ON THE INTEGRODIFFERENTIAL EQUATION OF TAKÁCS. I.

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1. Introduction. This paper is devoted to a study of certain aspects of the mixed-type Markov process $\eta(t)$, originally treated by Takács [8]. It extends and unifies a number of results of previous workers.

Let $N(t)$, $N(0) = 0$, $t \geqq 0$ denote max $\{n \mid t_n \leqq t\}$. We shall be especially interested in the case where $0 < t_1 < t_2 < \cdots$ are the events of an (in general) non-homogeneous Poisson process of density $\lambda(t) \geqq 0$. We assume that $\lambda(t)$ is Riemann integrable over all finite intervals. (The homogeneous Poisson process corresponds to $\lambda(t) = \text{const.}$) Let $\chi_0, \chi_1, \chi_2, \cdots$ be a sequence of non-negative random variables. Except in a part of Section 5, they are mutually independent, and independent of $N(t)$; moreover, $H(x) = \Pr\{\chi_i \leqq x\}$ is the same for $i = 1, 2, \cdots$. Introducing the notations

$$\int_{-\infty}^{t} \chi(t)\, dN(t) = \chi_0 + \sum_{i=1}^{N(t)} \chi_i, \quad L(x) = \begin{cases} 0, & x \leqq 0 \\ 1, & x > 0 \end{cases},$$

one may define (See Fig. 1)

$$\eta(t) = \int_{-\infty}^{t} \chi(u)\, dN(u) - \int_0^t L(\eta(u))\, du. \tag{1.1}$$

It is sometimes instructive to formally redefine $\chi(t)$ as a stochastic process with $\chi(t)$, $\chi(t')$, $(t \neq t')$, independent, $\Pr\{\chi(t) \leqq x\} = H(x)$, $t > 0$. One then concludes immediately, from the functional form of (1.1) that $\eta(t)$ is a Markov process. Note that var $(\eta(t + \Delta t) - \eta(t)) = O((\Delta t)^2)$, $t_i < t < t_{i+1}$, so that Feller's [5] function $a(t, x) = 0$.

In Section 2, the problem of finding the distribution of $\eta(t)$ will be reduced to finding the unique solution of a Volterra equation of the second kind. In Section 3, the corresponding result is found for the process $\eta^*(t)$, where, if t' is the first zero of $\eta(t)$,

$$\eta^*(t) = \begin{cases} \eta(t), & t < t' \\ 0, & t \geqq t'. \end{cases}$$

The work in Sections 2–4 generalizes results of Beneš [2] who treated the Takács process when $\lambda(t) = \text{const}$ (under somewhat milder restrictions on H). Section 5 contains some results on the asymptotic nature of $\eta(t)$, derived from a more general point of view than that employed in the preceding sections.

2. The Volterra equation for $\Pr\{\eta(t) = 0\}$.

Define $\Lambda(t) = \int_0^t \lambda(u)\, du$, $F(t,x) = \Pr\{\eta(t) \leqq x\}$, $F(t) = F(t,0) = \Pr$

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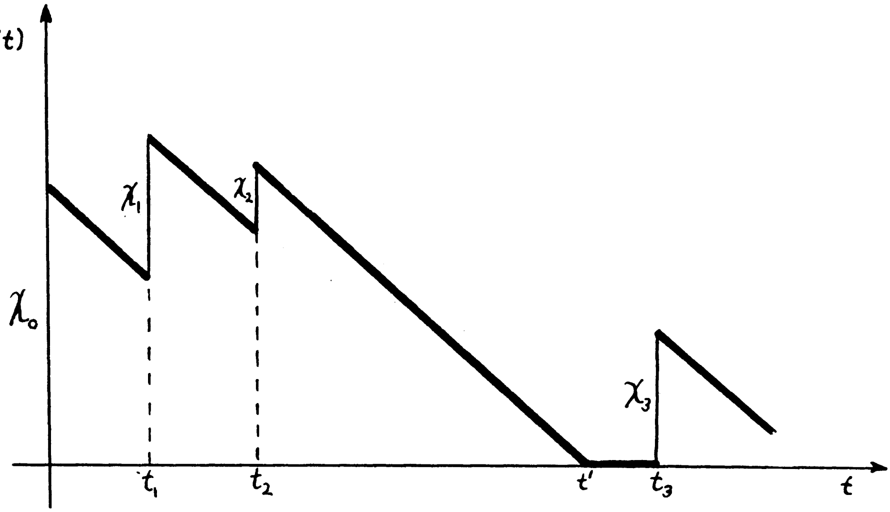

FIG. 1

$\{\eta(t) = 0\}$, $\psi(s) = Ee^{-s\chi_i}$, $i = 1, 2, \cdots$, $\Phi(t, s) = Ee^{-s\eta(t)}$, $\Phi(s) = \Phi(0, s) = Ee^{-s\chi_0}$, $(\Re s \geqq 0)$. It then follows [8] that $F(t, x)$ is continuous, $t \geqq 0$, $x > 0$, and[2]

$$\frac{\partial F(t, x)}{\partial t} = \frac{\partial F(t, x)}{\partial x} - \lambda(t)F(t, x) + \lambda(t) \int_{0-}^{x} H(x - y)\, d_y F(t, y).$$

Consequently,

$$\text{(2.1)}\qquad \begin{aligned} \Phi(t, s) + s \int_0^t e^{s(t-u)-[1-\psi(s)][\Lambda(t)-\Lambda(u)]} F(u)\, du \\ = \Phi(s)e^{st-[1-\psi(s)]\Lambda(t)}, \qquad \Re s \geqq 0. \end{aligned}$$

Thus, if $F(t)$ is known, $F(t, x)$ can be computed by quadratures. Equation (2.1) contains two unknown functions, $F(t)$, and $\Phi(t, s)$, which might *a priori* lead one to believe that, unless the explicit relation between the two functions were brought into the picture, neither could be uniquely determined from (2.1) alone. However, by taking advantage of the regularity properties of Φ, (and certain additional regularity properties of λ, H) it turns out that (2.1) actually determines $F(t)$, and hence also $\Phi(t, s)$, uniquely. (Cf. Bailey [1] where regularity properties are used to solve a functional equation containing two unknown functions. See also [9], pp. 52–53.)

THEOREM 1. *Suppose* (i) $\lambda(t) \in \mathfrak{L}^2$ *for every finite interval,* (ii) $H(x) = \int_0^x h(\xi)\, d\xi$, $e^{-cx}h(x) \in \mathfrak{L}^2(0, \infty)$ *for some* $c \geqq 0$. *Then* $F(t)$ *is the unique continuous solution*

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[2] Two functions of t will be written as equal, if they exist and are the same for almost all $t \geqq 0$.

of the Volterra equation of the second kind

$$g'(t) = F(t) + \int_0^t K(t, u)F(u)\, du, \quad \textit{where} \tag{2.2}$$

$$K(t, u) = \frac{1}{2\pi i}\frac{d}{dt}\text{P.V.}\int_{x-i\infty}^{x+i\infty} e^{(t-u)s-[\Lambda(t)-\Lambda(u)][1-\psi(s)]}\frac{ds}{s},$$

$$g'(t) = \frac{1}{2\pi i}\frac{d}{dt}\int_{x-i\infty}^{x+i\infty} \Phi(s)e^{ts-[1-\psi(s)]\Lambda(t)}\frac{ds}{s^2}, \qquad (x > c).$$

LEMMA 2.1. *If* $x > 0$, $M > 0$, $-\infty < \gamma \leqq \gamma_0$, *then* $1/2\pi i \int_{x-iM}^{x+iM} e^{\gamma s}/s\, ds$ *is bounded uniformly with respect to* M, γ.

PROOF. Let C_γ be the rectangular contour bounded by $s = x \pm iM$, and parallel lines extending to ∞ in the right (left) half plane if $\gamma < 0$ ($\gamma > 0$). Then

$$\frac{1}{2\pi i}\int_{C_\gamma}\frac{e^{\gamma s}}{s}ds = \frac{1 + \operatorname{sgn}\gamma}{2}$$

$$\therefore \left|\frac{1}{2\pi i}\int_{x-iM}^{x+iM}\right| - 1 \leqq \left|\frac{1}{2\pi i}\int_{C_\gamma} - \frac{1}{2\pi i}\int_{x-iM}^{x+iM}\right|$$

$$\leqq \frac{K_1}{|\gamma|M} \leqq K_1, \quad \text{if} \quad |\gamma M| \geqq 1.$$

On the other hand, if $|\gamma M| \leqq 1$, then

$$\left|\frac{1}{2\pi i}\int_{x-iM}^{x+iM}\right| - \frac{1}{2} \leqq \left|\frac{1}{2\pi i}\int_{x-iM}^{x+iM}\frac{e^{\gamma s}-1}{s}ds\right|$$

$$\leqq K_2\int_{x-iM}^{x+iM}\frac{|\gamma s|}{|s|}|ds| = 2K_2|\gamma|M \leqq 2K_2.$$

LEMMA 2.2. *If* $x > 0$, $M > 0$, $0 \leqq \alpha \leqq \alpha_0$, $r(t) \in \mathfrak{L}^1(0, \infty)$, $R(s) = \int_0^\infty e^{-st}r(t)\, dt$, then $1/2\pi i \int_{x-iM}^{x+iM} R(s)e^{\alpha s}\, ds/s$ is bounded uniformly with respect to M, α.

PROOF. Since the integral for $R(s)$ converges uniformly on the line $\Re s = x$,

$$L = \frac{1}{2\pi i}\int_{x-iM}^{x+iM} R(s)e^{\alpha s}\frac{ds}{s} = \int_0^\infty r(t)\left\{\frac{1}{2\pi i}\int_{x-iM}^{x+iM} e^{(\alpha-t)s}\frac{ds}{s}\right\}dt.$$

Hence, by Lemma 2.1, $|L| \leqq K_3\int_0^\infty |r(t)|\, dt$.

PROOF OF THEOREM 1. Dividing both sides of (2.1) by s^2, and integrating along the line $\Re s = x > c$ from $s = x - iM_1$ to $s = x + iM_2$, $(M_1, M_2 > 0)$, we have

$$\frac{1}{2\pi i}\int_{x-iM_1}^{x+iM_2}\Phi(t, s)\frac{ds}{s^2} + \int_0^t\left\{\frac{1}{2\pi i}\int_{x-iM_1}^{x+iM_2} e^{(t-u)s-[\Lambda(t)-\Lambda(u)][1-\psi(s)]}\frac{ds}{s}\right\}F(u)\, du$$

$$= \frac{1}{2\pi i}\int_{x-iM_1}^{x+iM_2}\Phi(s)e^{ts-\Lambda(t)[1-\psi(s)]}\frac{ds}{s^2}, \qquad t \geqq 0.$$

Since $\Phi(t, s)$ is regular, $|\Phi(t, s)| \leqq 1$, when $\Re s > 0$, the first integral on the left side converges (absolutely) to zero as $M_1, M_2 \to \infty$, and the integral on the right side converges absolutely to the function

$$g(t) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} \Phi(s) e^{ts - \Lambda(t)[1-\psi(s)]} \frac{ds}{s^2}.$$

Hence

$$\lim_{M_1, M_2 \to \infty} \int_0^t \left\{ \frac{1}{2\pi i} \int_{x-iM_1}^{x+iM_2} e^{(t-u)s - [\Lambda(t)-\Lambda(u)][1-\psi(s)]} \frac{ds}{s} \right\} F(u)\, du = g(t), \qquad t \geqq 0, x > c. \tag{2.3}$$

In particular, we shall henceforth take $M_1 = M_2 = M$, in order to make it possible to be able to invert the order of integration in (2.3). We can write

$$\frac{1}{2\pi i} \int_{x-iM}^{x+iM} e^{\alpha s - \beta[1-\psi(s)]} \frac{ds}{s} = \frac{1}{2\pi i} \int_{x-iM}^{x+iM} e^{\alpha s - \beta}[e^{\beta\psi(s)} - 1 - \beta\psi(s)] \frac{ds}{s}$$
$$+ \frac{e^{-\beta}}{2\pi i} \int_{x-iM}^{x+iM} e^{\alpha s} \frac{ds}{s} + \frac{\beta e^{-\beta}}{2\pi i} \int_{x-iM}^{x+iM} \psi(s) e^{\alpha s} \frac{ds}{s} = I + II + III$$

By the Riemann-Lebesgue Lemma, $\lim_{|y| \to \infty} |\psi(x + iy)| = 0$. By Parseval's equality,

$$\int_{-\infty}^{\infty} |\psi(x + iy)|^2 dy = \int_0^{\infty} e^{-2cx}[h(x)]^2 dx < \infty. \quad \text{Hence,}$$

$$|I| \leqq \beta K e^{\alpha x} \int_{x-i\infty}^{x+i\infty} |\psi(s)|^2 \frac{ds}{s} < \infty.$$

Therefore, as $M \to \infty$, I converges absolutely, and uniformly with respect to α, β, $|\alpha| \leqq \alpha_0$, $|\beta| \leqq \beta_0$. By Lemma 2.1, $\lim_{M\to\infty} II = e^{-\beta}$, boundedly with respect $M > 0$, $0 \leqq \alpha \leqq \alpha_0$. By Lemma 2.2, $\lim_{M\to\infty} III = \beta e^{-\beta} H(\alpha)$, boundedly with respect to $M > 0$, $0 \leqq \alpha \leqq \alpha_0$. Thus we may rewrite (2.3) as the Volterra equation of the first kind,

$$\int_0^t G(t, u) F(u)\, du = g(t), \qquad t \geqq 0, \qquad (x > c), \qquad \text{where} \quad G(t, u)$$
$$= \text{P.V.} \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{(t-u)s - [\Lambda(t)-\Lambda(u)][1-\psi(s)]} \frac{ds}{s} = \rho(\alpha, \beta) - \sigma(\alpha, \beta), \tag{2.4}$$

$$\rho(\alpha, \beta) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{\alpha s - \beta}[e^{\beta\psi(s)} - 1 - \beta\psi(s)] \frac{ds}{s} + e^{-\beta},$$
$$\sigma(\alpha, \beta) = \beta e^{-\beta} H(\alpha), \qquad \alpha = t - u \geqq 0, \qquad \beta = \Lambda(t) - \Lambda(u). \tag{2.5}$$

Next we deal with the question of the existence and nature of the derivative $g'(t)$ for almost all $t \geqq 0$. First we focus on the existence and nature of

$$\frac{d}{dt} \int_0^t \rho(\alpha, \beta) F(u)\, du.$$

Since $\rho(\alpha, \beta)F(u)$ is continuous in t, u, $\rho(0, 0) = 1$,

$$\frac{d}{dt}\int_0^t \rho(\alpha, \beta)F(u)\,du = F(t) + \left[\frac{d}{dt}\int_0^\tau \rho(\alpha, \beta)F(u)\,du\right]_{\tau=t}$$

The partial derivatives ρ_α, ρ_β exist and are uniformly continuous in $0 \leqq \alpha \leqq \alpha_0$, $0 \leqq \beta \leqq \beta_0$. Thus $\Delta\rho = \rho_\alpha\Delta\alpha + \rho_\beta\Delta\beta + \epsilon_1\Delta\alpha + \epsilon_2\Delta\beta$, $\lim_{\Delta\alpha,\Delta\beta\to 0}\epsilon_i = 0$, ϵ_i uniformly bounded with respect to α, β, $0 \leqq \alpha \leqq \alpha_0$, $0 \leqq \beta \leqq \beta_0$. We have

$$\frac{\Delta\rho}{\Delta t} = \rho_\alpha + \epsilon_1 + \frac{(\rho_\beta + \epsilon_2)}{\Delta t}\int_t^{t+\Delta t} \lambda(v)\,dv.$$

Let $E = \{t \mid \Lambda'(t) = \lambda(t)\}$. We see that for $t \,\varepsilon\, E$, $\Delta\rho/\Delta t$ is bounded uniformly with respect to u, and $d\rho/dt = \rho_\alpha + \rho_\beta\lambda(t)$. Hence, by the bounded convergence theorem,

$$\frac{d}{dt}\int_0^\tau \rho F(u)\,du = \int_0^\tau \frac{\partial\rho}{\partial t}F(u)\,du.$$

and therefore

$$\begin{aligned}\frac{d}{dt}\int_0^t \rho(\alpha, \beta)F(u)\,du &= F(t) + \int_0^t \frac{\partial\rho}{\partial t}F(u)\,du \\ &= \int_0^t \rho_\alpha F(u)\,du + \lambda(t)\int_0^t \rho_\beta F(u)\,du.\end{aligned} \tag{2.6}$$

We see, by (2.6), that

$$\frac{\partial\rho}{\partial t}\,\varepsilon\mathfrak{L}^2(\Delta), \qquad \Delta = \{(t, u) \mid 0 \leqq u \leqq t\}.$$

Also, $\int_0^t \partial\rho/\partial t\, F(u)\,du \;\varepsilon\; \mathfrak{L}^2$ (for every finite interval). Next, consider

$$\int_0^t \sigma(\alpha, \beta)F(u)\,du.$$

By noting that fact ([4], pp. 111 ff.) that for continuous $Q(u)$, and $h \,\varepsilon\, \mathfrak{L}^2$,

$$d/dt\int_0^t Q(u)H(t-u)\,du = \int_0^t Q(u)h(t-u)\,du = \text{continuous function of } t,$$

for all t, one finds that $d/dt\int_0^t \sigma(\alpha, \beta)F(u)\,du = \int_0^t \partial\sigma/\partial t\; F(u)\,du \;\varepsilon\; \mathfrak{L}^2$, with $\partial\sigma/\partial t \;\varepsilon\; \mathfrak{L}^2(\Delta)$. Thus (2.2) holds for almost all t, with

$$K(t, u) = \frac{\partial G(t, u)}{\partial t}\;\varepsilon\;\mathfrak{L}^2(\Delta),$$

and $g'(t) \;\varepsilon\; \mathfrak{L}^2$ over every finite interval. Under these conditions it is known [7] that (2.2) has a unique $\mathfrak{L}^2$ solution, $F(t)$; in particular, there is a unique continuous solution.

3. The Volterra equation for Pr $\{\eta^*(t) = 0\}$.

Define $B(t, x) = \Pr\{\eta^*(t) \leqq x\}$, $B(t) = \Pr\{\eta^*(t) = 0\} = \Pr\{t' \leqq t\}$, $\zeta(t, s) = Ee^{-s\eta^*(t)}$, $\Phi(s) = Ee^{-s\chi_0} = Ee^{-s\eta^*(0)}$. Then [2]

$$\frac{\partial B(t,x)}{\partial t} = \frac{\partial B(t,x)}{\partial x} - \lambda(t)B(t,x) + \lambda(t)\int_{0-}^{x} B(x-y,t)\,dH(y) + \lambda(t)[1-H(x)]B(t).$$

Hence

$$\zeta(t,s) + \int_0^t \{s - \lambda(u)[1-\psi(s)]\}e^{s(t-u)-[1-\psi(s)][\Lambda(t)-\Lambda(u)]}B(u)\,du = \Phi(s)e^{st-[1-\psi(s)]\Lambda(t)}, \qquad \Re s \geqq 0. \tag{3.1}$$

THEOREM 2. *Under the same assumptions on* λ *and* H *as for Theorem* 1, $B(t)$ *is the unique continuous solution of the Volterra equation of the second kind*

$$g'(t) = B(t) + \int_0^t K^*(t,u)B(u)\,du, \tag{3.2}$$

where

$$K^*(t,u) = \frac{1}{2\pi i}\frac{d}{dt}\,\text{P.V.}\int_{x-i\infty}^{x+i\infty} \cdot\{s - \lambda(u)[1-\psi(s)]e^{s(t-u)-[1-\psi(s)][\Lambda(t)-\Lambda(u)]}\frac{ds}{s^2}, \qquad x > c.$$

PROOF. The proof proceeds as for Theorem 1, except that, before differentiation, the kernel now contains an additional term of the type

$$\begin{aligned}\rho^*(\alpha,\beta) &= \frac{1}{2\pi i}\,\text{P.V.}\int_{x-i\infty}^{x+i\infty}[1-\psi(s)]e^{\alpha s-\beta[1-\psi(s)]}\frac{ds}{s^2} \\ &= \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} e^{\alpha s-\beta}\{[1-\psi(s)][e^{\beta\psi(s)}-1] - \beta\psi(s)\}\frac{ds}{s^2} \\ &\quad + (\beta-1)e^{-\beta}\int_0^{\alpha} H(\tau)\,d\tau.\end{aligned}$$

This expression is treated in the same manner as $\rho(\alpha,\beta)$ was treated.

4. $\psi(s)$ Regular at infinity. We shall briefly remark on the practically important case when $\psi(s)$ is regular at infinity (e.g. when $\psi(s)$ is rational). This assumption regarding ψ is more restrictive than the assumptions in the hypotheses of Theorems 1 and 2, because by Pincherle's Theorem ([4], pg. 263),

$$\psi(s) = a_1 s^{-1} + a_2 s^{-2} + \cdots$$

is the Laplace transform of a density

$$h(t) = \frac{1}{2\pi i} = \int_\Gamma e^{ts}\psi(s)\,ds, \qquad t > 0,$$

where Γ is a contour, on and outside of which $\psi(s)$ is regular. In particular, if ψ is a rectangle on which $\Re s \leqq \delta > 0$, then we see that $|h(t)| \leqq Ke^{\delta t}$. Thus one may choose $c = 0$.

Instead of multiplying (2.1) (or (3.1)) by the "convergence factor" s^{-2}, it is sometimes more convenient to use $(s + \mu)^{-2}$, $\mu > 0$. For example, the kernel $G(t, u)$ of (2.4) then becomes

$$G(t, u) = \frac{1}{2\pi i} \int_\Gamma s(s + \mu)^{-2} e^{(t-u)s - [\Lambda(t) - \Lambda(u)][1 - \psi(s)]} \, ds, \tag{4.1}$$

and $K(t, u) = (\partial/\partial t)G(t, u)$. For instance, if $\psi(s) = (1 + s)^{-1}$, and if we choose $\mu = 1$, then, if I_ν is the modified Bessel function,

$$G(t, u) = \begin{cases} e^{-(t-u)-[\Lambda(t)-\Lambda(u)]} I_0[2(t-u)^{1/2}(\Lambda(t) - \Lambda(u))^{1/2}] - (t-u)^{1/2} \\ (\Lambda(t) - \Lambda(u))^{-1/2} I_1[2(t-u)^{1/2}(\Lambda(t) - \Lambda(u))^{1/2}], \\ \text{if } \Lambda(t) \neq \Lambda(u), \\ e^{-(t-u)}[1 - (t - u)], \text{ if } \Lambda(t) = \Lambda(u). \end{cases}$$

This is rather similar to the kernel encountered by Clarke [3] by a completely different approach.

5. Asymptotic behavior of $\eta(t)$. Unless specifically stated, no restrictions regarding the distribution, or *independence* of the sequences $\{\chi_i\}$, $\chi_i \geqq 0$, and $\{t_n\}$, $0 < t_1 < t_2, \cdots$, shall be made in this section. Therefore, we cannot use the results of Sections 2–4, but must return to the fundamental relation (1.1).

LEMMA 5.1.

$$\eta(t) = \sup_{x>0} \left[\int_{t-x}^{t} \chi(u) \, dN(u) - x \right]$$

PROOF. Let $y = \{\max u \mid u \leqq t, \eta(u) = 0\}$. Then

$$\eta(t) = \int_y^t \chi(u) \, dN(u) - (t - y).$$

On the other hand,

$$\eta(t) = \eta(t - x) + \int_{t-x}^{t} \chi(u) \, dN(u) - \int_{t-x}^{t} L(\eta(u)) \, du \geqq \int_{t-x}^{t} \chi(u) \, dN(u) - x.$$

THEOREM 3. *If* $N(t) = \lambda t + o(t)$ *as* $t \to \infty$, $\sum_{i=1}^{n} \chi_i = \alpha n + o(n)$, *as* $n \to \infty$, $\lambda\alpha \leqq 1$, *then* $\eta(t) = o(t)$.

PROOF. We note first that the hypothesis implies that if $0 < \gamma_1 \leqq \gamma \leqq \gamma_2$, then

$$\lim_{t\to\infty} t^{-1} \left[\int_{-\infty}^{\gamma t} \chi(u) \, dN(u) - \gamma\alpha\lambda t \right] = 0,$$

uniformly with respect to γ. Let $\delta, \epsilon > 0$, be given. Then if $0 < x \leqq (1 - \delta)t$, there exists a $T_{\epsilon,\delta}$, such that

$$t^{-1} \left[\int_{t-x}^{t} \chi(u) \, dN(u) - x \right] = t^{-1} \left[\int_{-\infty}^{t} \chi(u) \, dN(u) - \int_{-\infty}^{(1-x/t)} \chi(u) \, dN(u) \right]$$
$$- (x/t) \leqq \alpha\lambda - (1 - x/t)\alpha\lambda + \epsilon - x/t \leqq \epsilon,$$

if $t > T_{\epsilon,\delta}$. On the other hand, if $x > (1 - \delta)t$, there exists a T_ϵ such that

$$t^{-1}\left[\int_{t-x}^{t} \chi(u)\, dN(u) - x\right] \leqq t^{-1}\left[\int_{-\infty}^{t} \chi(u)\, dN(u)\right] - x/t \leqq \alpha\lambda + \epsilon - (1 - \delta) \leqq \epsilon + \delta,$$

if $t > T_\epsilon$. Hence $\eta(t) / t \leqq \epsilon + \delta$ if $t > \max(T_\epsilon, T_{\epsilon,\delta})$.

COROLLARY. *If $N(t)$ is a Poisson process with cumulative mass $\Lambda(t) = \lambda t + o(t)$ as $t \to \infty$, $\sum_{i=1}^{n} \chi_i = cn + o(n)$, as $n \to \infty$, $\lambda\alpha \leqq 1$, then* $\eta(t) = o(t)$ with probability 1.

PROOF. If $\Lambda(\infty) < \infty$, the result is trivial, as then $\eta(t) = O(1)$, with probability one. Assume $\Lambda(\infty) = \infty$. Let $N^*(t, \omega)$, $\omega \varepsilon \Omega$, be a homogeneous Poisson process with unit density. Then $N(t, \omega) = N^*(\Lambda(t), \omega)$ is a Poisson process with density $\lambda(t)$. Hence

$$\lim_{t\to\infty} \frac{N(t, \omega)}{t} = \lim_{t\to\infty} \frac{N^*(\Lambda(t), \omega)}{\Lambda(t)} \frac{\Lambda(t)}{t} = \lambda, \qquad \text{a.a.}\omega.$$

The following result follows from results of Kiefer and Wolfowitz [6], after some elementary transformations.

THEOREM 4. *Suppose $\Lambda(t) = \lambda t + O(1)$, as $t \to \infty$. If $\{\chi_i\}$ are independent of each other, and $N(t)$, and are equidistributed, and if $\overline{\chi_i}\lambda < 1$, $\overline{\chi_i^2} < \infty$, then*

$$E\eta(t_n + O) = O(1),$$

as $n \to \infty$.

The hypothesis on $\Lambda(t)$ is satisfied, e.g., if $\lambda(t)$ is periodic with mean λ. It may be shown, by counterexample, that the conclusion of Theorem 4 becomes false if the hypothesis is weakened to $\Lambda(t) = \lambda t + o(t)$.

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