^0)$ and $x'_2(\alpha_2)$, $x'_3(\alpha_2)$ defined by $h_3(\omega_2^0) = \alpha_2 = h_3(\omega_3^0)$, where α_1 and α_2 are chosen small enough to insure that $x_2(\alpha_1) < x'_2(\alpha_2)$. By Lemma 5 this is possible. Increase α_1 and α_2 until $x_2(\alpha_1) = x'_2(\alpha_2)$. The monotone procedure defined by $(x_1(\alpha_1), x_2(\alpha_1), x'_3(\alpha_2))$ has the property of M.P. Again the monotone M.P. procedures form a one parameter family since any point $y \varepsilon (x_1^*, x_2^*)$ will determine α_1 and α_2 by the condition that $x_2(\alpha_1) = y = x'_2(\alpha_2)$.

For the case of 5 actions the same method of construction is employed and a one parameter family of monotone M.P. decision procedures is designated. Define

$$x_1(\alpha_1),\ x_2(\alpha_1) \text{ by } h_2(\omega_1^0) = \alpha_1 = h_2(\omega_2^0),$$

$$x'_2(\alpha_2),\ x'_3(\alpha_2) \text{ by } h_3(\omega_2^0) = \alpha_2 = h_3(\omega_3^0),$$

$$x''_3(\alpha_3),\ x''_4(\alpha_3) \text{ by } h_4(\omega_3^0) = \alpha_3 = h_4(\omega_4^0),$$

where α_1, α_2, α_3 are chosen so small that $x_2(\alpha_1) < x'_2(\alpha_2)$ and $x'_3(\alpha_2) < x''_3(\alpha_3)$. Increase α_1 and α_3 until $x_2(\alpha_1) = x'_2(\alpha_2)$ and $x'_3(\alpha_2) = x''_3(\alpha_3)$. The monotone procedure $(x_1(\alpha_1), x_2(\alpha_1), x'_3(\alpha_2), x''_4(\alpha_3))$ has the property of maximum probabilities.

The family has only one parameter since the point $y \varepsilon (x_1^*, x_2^*)$ determines α_1, α_2, and α_3 through the relation $x_2(\alpha_1) = y = x_2'(\alpha_2)$. (Note that some values of y in the interval may not be legitimate parameter points. This will happen when the condition $y = x_2'(\alpha_2)$ is satisfied by an α_2 for which $x_3'(\alpha_2) > x_3^*$.)

When $n = 6$, the reader may verify that this method of construction breaks down. The difficulty is that $x_i(\alpha)$ does not have to decrease at the same rate at which $x_{i+1}(\alpha)$ increases. It may not be possible to choose α_2 and α_3 such that $x_3'(\alpha_2) = x_3''(\alpha_3)$ and still have $x_1^* < x_2'(\alpha_2)$ and $x_4''(\alpha_3) < x_4^*$.

For the cases $n = 3, 4$, and 5, note what has been accomplished by introducing the principle of M.P. The statistician, instead of having to choose a procedure from the class of all monotone procedures which is defined by $n - 1$ parameters, has only to choose from a class of procedures defined by only one parameter, those monotone procedures which have the additional property of maximum probabilities.

If the unknown parameter occurs in the density in the form of a translation parameter, that is $p(\xi, \omega) = p(\xi - \omega)$, $d\mu(\xi) = d\xi$, and $p(\cdot)$ is a symmetric function with respect to the origin, then any monotone procedure φ^0 defined by the critical numbers $x_1 < x_2 < \cdots < x_{n-1}$ such that

$$\frac{x_i + x_{i+1}}{2} = \frac{\omega_i^0 + \omega_{i+1}^0}{2} \text{ for } i = 1, 2, \cdots, n - 2$$

satisfies the property of M.P. The proof of this statement is straightforward and is omitted.

3. Unbiasedness in the sense of Lehmann— A decision procedure $\varphi(x)$ is said to be unbiased (in the sense of Lehmann or risk unbiased) if

$$E_\theta[L(\omega, \varphi(x))] \geqq E_\theta[L(\theta), \varphi(x))] \tag{11}$$

for all ω and θ, where $E_\theta(\cdot)$ represents the expected value given that the state of nature is θ. By specializing the loss function $L(\omega, a)$, it can be readily verified that this general definition of unbiasedness reduces to some of the classical notions. For a full discussion of the significance of this concept, the reader is referred to [7].

We search in this analysis to discover when unbiased procedures exist within the class of monotone procedures for the case of multiaction problems. An effective method of explicit construction of such procedures would also be desirable. Unfortunately, in general unbiased procedure need not exist. However, Theorem 1 below provides an affirmative answer for a substantial class of loss functions satisfying assumptions (a) and (b).

It should be emphasized that in contrast to the principle of M.P., which also embodies a generalization of the notion of unbiasedness in testing hypotheses, the present extension involves the specific loss functions in a fundamental way.

(a) $$L_j(\omega) = L_{ij} \qquad \text{for all } \omega \text{ in } S_i = (\omega_{i-1}^0, \omega_i^0],$$

$$i = 1, \cdots, n, \quad j = 1, \cdots, n.$$

Let $L_{ij} - L_{i+1,j} = a_{i,j}$.

(b) $0 \geqq a_{i1} \geqq a_{i2} \geqq \cdots \geqq a_{ii}$ and $a_{ii} < 0$;

$$a_{i,i+1} \geqq a_{i,i+2} \geqq \cdots a_{i,n} \geqq 0 \text{ and } a_{i,i+1} > 0 \text{ for } i = 1, 2, \cdots, n-1.$$

$$\text{Let } b_{ij} = \begin{cases} -a_{ij} & j = 1, \cdots, i \\ a_{ij} & j = i+1, \cdots, n. \end{cases} \qquad i = 1, \cdots, n-1$$

For $j \leqq$ i, $k \geqq i + 2$, $\qquad i = 1, \cdots, n-1,$

(c)
$$\begin{vmatrix} b_{i,j} & b_{i,k} \\ b_{i+1,j} & b_{i+1,k} \end{vmatrix} \geqq 0.$$

Two important examples of decision problems whose loss functions satisfy conditions (b) and (c) are worth noting.

(I)
$$L_j(\omega) = c \mid i - j \mid \qquad \text{for } \omega \text{ in } S_i.$$

This case is referred to as the discrete absolute error loss function.

(II)
$$L_j(\omega) = \begin{cases} 0 & \omega \varepsilon S_j, \\ c & \omega \not\varepsilon S_j. \end{cases}$$

The second example corresponds to the case where one assigns a constant loss c for any error and zero loss for a correct decision.

The fact that, if it exists, the monotone unbiased procedure is unique lends greater significance to this principle.

Examples I and II above are special cases of loss structures having the form $L_{ij} = f(\mid i - j \mid) = L_{\mid i-j \mid}$. Loss structures of this general pattern possess considerable interest since many practical problems arise in which the incurred losses can be assumed to be proportional to the magnitude of the error and unrelated to the type of error. In the event that $L_{ij} = L_{\mid i-j \mid}$ (we say L_{ij} has a convolution form), condition (b) implies that $L_{\mid i-j \mid}$ is a concave function of $\mid i - j \mid$, i.e., $L_{r+1} \geqq \frac{1}{2}(L_r + L_{r+2})$, $r = 0, 1, \cdots, n-2$. This is to say the loss increases concavely as the action actually taken diverges from the correct action. That concavity implies condition (b) is also true, so condition (b) is fully equivalent to the concavity of $L_{\mid i-j \mid}$ as a function of $\mid i - j \mid$. Moreover, condition (c) is automatically satisfied if $L_{\mid i-j \mid}$ is concave since $b_{ij} \geqq b_{i+1,j}$ for $j \leqq i$ and $b_{i,k} \leqq b_{i+1,k}$ for $k \geqq i + 2$. Therefore, for this convolution case, the hypotheses of Theorem 1 are equivalent to the statement that $L_{\mid i-j \mid}$ is a concave function of its argument.

It should be noted that condition (c) is not the same as condition (II) of [3]. However, in the important case $L_{ij} = L_{\mid i-j \mid}$, the two conditions are equivalent since the two b_{ij} matrices are identical. Consequently, when the loss function $L_{ij} = L_{\mid i-j \mid}$ is concave, all non-degenerate monotone procedures are admissible.

In particular, the unique unbiased procedure guaranteed by Theorem 1 which is also shown to be non-degenerate (Corollary 4) is necessarily admissible in the case where L_{ij} is of the convolution form.

(The proof of Theorem 2 of [3] is easily seen to apply in the case of loss functions of convolution form satisfying (b) and (c), above.)

The principle theorem concerning unbiased procedures is the following:

THEOREM 1. *If assumptions* (a), (b), *and* (c) *are satisfied, then there exists a unique monotone procedure which is unbiased in the sense of Lehmann.*

To avoid inessential tedious details we assume that $p(x, \omega)$ is strictly Pólya type 2, and μ is a continuous measure whose spectrum is an interval. The analogous results when the assumption on μ is relaxed are immediate.

The proof of Theorem 1 is more elaborate and will be presented in Sec. 4. We dwell in this section on the important special case of (I) where the proofs are considerably simpler and for which some additional results are obtained (Theorem 2).

Proof of Theorem 1 *for the special case* (I). For a monotone procedure $(x_1, \cdots, x_{n-1})$ define

$$A_1(\omega) = c\int_{x_1}^{x_2} p(x, \omega)\, d\mu(x) + 2c\int_{x_2}^{x_3} p(x, \omega)\, d\mu(x) + \cdots + (n-1)c\int_{x_{n-1}}^{\infty} p(x, \omega)\, d\mu(x)$$

$$A_2(\omega) = c\int_{-\infty}^{x_1} p(x, \omega)\, d\mu(x) + c\int_{x_2}^{x_3} p(x, \omega)\, d\mu(x) + \cdots + (n-2)c\int_{x_{n-1}}^{\infty} p(x, \omega)\, d\mu(x)$$

$\vdots$

$$A_n(\omega) = (n-1)c\int_{-\infty}^{x_1} p(x, \omega)\, d\mu(x) + (n-2)c\int_{x_1}^{x_2} p(x, \omega)\, d\mu(x) + \cdots + c\int_{x_{n-2}}^{x_{n-1}} p(x, \omega)\, d\mu(x).$$

For $\omega \in S_i$, $i = 1, \cdots, n$, $\rho(\omega, \varphi) = A_i(\omega)$. Define

$$B_i(\omega) = A_i(\omega) - A_{i+1}(\omega), \qquad i = 1, \cdots, n-1.$$

It is immediate that

$$B_i(\omega) = -c\int_{-\infty}^{x_i} p(x, \omega)\, d\mu(x) + c\int_{x_i}^{\infty} p(x, \omega)\, d\mu(x), \qquad i = 1, \cdots, n-1.$$

In order that the monotone procedure be unbiased it is necessary and sufficient that $B_j(\omega) \geqq 0$, $j = 1, \cdots, i-1$; $B_j(\omega) \leqq 0$, $j = i, \cdots, n-1$ for $\omega \in S_i$, $i = 1, \cdots, n-1$. Choose the unique $x_1 = x_1^0$ which satisfies $B_1(\omega_1^0) = 0$. Then $B_1(\omega)$ $(\gtrless)$ 0 for all ω $(\gtrless)$ ω_1^0. Since $x_i \geqq x_1^0$ for $i = 2, \cdots, n-1$, $B_i(\omega) < 0$ for $\omega < \omega_1^0$, $i = 2, \cdots, n-1$. Unbiasedness further requires that for $\omega \in S_2$ $B_1(\omega) \geqq 0$ and $B_i(\omega) \leqq 0$ for $i = 2, \cdots, n-1$. Determine the unique $x_2 = x_2^0$ such that $B_2(\omega_2^0) = 0$. $x_2^0 > x_1^0$ since $\omega_2^0 > \omega_1^0$, and $B_i(\omega) < 0$ for $\omega \in S_2$ and $i =$

$2, \cdots, n-1$. The continuation of this construction will produce the unique monotone unbiased procedure $(x_1^0, \cdots, x_{n-1}^0)$.

For the special loss function under consideration this unique unbiased procedure is uniformly most powerful within the class of all unbiased procedures. This is the substance of the following theorem which is a special case of Theorem 2 of [8]. The proof is included by merit of its simplicity and because it also illustrates on a small scale some of the ideas necessary in carrying out the arguments of Theorem 1.

THEOREM 2. *If $L_i(\omega) = c \mid i - j \mid$ for ω in S_j, then any unbiased procedure $\varphi = (\varphi_1, \cdots, \varphi_n)$ is everywhere improved upon by the unique monotone unbiased procedure, except possibly at $\omega_1^0, \cdots, \omega_{n-1}^0$.*

Proof. By definition,

$$B_1(\omega) = A_1(\omega) - A_2(\omega) = -c \int_{-\infty}^{\infty} \varphi_1(x)p(x, \omega)\, d\mu(x) + c \int_{-\infty}^{\infty} (1 - \varphi_1(x))p(x, \omega)\, d\mu(x)$$

$$= c - 2c \int_{-\infty}^{\infty} \varphi_1(x)p(x, \omega)\, d\mu(x),$$

and for $k = 2, \cdots, n-1$,

$$B_k(\omega) = A_k(\omega) - A_{k+1}(\omega) = c - 2c \int_{-\infty}^{\infty} [\varphi_1(x) + \cdots + \varphi_k(x)]p(x, \omega)\, d\mu(x).$$

Consider any other decision procedure φ^* which is not necessarily unbiased. For $k = 1, \cdots, n-1$,

$$B_k^{\varphi}(\omega) - B_k^{\varphi^*}(\omega) = 2c \int_{-\infty}^{\infty} [(\varphi_1^*(x) + \cdots + \varphi_k^*(x)) - (\varphi_1(x) + \cdots + \varphi_k(x)]p(x, \omega)\, d\mu(x).$$

If φ^* is the monotone procedure constructed so that it improves upon φ according to Lemma 4 of [4], then φ^* satisfies

$$B_k^{\varphi}(\omega) - B_k^{\varphi^*}(\omega) \begin{cases} \geqq 0 & \text{for } \omega \leqq \omega_k^0 \\ \leqq 0 & \text{for } \omega \geqq \omega_k^0 \end{cases}$$

for $k = 1, \cdots, n-1$. But $B_k^{\varphi}(\omega_k^0) = 0$. Therefore, $B_k^{\varphi^*}(\omega_k^0) = 0$ which implies that φ^* is unbiased. Since there is only one monotone unbiased procedure, φ^* must be identical with the φ^0 of Theorem 1.

The limiting case of an n-action problem as $n \to +\infty$ is an estimation problem. Suppose that for the problem under consideration the limit is taken in such a manner that as $n \to \infty$, $\omega_1^0 \to -\infty$, $\omega_{n-1}^0 \to +\infty$, $\mid \omega_i^0 - \omega_{i-1}^0 \mid \to 0$, $i = 2, \cdots, n-1$, and $L_n(\omega, i_n(a)) \to c \mid a - \omega \mid$, where $i_n(a)$ is defined by $a \varepsilon S_{i_n(a)}$. The resulting problem is an estimation problem with absolute error loss function. It is easily verified that the estimate $\delta(x)$, which is the limit of the unique monotone procedures which are unbiased in the sense of Lehmann, is defined by the relation

$$\int_{-\infty}^{x} p(y, \delta(x))\, d\mu(y) = \int_{x}^{\infty} p(y, \delta(x))\, d\mu(y). \tag{12}$$

This, of course, states the well-known fact that the median unbiased estimate of θ is the function $\delta(x)$ which satisfies (12) when x is observed.

4. *Proof of Theorem* 1. For purposes of clarity the proof of the theorem is divided into a series of separate steps. First, we introduce the relevant quantities entering into the analysis. For a procedure $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_n)$, let

$$A_i^{\varphi}(\omega) = L_{i1}\int_{-\infty}^{\infty} \varphi_1(x)p(x, \omega)\, d\mu(x) + \cdots + L_{in}\int_{-\infty}^{\infty} \varphi_n(x)p(x, \omega) d\mu(x)$$

for $i = 1, \cdots, n$. When ω ranges over S_i the function $A_i^{\varphi}(\omega)$ coincides with $\rho(\omega, \varphi)$, the expected risk. Also for $i = 1, 2, \cdots, n - 1$, we define

$$\begin{aligned} B_i^{\varphi}(\omega) &= A_i^{\varphi}(\omega) - A_{i+1}^{\varphi}(\omega) \\ &= a_{i1}\int_{-\infty}^{\infty} \varphi_1(x)p(x, \omega)\, d\mu(x) + \cdots + a_{in}\int_{-\infty}^{\infty} \varphi_n(x)p(x, \omega)\, d\mu(x) \\ &= -b_{i1}\int_{-\infty}^{\infty} \varphi_1(x)p(x, \omega)\, d\mu(x) - \cdots - b_{ii}\int_{-\infty}^{\infty} \varphi_i(x)p(x, \omega)\, d\mu(x) \\ &\quad + b_{i,i+1}\int_{-\infty}^{\infty} \varphi_{i+1}(x)p(x, \omega)\, d\mu(x) + \cdots + b_{in}\int_{-\infty}^{\infty} \varphi_n(x)p(x, \omega)\, d\mu(x). \end{aligned} \tag{13}$$

If a decision procedure φ satisfies the system of inequalities

$$\begin{aligned} B_k^{\varphi}(\omega) &\geqq 0 \qquad 1 \leqq k \leqq i - 1 \\ B_k^{\varphi}(\omega) &\leqq 0 \qquad i \leqq k \leqq n - 1 \end{aligned} \qquad \text{and } \omega \text{ in } S_i, \tag{14}$$

then φ is clearly unbiased in the sense of Lehmann. In general, the converse is not valid. However, it is true that for monotone procedures the property of unbiasednesss implies that this system of inequalities is satisfied. The inequalities are fulfilled for a monotone procedure $\varphi = (x_1, x_2, \cdots, x_{n-1})$ if and only if

$$B_i^{\varphi}(\omega_i^0) = 0, \qquad i = 1, 2, \cdots, n - 1. \tag{15}$$

In fact, the variation diminishing properties of the density $p(x, \omega)$ imply that $B_i^{\varphi}(\omega) < 0$ for $\omega < \omega_i^0$ and $B_i^{\varphi}(\omega) > 0$ for $\omega > \omega_i^0$ which in turn are equivalent to the system of inequalities (14). Our problem reduces to the demonstration of the existence and uniqueness of a set of values $x = (x_1, x_2, \cdots, x_{n-1})$ where $x_1 \leqq x_2 \leqq \cdots \leqq x_{n-1}$ which are a solution to the system of non-linear equations:

$$\begin{aligned} B_i^{x}(\omega_i^0) = &-b_{i1}\int_{-\infty}^{x_1} p(\xi, \omega_i^0)\, d\mu(\xi) - \cdots - b_{ii}\int_{x_{i-1}}^{x_i} p(\xi, \omega_i^0)\, d\mu(\xi) \\ &+ b_{i,i+1}\int_{x_i}^{x_{i+1}} p(\xi, \omega_i^0)\, d\mu(\xi) + \cdots + b_{i,n}\int_{x_{n-1}}^{\infty} p(\xi, \omega_i^0)\, d\mu(\xi) = 0. \end{aligned} \tag{16}$$

Turning to this task we start by showing that the mapping $x \to y$ which is defined coordinate-wise by $y_i = B_i^x(\omega_i^0)$, $i = 1, \cdots, n-1$, and which maps the $n-1$ dimensional simplex of all $n-1$ tuples $x = (x_1, x_2, \cdots, x_{n-1})$ satisfying $x_1 \leqq x_2 \leqq \cdots \leqq x_{n-1}$ into Euclidean $n-1$ dimensional space (E^{n-1}) is a one-to-one mapping. Precisely:

LEMMA 6. *The mapping* $y_i = B_i^x(\omega_i^0)$, $i = 1, \cdots, n-1$, *defined on the set of all monotone procedures by means of the formulas* (16) *with image in* E^{n-1} *space is a one-to-one transformation.*

Proof (by contradiction). Suppose there exist two different monotone procedures $\varphi \sim x = (x_1, x_2, \cdots, x_{n-1})$ and $\varphi' \sim x' = (x_1', x_2', \cdots, x_{n-1}')$ with the property that $B_i^{x'}(\omega_i^0) - B_i^x(\omega_i^0) = 0$ for $i = 1, \cdots, n-1$. Without loss of generality assume $x_1' \geqq x_1$. $B_i^{x'}(\omega_i^0) - B_i^x(\omega_i^0) = 0$, $i = 1, \cdots, n-1$, yields the system of equations

$$0 = -(b_{11} + b_{12}) \int_{x_1}^{x_1'} p(x, \omega_1^0)\, d\mu(x) + (b_{12} - b_{13}) \int_{x_2}^{x_2'} p(x, \omega_1^0)\, d\mu(x) + \cdots + (b_{1,n-1} - b_{1n}) \int_{x_{n-1}}^{x_{n-1}'} p(x, \omega_1^0)\, d\mu(x)$$

$$0 = (b_{22} - b_{21}) \int_{x_1}^{x_1'} p(x, \omega_2^0)\, d\mu(x) - (b_{22} + b_{23}) \int_{x_2}^{x_2'} p(x, \omega_2^0)\, d\mu(x) + \cdots + (b_{2,n-1} - b_{2,n}) \int_{x_{n-1}}^{x_{n-1}'} p(x, \omega_2^0)\, d\mu(x)$$

$$0 = (b_{n-1,2} - b_{n-1,1}) \int_{x_1}^{x_1'} p(x, \omega_{n-1}^0)\, d\mu(x) + \cdots + (b_{n-1,n-1} - b_{n-1,n-2}) \int_{x_{n-2}}^{x_{n-2}'} p(x, \omega_{n-1}^0)\, d\mu(x) - (b_{n-1,n-1} + b_{n-1,n}) \int_{x_{n-1}}^{x_{n-1}'} p(x, \omega_{n-1}^0)\, d\mu(x).$$

Since $(b_{11} + b_{12}) > (b_{12} - b_{13} + \cdots + (b_{1,n-1} - b_{1,n})$, it follows that there exists a k, $1 < k \leqq n-1$, such that

$$\int_{x_l}^{x_l'} p(x, \omega_1^0)\, d\mu(x) < \int_{x_k}^{x_k'} p(x, \omega_1^0)\, d\mu(x)$$

for $1 \leqq l < k$. If k is not unique, choose the largest k which satisfies this property. Consider the kth equation. For $1 \leqq l < k$,

$$\int_{x_l}^{x_l'} p(x, \omega_k^0)\, d\mu(x) < \int_{x_k}^{x_k'} p(x, \omega_k^0)\, d\mu(x)$$

by the fundamental change of sign property for strictly Pólya type 2 densities since $x_l \leqq x_k$ and $x_l' < x_k'$. But $(b_{kk} + b_{k,k+1}) \geqq (b_{k2} - b_{k1}) + \cdots + (b_{kk} - b_{k,k-1})$

$+ (b_{k,k+1} - b_{k,k+2}) + \cdots + (b_{k,n-1} - b_{k,n})$. Therefore on examination of the kth equation, if $k < n - 1$, there exists an $h > k$ for which

$$\int_{x_l}^{x'_l} p(x, \omega_k^0)\, d\mu(x) < \int_{x_h}^{x'_h} p(x, \omega_k^0)\, d\mu(x)$$

for $1 \leqq l < h$. If h is not unique, choose the largest h.

Continue this argument until at the last step it has been established that

$$\int_{x_l}^{x'_l} p(x, \omega_{n-1}^0)\, d\mu(x) < \int_{x_{n-1}}^{x'_{n-1}} p(x, \omega_{n-1}^0)\, d\mu(x)$$

for $1 \leqq l < n - 1$. But this contradicts the fact that $B_{n-1}^{x'}(\omega_{n-1}^0) - B_{n-1}^{x}(\omega_{n-1}^0) = 0$ since $(b_{n-1,n-1} + b_{n-1,n}) > (b_{n-1,2} - b_{n-1,1}) + \cdots + (b_{n-1,n-1} - b_{n-1,n-2})$.

COROLLARY 1. *There exists at most one monotone unbiased procedure.*

The proof is immediate. We shall need the following slight extension of Lemma 6.

Corollary 2. *If* φ *is the monotone procedure* $x = (x_1, x_2, \cdots, x_{n-1})$ *and* $\varphi' \sim x' = (x'_1, x'_2, \cdots, x'_{n-1})$ *with* $x'_{n-1} \geqq x_{n-1}$ *and* $B_i^{x'}(\omega_i^0) - B_i^{x}(\omega_i^0) \geqq 0$ *for* $i = 1, 2, \cdots, n-1$, *then* $x'_i = x_i$ *for* $i = 1, 2, \cdots, n - 1$.

The proof of Corollary 2 is essentially a paraphrase of that of Lemma 6. We sketch the details. Let k be the first index where $x'_k \geqq x_k$ $(k \leqq n - 1)$. By examining the kth relation $B_k^{x'}(\omega_k^0) - B_k^{x}(\omega_k^0) \geqq 0$ as in the proof of the lemma, when $k < n - 1$, we may find a larger index $h > k$ such that for $i < h$,

$$\int_{x_i}^{x'_i} p(\xi, \omega_k^0)\, d\mu(\xi) < \int_{x_h}^{x'_h} p(\xi, \omega_k^0)\, d\mu(\xi).$$

From the variation diminishing properties of $p(\xi, \omega)$ we may conclude that for $i < h$,

$$\int_{x_i}^{x'_i} p(\xi, \omega_h^0)\, d\mu(\xi) < \int_{x_h}^{x'_h} p(\xi, \omega_h^0)\, d\mu(\xi).$$

On continued inspection of the hth relation, we find a larger index until we reach the $(n - 1)^{\text{th}}$ index with the property that

$$\int_{x_i}^{x'_i} p(\xi, \omega_{n-1}^0)\, d\mu(\xi) < \int_{x_{n-1}}^{x'_{n-1}} p(\xi, \omega_{n-1}^0)\, d\mu(\xi), \qquad i = 1, 2, \cdots, n - 2.$$

The last inequality

$$B_{n-1}^{x'}(\omega_{n-1}^0) - B_{n-1}^{x}(\omega_{n-1}^0) \geqq 0$$

is evidently contradicted.

One final extension in the same direction is the following:

COROLLARY 3. *If* $\varphi \sim (x_1, x_2, \cdots, x_{n-1})$ *and* $\varphi' \sim (x'_1, x'_2, \cdots, x'_{k-1}, \gamma, \cdots, \gamma)$

are two monotone procedures such that $B_i^{\varphi'}(\omega_i^0) - B_i^{\varphi}(\omega_i^0) \geqq 0$ *for* $i = 1, \cdots, k-1$ *and* $\gamma \geqq x_{n-1}$, *then*

$$B_k^{\varphi'}(\omega_k^0) - B_k^{\varphi}(\omega_k^0) \leqq 0.$$

The proof follows the same line of reasoning as the preceding.

In view of Corollary 1 it remains to prove the existence part of Theorem 1. We require the following lemma.

LEMMA 7. *Let the* $2 \times m$ *matrix* (e_{ij}), $i = 1, 2, j = 1, \cdots, m$, *consist of non-negative elements, and let* $\lambda_1, \cdots, \lambda_m$ *be non-negative constants. Let condition* (E) *be satisfied*:

$$\text{(E)} \qquad \begin{vmatrix} e_{1j} & e_{1k} \\ e_{2j} & e_{2k} \end{vmatrix} \geqq 0$$

for $1 \leqq j \leqq l$, $l + 2 \leqq k \leqq m$. *If* $0 < e_{11}\lambda_1 + \cdots + e_{1l}\lambda_l \leqq e_{1,l+2}\lambda_{l+2} + \cdots + e_{1,m}\lambda_m$, *then* $e_{21}\lambda_1 + \cdots + e_{2l}\lambda_l \leqq e_{2,l+2}\lambda_{l+2} + \cdots + e_{2m}\lambda_m$.

Proof. By (E), $\sum_{j=1}^{l} (e_{2k}e_{1j} - e_{1k}e_{2j})\lambda_j \geqq 0$ for $k \geqq l + 2$. Therefore, $e_{2k} \sum_{j=1}^{l} e_{1j}\lambda_j \geqq e_{1k} \sum_{j=1}^{l} e_{2j}\lambda_j$, and $\sum_{k=l+2}^{m} e_{2k}\lambda_k \cdot \sum_{j=1}^{l} e_{1j}\lambda_j \geqq \sum_{k=l+2}^{m} e_{1k}\lambda_k \cdot \sum_{j=1}^{l} e_{2j}\lambda_j$. For $0 < \sum_{j=1}^{l} e_{1j}\lambda_j \leqq \sum_{k=l+2}^{m} e_{1k}\lambda_k$, $\sum_{k=l+2}^{m} e_{2k}\lambda_k \geqq \sum_{j=1}^{l} e_{2j}\lambda_j$.

Proof of existence. It suffices to show there exists a monotone φ for which $B_i^{\varphi}(\omega_i^0) = 0$, $i = 1, \cdots, n-1$. This holds trivially for $n = 2$. Suppose it is true for the case of n actions. The argument is inductive. For $n + 1$ actions and a monotone procedure, let

$$B_i^{\varphi}(\omega) = -b_{i1} \int_{-\infty}^{x_1} p(x, \omega)\, d\mu(x) - \cdots - b_{ii} \int_{x_{i-1}}^{x_i} p(x, \omega)\, d\mu(x)$$
$$+ b_{i,i+1} \int_{x_i}^{x_{i+1}} p(x, \omega)\, d\mu(x) + \cdots + b_{i,n+1} \int_{x_n}^{\infty} p(x, \omega)\, d\mu(x)$$

for $i = 1, \cdots, n$.

(1) Choose $x_n = \infty$. The conditions (a), (b), and (c) are fulfilled so by the induction hypothesis there exists a solution $\varphi^{\infty} \sim (x_1^{\infty}, \cdots, x_{n-1}^{\infty}, \infty)$ of the system of equations $B_i^{\varphi}(\omega_i^0) = 0$, $i = 1, \cdots, n-1$. For this solution obviously $B_n^{\infty}(\omega_n^0) \leqq 0$.

(2) Choose $x_{n-1} = x_n$. By the induction hypothesis there exists a solution $\varphi^{x^0} \sim (x_1^0, \cdots, x_{n-1}^0 = x^0, x_n^0 = x^0)$ of $B_i^{\varphi}(\omega_i^0) = 0$, $i = 1, \cdots, n-1$. Since $B_{n-1}^{x^0}(\omega_{n-1}^0) = 0$, the variation diminishing properties of densities possessing a strict monotone likelihood ratio lead to the conclusion that $B_{n-1}^{x^0}(\omega_n^0) \geqq 0$. If $B_{n-1}^{x^0}(\omega_n^0) = 0$, then it follows that $x^0 = -\infty$ which in turn implies that $B_n^{x^0}(\omega_n^0) > 0$. On the other hand, if $B_{n-1}^{x^0}(\omega_n^0) > 0$, let $l = n - 1$, $m = n$, $e_{1j} = b_{n-1,j}$ for $j = 1, \cdots, n-1$, $e_{1n} = b_{n-1,n+1}$, $e_{2j} = b_{n,j}$ for $j = 1, \cdots, n-1$ and $e_{2n} = b_{n,n+1}$ in Lemma 7. Then, by Lemma 7 $B_{n-1}^{x^0}(\omega_n^0) > 0$ implies $B_n^{x^0}(\omega_n^0) \geqq 0$.

It has been shown thus far that there exists a strategy $(x_1^{\infty}, \cdots, x_{n-1}^{\infty}, \infty)$ such that $B_i^{\varphi}(\omega_i^0) = 0$, $i = 1, \cdots, n-1$, and $B_n^{\varphi}(\omega_n^0) \leqq 0$ and a strategy $(x_1^0, \cdots, x_{n-1}^0 = x^0, x_n^0 = x^0)$ such that $B_i^{\varphi}(\omega_i^0) = 0$, $i = 1, \cdots, n-1$ and $B_n^{\varphi}(\omega_n^0) \geqq 0$. If it can be shown that for every x_n satisfying $x^0 < x_n < \infty$ there exists a solution $(x_1, \cdots, x_{n-1})$ to $B_i^\varphi(\omega_i^0) = 0$ $i = 1, \cdots, n-1$, then by continuity a solution exists satisfying $B_i^\varphi(\omega_i^0) = 0$, $i = 1, \cdots, n$; the continuity of the solution as a function of x_n being a simple consequence of Lemma 6.

The proof that for every z, $x^0 < z < \infty$, there exist $(x_1(z), \cdots, x_{n-1}(z))$ such that $\varphi \sim (x_1(z), \cdots, x_{n-1}(z), z)$ satisfies $B_i^\varphi(\omega_i^0) = 0$, $i = 1, \cdots, n-1$, proceeds in a stepwise manner.

(3) Let $x_1 \leqq x_2 = \cdots = x_{n-1} = z$.

(a) Choose $x_1 = z$. Since $b_{12} \geqq b_{13} \geqq \cdots \geqq b_{1,n+1}$,

$$-b_{11} \int_{-\infty}^{x_1^0} p(x, \omega_1^0)\, d\mu(x) + b_{1,n+1} \int_{x_1^0}^{\infty} p(x, \omega_1^0)\, d\mu(x) \leqq 0$$

which implies

$$-b_{11} \int_{-\infty}^{z} p(x, \omega_1^0)\, d\mu(x) + b_{1,n+1} \int_{z}^{\infty} p(x, \omega_1^0)\, d\mu(x) \leqq 0$$

since $x_1^0 \leqq x_n^0 < z$.

(*b*) Choose $x_1 = -\infty$.

$$b_{12} \int_{-\infty}^{z} p(x, \omega_1^0)\, d\mu(x) + b_{1,n+1} \int_{z}^{\infty} p(x, \omega_1^0)\, d\mu(x) \geqq 0.$$

(c) Thus by continuity there must exist an $x_1^1 = x_1^1(z)$ which satisfies

$$-b_{11} \int_{-\infty}^{x_1^1} p(x, \omega_1^0)\, d\mu(x) + b_{12} \int_{x_1^1}^{z} p(x, \omega_1^0)\, d\mu(x) + b_{1,n+1} \int_{z}^{\infty} p(x, \omega_1^0)\, d\mu(x) = 0.$$

(4) Let $x_1 \leqq x_2 \leqq x_3 = \cdots = x_{n-1} = z$. Consider the two expressions

$$\begin{aligned} c_1(\omega; x_1, x_2) = {} & -b_{11} \int_{-\infty}^{x_1} p(x, \omega)\, d\mu(x) + b_{12} \int_{x_1}^{x_2} p(x, \omega)\, d\mu(x) \\ & + b_{13} \int_{x_2}^{z} p(x, \omega)\, d\mu(x) + b_{1,n+1} \int_{z}^{\infty} p(x, \omega)\, d\mu(x), \\ c_2(\omega; x_1, x_2) = {} & -b_{21} \int_{-\infty}^{x_1} p(x, \omega)\, d\mu(x) - b_{22} \int_{x_1}^{x_2} p(x, \omega)\, d\mu(x) \\ & + b_{23} \int_{x_2}^{z} p(x, \omega)\, d\mu(x) + b_{2,n+1} \int_{z}^{\infty} p(x, \omega)\, d\mu(x). \end{aligned}$$

Of course $c_j(\omega, x_1, x_2) = B_j^\varphi(\omega)$, $j = 1, 2$, for the special procedure $\varphi \sim (x_1, x_2, z, \cdots z)$. Our immediate object now is to show that x_1 and x_2 exist satisfying $(x_1 \leqq x_2 \leqq z)$ such that $c_1(\omega_1^0; x_1, x_2) = 0$ and $c_2(\omega_2^0; x_1, x_2) = 0$.

(a) Choose $x_2 = z$. By (3) above there exists an $x_1'(z)$ for which

$$c_1(\omega_1^0; x_1'(z), z) = 0.$$

We assert that $c_2(\omega_2^0; x_1'(z), z) \leqq 0$. Comparing for $i = 1, 2$ $B_i^{\varphi^0}(\omega_i^0)$ and $B_i^\varphi(\omega_i^0)$ where $\varphi^0 \sim (x_1^0, x_2^0, \cdots, x_{n-1}^0 = x^0, x_n^0 = x^0)$ of (2) above and

$\varphi \sim (x_1'(z), z, z, \cdots, z)$ with $z > x^0$, we see the conditions of Corollary 3 are met and therefore we may conclude $c_2(\omega_2^0, x_1'(z), z) \leqq 0$ as stated.

(b) Choose $x_1 = x_2$; then $c_1(\omega_1^0; -\infty, -\infty) \geqq 0$ and $c_1(\omega_1^0; z, z) \leqq 0$ by (3a). Thus there exists a $u = x_1 = x_2$ such that $c_1(\omega_1^0; u, u) = 0$ which implies $c_1(\omega_2^0; u, u) \geqq 0$ If $c_1(\omega_2^0; u, u) = 0$, then $u = -\infty$ which in turn implies $c_2(\omega_2^0; u, u) \geqq 0$. If in the other circumstance $c_1(\omega_2^0; u, u) > 0$, then by Lemma 7 we infer that $c_2(\omega_2^0; u, u) \geqq 0$.

(c) We next prove that there exists an $x_1^* = x_1^*(y)$ such that $c_1(\omega_1^0; x_1^*, y) = 0$ for every $u < y < z$. (This is like the larger problem we are trying to solve for the special case when $n = 2$. The quantity z plays the role of ∞ and u adopts the role of z.) When $x_1 = y$, $c_1(\omega_1^0; y, y) < 0$ because $c_1(\omega_1^0; u, u) = 0$ and $y > u$. Obviously $c_1(\omega_1^0; -\infty, y) > 0$. By continuity there exists an x_1^* such that $c_1(\omega_1^0; x_1^*, y) = 0$.

Since $c_1(\omega_1^0; x_1, y) = 0$ has a solution x_1^* for every y in the interval $[u, z]$ and $c_2(\omega_2^0; x_1'(z), z) \leqq 0$, $c_2(\omega_2^0; u, u) \geqq 0$, by continuity there must exist an $x_2^2(z) = y \,\varepsilon\, [u, z]$ and $x_1^2(z)$ such that $c_1(\omega_1^0; x_1^2, x_2^2) = c_2(\omega_2^0; x_1^2, x_2^2) = 0$.

(5) Let $x_1 \leqq x_2 \leqq x_3 \leqq x_4 = \cdots = z$. Consider the three expressions

$$\begin{aligned} D_i(\omega; x_1, x_2, x_3) = &- \sum_{j=1}^{i} b_{ij} \int_{x_{j-1}}^{x_j} p(x, \omega)\, d\mu(x) \\ &+ \sum_{j=i+1}^{3} b_{ij} \int_{x_{j-1}}^{x_j} p(x, \omega)\, d\mu(x) + b_{i4} \int_{x_3}^{z} p(x, \omega)\, d\mu(x) \\ &+ b_{i,n+1} \int_{z}^{\infty} p(x, \omega)\, d\mu(x), \end{aligned}$$

$i = 1, 2, 3$, where $x_0 = -\infty$. Of course $D_i(\omega; x_1, x_2, x_3) = B_i^{\varphi}(\omega)$ where $\varphi \sim (x_1, x_2, x_3, z, z, \cdots, z)$. The next step is to try to solve $D_i(\omega_i^0; x_1, x_2, x_3) = 0$, $i = 1, 2, 3$.

(a) Choose $x_3 = z$. By (4) above there exists a couple $(x_1^2(z), x_2^2(z))$ such that $D_1(\omega_1^0; x_1^2(z), x_2^2(z), z) = D_2(\omega_2^0; x_1^2(z), x_2^2(z), z) = 0$. Corollary 3 may be applied and we find that on comparison with the relations $B_i^{\varphi^0}(\omega_i) = 0$, $i = 1, 2, 3$, for $\varphi^0 \sim (x_1^0, x_2^0, \cdots, x_n^0)$ of (2), $D_3(\omega_3^0; x_1^2(z), x_2^2(z), z) \leqq 0$.

(b) Choose $x_2 = x_3$. By (4) there exists a solution $(\bar{x}_1(w), w)$ where $x_2 = x_3 = w$ to the equations $D_1(\omega_1^0; x_1, x_2, x_2) = 0$, $D_2(\omega_2^0; x_1, x_2, x_2) = 0$. $D_3(\omega_3^0; \bar{x}_1, w, w) \geqq 0$ is a consequence of Lemma 7.

(c) There exists a couple $(x_1^{**}(y), x_2^{**}(y))$ such that

$$D_1(\omega_1^0; x_1^{**}(y), x_2^{**}(y), y) = D_2(\omega_2^0; x_1^{**}(y), x_2^{**}(y), y) = 0$$

for every $y \in [w, z]$.

The proof of this step requires a repetition of the previous arguments as carried out for the function c_i with y taking the role of ∞. To this end, we establish

(c.1) Choose $x_1 = x_2$. For $x_1 = -\infty$, $D_1(\omega_1^0; -\infty, -\infty, y) \geqq 0$. For $x_1 = y$, $D_1(\omega_1^0; y, y, y) \leqq 0$ since $D_1(\omega_1^0; \bar{x}_1(w), w, w) = 0$ implies $D_1(\omega_1^0; w, w, w) \leqq 0$ and $y > w$. Therefore, there exists $x_1 = v$ such that $D_1(\omega_1^0; v, v, y) = 0$. It can be shown by applying Lemma 7 that $D_2(\omega_2^0; v, v, y) \geqq 0$.

(c.2) Choose $x_2 = y$. $D_1(\omega_1^0; -\infty, y, y) \geqq 0$ and $D_1(\omega_1^0; y, y, y) \leqq 0$. Thus, there exists $x_1(y)$ such that $D_1(\omega_1^0; x_1(y), y, y) = 0$. $D_2(\omega_2^0; x_1(y), y, y) \leqq 0$. The last inequality may be deduced from Corollary 3 by comparing the procedures $\varphi' \sim (x_1(y), y, y, z, z, \cdots, z)$ and $\varphi \sim (\bar{x}_1(w), w, w, z, z, \cdots, z)$.

In fact, suppose the inequality $D_2(\omega_2^0; x_1(y), y, y) \leqq 0$ is violated. Consider the solution $(\bar{x}_1(w), w, w)$ to the system of equations

$$D_1(\omega_1^0; x_1, x_2, x_2) = D_2(\omega_2^0; x_1, x_2, x_2) = 0.$$

$$\begin{aligned} &D_1(\omega_1^0; x_1(y), y, y) - D_1(\omega_1^0; \bar{x}_1(w), w, w) \\ &\quad = -(b_{11} + b_{12}) \int_{\bar{x}_1(w)}^{x_1(y)} p(x, \omega_1^0)\, d\mu(x) + (b_{12} - b_{14}) \int_w^y p(x, \omega_1^0)\, d\mu(x) = 0, \end{aligned} \tag{17}$$

$$\begin{aligned} &D_2(\omega_2^0; x_1(y), y, y) - D_2(\omega_2^0; \bar{x}_1(w), w, w) \\ &\quad = (b_{22} - b_{21}) \int_{\bar{x}_1(w)}^{x_1(y)} p(x, \omega_2^0)\, d\mu(x) - (b_{22} + b_{24}) \int_w^y p(x, \omega_2^0)\, d\mu(x) > 0. \end{aligned} \tag{18}$$

Eq. (17) implies $\int_w^y p(x, \omega_1^0)\, d\mu(x) > \int_{\bar{x}_1(w)}^{x_1(y)} p(x, \omega_1^0)\, d\mu(x)$, but this contradicts (18). Therefore, $D_2(\omega_2^0; x_1(y), y, y) \leqq 0$.

(c.3) For every $x_2 \in [v, y]$, $D_1(\omega_1^0; x_1, x_2, y) = 0$ has a solution. By continuity, then, there exists an $x_1^{**}(y), x_2^{**}(y)$ such that $D_1(\omega_1^0; x_1^{**}(y), x_2^{**}(y), y) = D_2(\omega_2^0; x_1^{**}(y), x_2^{**}(y), y) = 0$.

(a), (b), and (c) of (5) show that there exists a 3-tuple $(x_1^3(z), x_2^3(z), x_3^3(z))$ which satisfies $D_i(\omega_i^0; x_1^3, x_2^3, x_3^3) = 0$, $i = 1, 2, 3$.

The steps for $x_1 \leqq x_2 \leqq x_3 \leqq x_4 \leqq x_5 = \cdots = z$ utilize the same principles as those employed above. The general pattern should now be clear to the reader.

The next step would consider the four functions $E_i(\omega; x_1, x_2, x_3, x_4) = B_i^{\varphi}(\omega)$, $i = 1, \cdots, 4$ where $\varphi \sim (x_1, x_2, x_3, x_4, z, z, \cdots, z)$. It is necessary to show that $E_i(\omega_i^0) = 0$, $i = 1, 2, 3, 4$, have a solution in x_1, x_2, x_3, and x_4. This entails repeating the entire preceding argument for the case of one, two, and three functions in each case using a suitable comparison monotone procedure. We sketch the argument. Setting $x_4 = z$ we obtain by (5) that there exists a tuple $(x_1(z), x_2(z), x_3(z), z)$ for which $E_i(\omega_i^0; x_1(z), x_2(z), x_3(z), z) = 0$ for $i = 1, 2, 3$. Corollary 3 may be applied by using the second procedure $\varphi^0 \sim (x_1^0, x_2^0, \cdots, x_n^0)$ to show that $E_4(\omega_4^0; x_1(z), x_2(z), x_3(z), z) \leqq 0$. Next put $x_3 = x_4 = t < z$ and again by (5) we obtain a tuple $(x_1(t), x_2(t), t, t)$ for which $E_i(\omega_i^0; x_1(t), x_2(t), t, t) = 0$ for $i = 1, 2, 3$. According to Lemma 7, $E_4(\omega_4^0; x_1(t), x_2(t), t, t) \geqq 0$. Given y, $t < y < z$, it would be enough to construct a solution to $E_i(\omega_i^0; x_1, x_2, x_3, y) = 0$, $i = 1, 2, 3$, for then by continuity there would exist a solution to $E_i(\omega_i^0) = 0$, $i = 1, 2, 3, 4$. The analysis of $E_i(\omega_i^0; x_1, x_2, x_3, y)$, $i = 1, 2, 3$, is similar to the arguments of (5) this time using the comparison procedure

$$\varphi \sim (x_1(t), x_2(t), t, t, z, z, \cdots, z)$$

as $\varphi \sim (\bar{x}_1(w), w, w, z, \cdots, z)$ was used in (5). For the final step we repeat this sequence of arguments $n - 1$ times. This completes the proof of Theorem 1.

COROLLARY 4. *The unique monotone unbiased procedure defined by Theorem* 1 *is non-degenerate.*

Remark. Since the density $p(x, \omega)$ is assumed to have a strict monotone likelihood ratio, the set $\sigma_\omega = \{x \mid p(x, \omega) > 0\}$ is independent of ω [4]. The concept of an interval (x_i, x_{i+1}) being degenerate should therefore be understood as taken with respect to $d\mu(x)$.

Proof. Suppose the unique unbiased procedure $x^0 = (x_0, x_1, x_2, \cdots, x_{n-1}, x_n)$ where $x_0 = -\infty$ and $x_n = +\infty$ possesses a degenerate interval. We shall prove that this assumption leads to an absurdity. First, observe that (x_0, x_1) must be non-degenerate. Otherwise, let j_0 be such that (x_{j_0}, x_{j_0+1}) is the first non-degenerate interval and $j_0 \geqq 1$. By condition (b), $B_{j_0-1}^{x^0}(\omega_{j_0-1}^0) > 0$, which contradicts the definition of x^0. Now let i_0 be the earliest interval where (x_{i_0}, x_{i_0+1}) is degenerate. Therefore by what has been established $i_0 \geqq 1$ and also $i_0 < n - 1$ for in the contrary case $B_{n-1}^{x^0}(\omega_{n-1}^0)$ would be negative, Let k_0 denote the smallest index larger than i_0 for which (x_{k_0}, x_{k_0+1}) is non-degenerate. A value of k must exist, for otherwise $B_{i_0}^{x^0}(\omega_{i_0}^0) < 0$.

The strict monotone likelihood ratio possessed by $p(x, \omega)$ implies that

$$(*) \qquad \int_{x_j}^{x_{j+1}} p(\xi, \omega_i^0)\, d\mu(\xi) \int_{x_r}^{x_{r+1}} p(\xi, \omega_k^0)\, d\mu(\xi) \geqq \int_{x_j}^{x_{j+1}} p(\xi, \omega_k^0)\, d\mu(\xi) \int_{x_r}^{x_{r+1}} p(\xi, \omega_i^0)\, d\mu(\xi)$$

for every $j < i_0$ and $r \geqq k_0$ with strict inequality valid for $j = i_0 - 1$ and $r = k_0$. Equation (*) in conjunction with conditions (b) and (c) and $B_{i_0}^{x^0}(\omega_{i_0}^0) = 0$ readily leads to the result

$$B_{k_0}^{x^0}(\omega_{k_0}^0) > 0,$$

which is impossible. This completes the proof.

In any special case this construction is considerably more facile than the general proof shows. We carry this out for the special case whose loss function is (II) of the preceding section. For any prescribed $x_{i-1} < x_i$ a value $x_{i+1}(x_{i+1} > x_i)$ is determined recursively, whenever possible, by

$$(19) \qquad \int_{x_{i-1}}^{x_i} p(\xi, \omega_i^0)\, d\mu(\xi) = \int_{x_i}^{x_{i+1}} p(\xi, \omega_i^0)\, d\mu(\xi)$$

for $i = 1, 2, \cdots, n-1$ where $x_0 = -\infty$. For x_1 sufficiently near $-\infty$, it is possible to solve (19) for each x_i such that $x_i > x_{i-1}$ and each is near $-\infty$. Allowing x_1 to increase, we observe that each x_i increases; and ultimately for $x_1 < \infty$, x_n reaches ∞. Let x_i^* be the solution of (19) where $x_n^* = +\infty$. The procedure $\varphi^* \sim (x_1^*, x_2^*, \cdots, x_{n-1}^*)$ is the unique monotone unbiased procedure for the case where

$$L_i(\omega) = \begin{cases} c & \omega \not\in S_i \\ 0 & \omega \in S_i \end{cases}.$$

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