ticular, if cov $\{y_k - t, t\} < 0$, a value of $\lambda > 0$ will provide lesser variance than for $\lambda = 0$.

From the result it is easy to show that:

(D) *If $F(q)$ is subexponential to the right, then no single order statistic, except possibly the righthandmost, is of minimum variance among orderly estimates of location.*

(Again, the analogs with "to the left . . . the lefthandmost" or "in both directions . . . statistic," follow by symmetry.) For if y_j were of minimum variance, and y_n the righthandmost, then by

$$(B_1)\ \text{cov}\ (y_n - y_j, y_j) = \text{cov}\ (q_n - q_j, q_j) < 0,$$

and by (C) y_j is not of minimum variance. It is reasonable to anticipate that, actually, all coefficients must be positive (particularly for distributions with monotone scores), but the elementary methods used here do not seem to show this easily.

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AN ELEMENTARY THEOREM CONCERNING STATIONARY ERGODIC PROCESSES

By Leo Breiman

University of California, Berkeley

1. Introduction. The purpose of this note is to state and prove a theorem concerning strictly stationary, ergodic processes and to give some of its applications. Although the theorem itself is a simple consequence of the ergodic theorem, its applications include a proof of the consistency of the maximum likelihood estimates for stationary distributions and an extension of the zero-one law for symmetric sets given by Hewitt and Savage [1].

Theorem 1. *Let $\cdots x_{-1}, x_0, x_1, \cdots$ be a strictly stationary process such that every set invariant under shifts has measure zero or one. Let $\{\phi_n\}$ be a sequence of real-valued functions, ϕ_n being a measurable function of $n + 1$ variables. Then if the sequence $\phi_n(x_0, \cdots, x_n)$ and the sequence $\phi_n(x_{-n}, \cdots, x_0)$ both converge in probability, their limits are almost surely constant and equal.*

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PROOF. We assume that $\phi_n(x_{-n}, \cdots, x_0) \xrightarrow{P} \phi^-$, $\phi_n(x_0, \cdots, x_n) \xrightarrow{P} \phi^+$. Let f be any differentiable function such that $|f| \leqq 1$, $|f'| \leqq 1$, and let

$$f_n^- = f(\phi_n(x_{-n}, \cdots, x_0)), \qquad f_n^+ = f(\phi_n(x_0, \cdots, x_n)).$$

Since $|f_n^- - f(\phi^-)| \leqq |\phi_n(x_{-n}, \cdots, x_0) - \phi^-|$, we have that $f_n^- \xrightarrow{P} f(\phi^-)$. From the uniform boundedness of the f_n^-, it follows that $\int |f_n^- - f(\phi^-)| \to 0$, and similarly $\int |f_n^+ - f(\phi^+)| \to 0$. We denote by T the shift operation, so that

$$T^n\phi_n(x_n, \cdots, x_0) = \phi_n(x_0, \cdots, x_n).$$

By the ergodic theorem,

$$\lim_n \int \left| Ef(\phi^-) - \frac{1}{n}\sum_{k=1}^n T^k f(\phi^-) \right| = 0.$$

Hence,

$$\int |Ef(\phi^-) - f(\phi^+)| \leqq \limsup_n \frac{1}{n}\sum_1^n \int |T^k f(\phi^-) - T^k f_k^-| + \limsup_n \frac{1}{n}\sum_1^n \int |f_k^+ - f(\phi^+)|.$$

Since T is measure preserving, the terms on the right may be reduced to

$$\limsup_n \frac{1}{n}\sum_1^n \int |f(\phi^-) - f_k^-| + \limsup_n \frac{1}{n}\sum_1^n \int |f_k^+ - f(\phi^+)| = 0.$$

We conclude that $f(\phi^+)$ is almost surely constant, which proves the theorem.

2. Applications. We use the above theorem first to prove a result concerning maximum likelihood ratios, which is a special case of a theorem due to C. Kraft [2].

THEOREM 2. *Let the* $\cdots x_{-1}, x_0, x_1, \cdots$ *process be distributed according to the stationary ergodic measure* P *with density functions* $p_n(\cdot, \cdots, \cdot)$ *and let* Q *be any other stationary measure with density functions* $q_n(\cdot, \cdots, \cdot)$ *such that* P *is not absolutely continuous with respect to* Q. *Then almost surely (a.s.),*

$$\lim_n \frac{q_n(x_0, \cdots, x_n)}{p_n(x_0, \cdots, x_n)} = 0.$$

PROOF. Let $\phi_n = q_n(\cdot, \cdots, \cdot)/p_n(\cdot, \cdots, \cdot)$; it is well known ([3], pp. 93, 348) that the sequence $-\phi_n(x_0, \cdots, x_n)$ forms a semi-martingale with respect to the fields B_{n-1}^+ generated by $x_0, \cdots, x_{n-1}$. Similarly, the sequence

$$-\phi_n(x_{-n}, \cdots, x_0)$$

forms a semi-martingale with respect to the fields B_{n-1}^- generated by $x_{-n+1}, \cdots, x_0$. Since in both cases the first absolute moments are bounded by one, both sequences converge a.s. From our main theorem we conclude that there is some constant α such that

$$\lim_n \frac{q_n(x_0, \cdots, x_n)}{p_n(x_0, \cdots, x_n)} = \alpha \quad \text{(a.s.)}.$$

Now for any finite dimensional cylinder set I, we have for all n sufficiently large,

$$Q(I) \geqq \int_I \phi_n(x_0, \cdots, x_n)\, dP.$$

Equality would obtain, except that p_n may vanish on some set of positive measure where q_n does not. Using the Fatou lemma,

$$Q(I) \geqq \liminf \int_I \phi_n(x_0, \cdots, x_n)\, dP \geqq \int_I \liminf \phi_n(x_0, \cdots, x_n)\, dP = \alpha P(I).$$

Since the above is true for any cylinder set, it is also true for any finite disjoint union of cylinder sets, and thus is true in general, contradicting our hypothesis unless $\alpha = 0$.

Another application of Theorem 1 results in an extremal property of random variables having symmetric distributions. While we do not know of an explicit statement of this theorem, it can also be proven using de Finetti's representation theorem (see, for example [1] and [4], p. 364), without too much difficulty.

THEOREM 3. *Let x_0, x_1, $\cdots$ be a sequence of random variables whose finite dimensional distribution functions are invariant under permutations of the arguments and such that every "tail" event has measure zero or one. Then the sequence is equivalent to a sequence of independent random variables.*

PROOF. From the symmetry of the x_0, x_1, $\cdots$ sequence follows its stationarity. By the usual procedure we extend the measure to the double-ended sequence $\cdots x_{-1}, x_0, x_1, \cdots$, noticing that the symmetry is preserved under this extension. We also verify that the zero-one hypothesis implies the process is ergodic so that we may apply Theorem 1. We define $\phi_n(x_0, x_1, \cdots, x_n) = p(x_{-1} \leqq a \mid x_0, \cdots, x_n)$. Then, by symmetry and stationarity

$$\phi_n(x_{-n}, \cdots, x_0) = p(x_1 \leqq a \mid x_0, \cdots, x_{-n}).$$

By the martingale convergence theorem, both

$$\phi_n(x_0, \cdots, x_n),\ \phi_n(x_{-n}, \cdots, x_0)$$

converge a.s. and we conclude that both

$$p(x_{-1} \leqq a \mid x_0, x_1, \cdots),\ p(x_1 \leqq a \mid x_0, x_{-1}, \cdots)$$

are a.s. constant, which proves the theorem.

A more specialized consequence of Theorem 1 runs as follows.

THEOREM 4. *Let x_0, x_1, $\cdots$ be a sequence of identically distributed, independent random variables, and $\{\phi_n\}$ a sequence of real-valued functions, ϕ_n a measurable function of $n + 1$ variables. Then if both $\phi_n(x_0, \cdots, x_n)$ and*

$$\phi_n(x_n, \cdots, x_0)$$

converge in probability, their limits are a.s. constant.

PROOF. By the usual procedure we extend the measure on x_0, x_1, $\cdots$ to a measure on the two-sided process $\cdots, x_{-1}, x_0, x_1, \cdots$. The set

$$[| \phi_m(x_{-m}, \cdots, x_0) - \phi_n(x_{-n}, \cdots, x_0)| > \epsilon]$$

has the same measure as the set $[| \phi_m(x_m, \cdots, x_0) - \phi_n(x_n, \cdots, x_0)| > \epsilon]$. Hence $\phi_n(x_{-n}, \cdots, x_0)$ converges in probability and Theorem 1 applies.

The above theorem is an extension of the Hewitt-Savage zero-one law for symmetric sets, as the following theorem makes clear.

THEOREM 5. *Let $x_0, x_1, \cdots$ be a sequence of identically distributed, independent random variables and f any integrable function on the process such that f is invariant under finite permutations of the coordinates. Then f is a.s. constant.*

PROOF. Let $\phi_n(x_0, \cdots, x_n) = E(f \mid x_0, \cdots, x_n)$. Then $\phi_n(x_n, \cdots, x_0) = \phi_n(x_0, \cdots, x_n)$ by the symmetry of f and the $\phi_n(x_0, \cdots, x_n)$ sequence forms a martingale which converges a.s. to f. The conclusion follows from Theorem 3.

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A TEST OF FIT FOR MULTIVARIATE DISTRIBUTIONS[1]

BY LIONEL WEISS

Cornell University

1. Summary and introduction. Suppose X is a chance variable taking values in k-dimensional Euclidean space. That is, $X = (Y_1, \cdots, Y_k)$, where Y_i is a univariate chance variable. The joint distribution of $(Y_1, \cdots, Y_k)$ has density $f(y_1, \cdots, y_k)$, say.

We shall call a function $h(y_1, \cdots, y_k)$ "piecewise continuous" if it is everywhere bounded, and k-dimensional Euclidean space can be broken into a finite number of Borel-measurable subregions, such that in the interior of each subregion $h(y_1, \cdots, y_k)$ is continuous, and the set of all boundary points of all subregions has Lebesgue measure zero.

We assume that $f(y_1, \cdots, y_k)$ is piecewise continuous. Let $h(y_1, \cdots, y_k)$ be some given nonnegative piecewise continuous function, and let $X_1, \cdots, X_n$ be independent chance variables, each with the density $f(y_1, \cdots, y_k)$. Choose a nonnegative number t, and for each i, construct a k-dimensional sphere with center at $X_i = (Y_{i1}, \cdots, Y_{ik})$ and of k-dimensional volume

$$\frac{th(Y_{i1}, \cdots, Y_{ik})}{n}.$$

Such a sphere will be called "of type s" if it contains exactly s of the $(n - 1)$

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