THE STAIRCASE DESIGN: THEORY

By Franklin A. Graybill and William E. Pruitt

Oklahoma State University

0. Summary and Introduction. One of the most popular designs in experimental work is the randomized block. These designs can be put into three broad classes viz. complete block design, balanced incomplete block design, and the partially balanced incomplete block design. These designs are all special cases of the general two way classification with unequal numbers in the subclasses, but since the analysis of this general classification is quite complex, these special cases have evolved which are adequate to fit most needs and the analysis of these special designs is relatively easy. [1], [2], [6]; [8].

However, most of the block designs considered to date have one feature in common—they require each block to contain an equal number of experimental units. The exceptions are given in [9], [10], where designs are considered in which the number of experimental units in blocks differ by one. The purpose of this paper is to extend the randomized block design to include the case where all blocks do not contain the same number of experimental units. We have called this the *staircase design.*

Suppose an experimenter, wishing to run an experiment using N treatments, decides to use a randomized block design, but after arranging his material into homogeneous groups he finds that he has blocks available which have varying number of experimental units. The experimenter has various courses open to him: (1) If enough blocks are available with N or more experimental units he can discard the extra units in these blocks, discard all the blocks which have less than N units, and use a randomized complete block design; (2) He can discard units in the blocks until he has enough units and blocks for a balanced incomplete block or a partially balanced incomplete block design; (3) He can use all the experimental units and use the staircase design proposed in this paper.

For example, if an experimenter has N treatments with which he wishes to experiment using a randomized block design, and if he has blocks of unequal size, then he must rank his N treatments in the order of their importance, i.e., $T_1, T_2, \cdots, T_N$, where he considers T_1 the most important and T_N the least important. Now suppose he has at his disposal b_1 blocks which each contain N experimental units. Then all N treatments are randomized in each of the b_1 blocks. Suppose further that he has b_2 blocks which each contain N_1 experimental units ($N_1 < N$). Then the first N_1 treatments are arranged at random in each of the b_2 blocks. This process is continued until all the blocks are used.

A particular example where this would be useful is an experiment involving animals as experimental units where a block consists of litter mates. Let us

Received July 16, 1956; revised December 26, 1957.

suppose that we have two litters of size seven, three of size five, and one of size four. Using the staircase design we can include seven treatments and still have the four we are most interested in replicated six times.

1. Notation. Consider a two-way classification model

$$Y_{ij} = \mu + \beta_i + \alpha_j + e_{ij}\,; \qquad i = 1, 2, \cdots, c_j\,; j = 1, 2, \cdots, N. \tag{1.1}$$

where μ, β_i, α_j are constants and the e_{ij} are normal independent variables with means zero and variances σ^2. Also the j's will be ordered in such a way that $c_j \geqq c_{j'}$ for $j < j'$. The purpose of this paper is,

1. to derive the least squares method for testing the hypothesis $\alpha_1 = \alpha_2 = \cdots = \alpha_N$ under the model given above and to give the power function of this test.
2. to derive the best, linear, unbiased estimates for $\alpha_j - \alpha_{j'}$, and the variances of these estimates.

First we will separate the j's into subsets such that j and j' will be in the same subset if and only if $c_j = c_{j'}$. Each of these subsets will be called a *step*. We will designate the number of steps as k.

Let

$$\begin{aligned} c_j &= M^1 \quad \text{for} \quad j = 1, 2, \cdots, N_1 \\ c_j &= M^2 \quad \text{for} \quad j = N_1 + 1, N_1 + 2, \cdots, N_1 + N_2, \\ &\vdots \\ c_j &= M^k \quad \text{for} \quad j = N_1 + N_2 + \cdots + N_{k-1} + 1, N_1 + N_2 + \cdots \\ &\qquad + N_{k-1} + 2, \cdots, N_1 + N_2 + \cdots + N_k, \end{aligned}$$

where

$$\sum_{t=1}^{k} N_t = N; \qquad \sum_{t=1}^{k} M^t N_t = N^*$$

Now, let

$$\begin{aligned} N^p &= \sum_{t=1}^{p} N_t, \qquad N^0 = 0, \qquad M^{k+1} = 0, \\ Y_{ij} &= Y_{ij}^s \quad \text{for} \quad i = 1, 2, \cdots, M^s, j = N^{s-1} + 1, N^{s-1} + 2, \cdots, N^s, \\ Y_{ij} &= Y_{ij}^{\prime s} \quad \text{for} \quad i = 1, 2, \cdots, M^{s+1}, j = 1, 2, \cdots, N^s \\ \alpha_j &= \alpha_j^s \quad \text{for} \quad j = N^{s-1} + 1, N^{s-1} + 2, \cdots N^s, \\ \alpha_j &= \alpha_j^{\prime s} \quad \text{for} \quad j = 1, 2, \cdots, N^s. \end{aligned}$$

Fig. 1 will serve to illustrate some of the notation. It will be noticed that c_j is the number of blocks in which treatment j appears; M^t is the number of blocks in the tth step; N_t is the number of treatments in the tth step. Also Y_{ij}^s is the observation of the jth treatment which appears in the ith block of the sth step. It may be helpful to note further that $Y_{ij}^{\prime 1}$ is a subset of Y_{ij}^1 ; $Y_{ij}^{\prime 2}$ is a subset of the union of Y^1_{ij} and Y^2_{ij}, Y'^3_{ij} is a subset of the union of Y^1_{ij} and Y^2_{ij} and Y^3_{ij}, etc.

A subscript replaced by a dot indicates the mean of the elements when summed over the range of the replaced subscript, eg.

$$Y'^2_{..} = \frac{\sum_{i=1}^{M^3} \sum_{j=1}^{N^2} Y'^2_{ij}}{M^3 N^2}.$$

Since superscripts are being used in abundance, a Y, M, N, or α that is raised to a power will always be enclosed in the appropriate brackets.

If, in a summation, the lower limit of summation should exceed the upper limit of summation, the sum will be zero.

The notation used in Section 3 is that used by Kempthorne [4], pages 79–82, with the following exceptions. To be consistent with the notation given above, the normal equations are divided by a constant to give them in terms of means instead of totals. Q^s_j will refer to only the Q_j where $j = N^{s-1} + 1, N^{s-1} + 2, \cdots, N^s$.

2. The test function and its distributional properties. The purpose of this section is to give a test of the hypothesis $\alpha_1 = \alpha_2 = \cdots = \alpha_N$ and to derive the

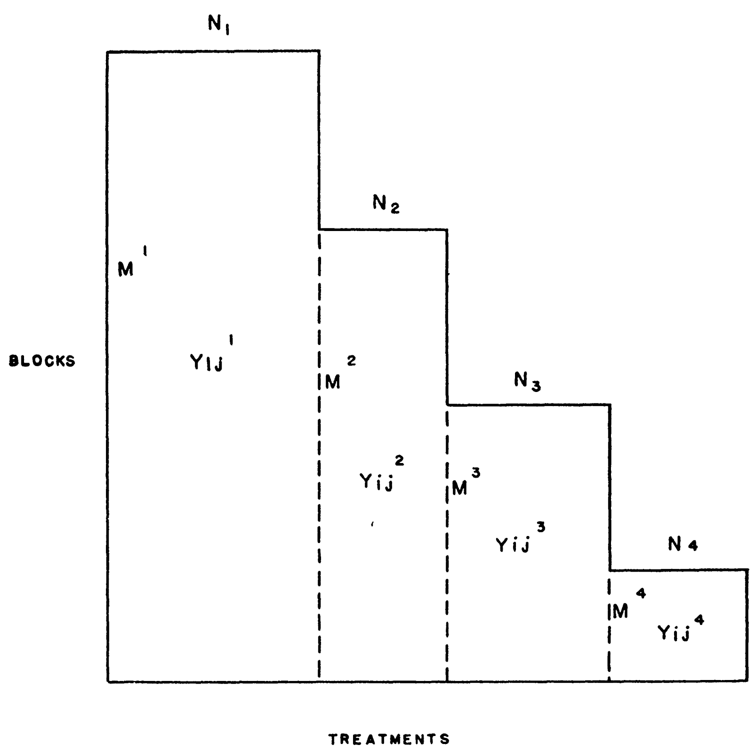

FIG. 1

distributional properties of the test function. The proof that this test is the same as that given by the method of least squares will be given in the next section.

Consider the following quadratic forms:

$$q_t^1 = \sum_{i=1}^{M^t} \sum_{j=N^{t-1}+1}^{N^t} (Y_{ij}^t - Y_{i.}^t - Y_{.j}^t + Y_{..}^t)^2, \qquad t = 1, 2, \cdots, k.$$

$$q_t^2 = \frac{N^t N_{t+1}}{N^{t+1}} \sum_{i=1}^{M^{t+1}} (Y_{i.}'^t - Y_{i.}^{t+1} - Y_{..}'^t + Y_{..}^{t+1})^2, \qquad t = 1, 2, \cdots, k-1,$$

$$q_t^3 = M^t \sum_{j=N^{t-1}+1}^{N^t} (Y_{.j}^t - Y_{..}^t)^2, \qquad t = 1, 2, \cdots, k.$$

$$q_t^4 = \frac{M^{t+1} N^t N_{t+1}}{N^{t+1}} (Y_{..}'^t - Y_{..}^{t+1})^2, \qquad t = 1, 2, \cdots, k-1.$$

$$q_t^5 = \frac{1}{N^t} \sum_{i=M^{t+1}+1}^{M^t} (N^{t-1} Y_{i.}'^{t-1} + N_t Y_{i.}^t)^2, \qquad t = 1, 2, \cdots, k.$$

$$q_t^6 = \sum_{i=1}^{M^t} \sum_{j=N^{t-1}+1}^{N^t} (Y_{ij}^t)^2, \qquad t = 1, 2, \cdots, k$$

We will prove the following:

THEOREM I. *If*

$$(2.1) \qquad v = \frac{\sum_{t=1}^k q_t^3 + \sum_{t=1}^{k-1} q_t^4}{\sum_{t=1}^k q_t^1 + \sum_{t=1}^{k-1} q_t^2} \cdot \frac{(M^1 - 1)(N - 1) - \sum_{t=2}^k (M^1 - M^t)(N_t)}{N - 1}$$

then v *is distributed as* $F'_{p,q,\lambda}$, *where* $F'_{p,q,\lambda}$ *represents the non-central* F *with degrees of freedom* p *and* q *and non-centrality* λ [7], *also*

$$(2.2) \qquad \begin{gathered} p = N - 1, \; q = (M^1 - 1)(N - 1) - \sum_{t=2}^k (M^1 - M^t) N_t, \\ \lambda = \sum_{t=1}^k \left[\frac{M^t}{2\sigma^2} \sum_{j=N^{t-1}+1}^{N^t} (\alpha_j^t - \alpha_.^t)^2 \right] + \sum_{t=1}^{k-1} \left[\frac{M^{t+1} N^t N_{t+1}}{2\sigma^2 N^{t+1}} (\alpha_.'^t - \alpha_.^{t+1})^2 \right] \end{gathered}$$

and $\lambda = 0$ *if and only if* $\alpha_1 = \alpha_2 = \cdots = \alpha_N$.

PROOF. It is clear that

$$(2.3) \qquad \sum_{t=1}^k q_t^1 + \sum_{t=1}^{k-1} q_t^2 + \sum_{t=1}^k q_t^3 + \sum_{t=1}^{k-1} q_t^4 + \sum_{t=1}^k q_t^5 = \sum_{t=1}^k q_t^6.$$

Now it is easily shown that the rank of q_t^1 is $(M^t - 1)(N_t - 1)$, the rank of q_t^2 is $(M^{t+1} - 1)$, the rank of q_t^3 is $(N_t - 1)$, the rank of q_t^4 is 1, and the rank q_t^5 is $(M^t - M^{t+1})$. Adding we see that

$$\sum_{t=1}^k (M^t - 1)(N_t - 1) + \sum_{t=1}^{k-1} (M^{t+1} - 1) + \sum_{t=1}^k (N_t - 1) + (k - 1) + \sum_{t=1}^{k-1} (M^t - M^{t+1}) + (M^k - M^{k+1}) = \sum_{t=1}^k M^t N_t$$

Thus we have the fact that the sum of the ranks of the quadratic forms on the left of (2.3) above is equal to the number of squared observations on the right. We may now invoke a theorem proved by Madow [5] showing the quadratic forms to be independent, and verifying the following distributions. (E will be used to denote mathematical expectation, and $\chi'^2_{p,\lambda}$ will represent a non-central chi square distribution with degrees of freedom p and non-centrality λ).

1. q_t^1/σ^2 is distributed as $\chi'^2_{p,\lambda}$, where $p = (M^t - 1)(N_t - 1)$, $\lambda = 0$, since

$$E(Y^t_{ij} - Y^t_{i.} - Y^t_{.j} + Y^t_{..}) = 0.$$

2. q_t^2/σ^2 is distributed as $\chi'^2_{p,\lambda}$, where $p = (M^{t+1} - 1)$, $\lambda = 0$, since

$$E(Y'^t_{i.} - Y^{t+1}_{i.} - Y'^t_{..} + Y^{t+1}_{..}) = 0.$$

3. q_t^3/σ^2 is distributed as $\chi'^2_{p,\lambda}$, where $p = (N_t - 1)$,

$$\lambda = \frac{M^t}{2\sigma^2} \sum_{j=N^{t-1}+1}^{N^t} (\alpha^t_j - \alpha^t_.)^2,$$

since

$$E(Y^t_{.j} - Y^t_{..}) = \alpha^t_j - \alpha^t_. \, .$$

4. q_t^4/σ^2 is distributed as $\chi'^2_{p,\lambda}$, where $p = 1$,

$$\lambda = \frac{M^{t+1}N^tN_{t+1}}{2\sigma^2 N^{t+1}} (\alpha'^t_. - \alpha^{t+1}_.)^2,$$

since

$$E(Y'^t_{..} - Y^{t+1}_{..}) = \alpha'^t_. - \alpha^{t+1}_. .$$

Therefore it follows that

$$\frac{1}{\sigma^2}\left[\sum_{t=1}^{k} q_t^1 + \sum_{t=1}^{k-1} q_t^2\right]$$

is distributed as $\chi'^2_{p,\lambda}$, where

$$p = (M^1 - 1)(N - 1) - \sum_{t=2}^{k} (M^1 - M^t)(N_t),$$

and $\lambda = 0$. Also we have

$$\frac{1}{\sigma^2}\left[\sum_{t=1}^{k} q_t^3 + \sum_{t=1}^{k-1} q_t^4\right]$$

is distributed as $\chi'^2_{p,\lambda}$, where

$$p = \sum_{t=1}^{k} (N_t - 1) + (k - 1) = N - 1,$$

$$\lambda = \sum_{t=1}^{k}\left[\frac{M^t}{2\sigma^2} \sum_{j=N^{t-1}+1}^{N^t} (\alpha^t_j - \alpha^t_.)^2\right] + \sum_{t=1}^{k-1} \frac{M^{t+1}N^tN_{t+1}}{2\sigma^2 N^{t+1}} (\alpha'^t_. - \alpha^{t+1}_.)^2$$

TABLE 3.1

Due to	df	Sum of Squares
Blocks ignoring treatments	M^1	$\Sigma N_{i.}(Y_{i.})^2$
Treatments eliminating blocks	$N - 1$	$\Sigma_j Q_j \tilde{\alpha}_j$
Error	$N^* - M^1 - N + 1$	By subtraction
Total	N^*	$\Sigma_{ij}(Y_{ij})^2$

Hence, we have v as defined in (2.1) above is distributed as $F'_{p,q,\lambda}$, where p, q, and λ are as defined in (2.2) above.

Now it is clear that $\lambda = 0$ if and only if $\alpha_1 = \alpha_2 = \cdots = \alpha_N$ since λ is a sum of non-negative terms and can be zero if and only if each term of the sum is zero. Therefore to test the hypothesis $\alpha_1 = \alpha_2 = \cdots = \alpha_N$ we use v as Snedecor's F with p degrees of freedom and q degrees of freedom, where p and q are as defined in (2.2).

3. The analysis of variance. In Section 2 it was shown that the test function v could be used to test the hypothesis $\alpha_1 = \alpha_2 = \cdots = \alpha_N$. We will now show that v can be derived by the method of least squares. The model can be considered as a two-way classification model with unequal numbers in the subclasses. In this case the conventional analysis is given in Table 3.1 [4]. If we now denote the mean square for treatments eliminating blocks by T and the mean square for error by E, then $W = T/E$ is the least squares test function used to test the hypothesis $\alpha_1 = \alpha_2 = \cdots = \alpha_N$. ($N_{i.}$ is the number of treatments in the ith block). We will now show that the function v in Section 2 is the test criterion given by least squares.

In the above table, the Total SS minus the Block ignoring treatments SS is equal to

$$\sum_{t=1}^{k} q_t^6 - \sum_{t=1}^{k} q_t^5 .$$

It remains only to show that

$$\sum_{j=1}^{N} Q_j \tilde{\alpha}_j = \sum_{t=1}^{k} q_t^3 + \sum_{t=1}^{k-1} q_t^4$$

and the rest follows by subtraction ($\tilde{\alpha}_j$ is the least squares estimate of α_j).

We have the following system of normal equations:

$$(3.1.1) \qquad Y_{i.} = \tilde{\mu} + \tilde{\beta}_i + \tilde{\alpha}_.'^1, \qquad i = M^2 + 1, M^2 + 2, \cdots, M^1$$

$$(3.1.2) \qquad Y_{i.} = \tilde{\mu} + \tilde{\beta}_i + \tilde{\alpha}_.'^2, \qquad i = M^3 + 1, M^3 + 2, \cdots, M^2$$

$$\vdots$$

$$(3.1.k-1) \qquad Y_{i.} = \tilde{\mu} + \tilde{\beta}_i + \tilde{\alpha}_.'^{k-1}, \qquad i = M^k + 1, M^k + 2, \cdots, M^{k-1}$$

$$(3.1.k) \qquad Y_{i.} = \tilde{\mu} + \tilde{\beta}_i + \tilde{\alpha}_. , \qquad i = 1, 2, \cdots, M^k$$

$$(3.2.1) \qquad Y_{.j}^1 = \tilde{\mu} + \tilde{\beta}_.^1 + \tilde{\alpha}_j^1 , \qquad j = 1, 2, \cdots, N^1$$

$$(3.2.2) \qquad Y^2_{.j} = \bar{\mu} + \bar{\beta}^2_. + \bar{\alpha}^2_j, \qquad j = N^1 + 1, N^1 + 2, \cdots, N^2$$

$$\vdots$$

$$(3.2.k-1) \qquad Y^{k-1}_{.j} = \bar{\mu} + \bar{\beta}^{k-1}_. + \bar{\alpha}^{k-1}_j, \qquad j = N^{k-2} + 1, N^{k-2} + 2, \cdots, N^{k-1}$$

$$(3.2.k) \qquad Y^k_{.j} = \bar{\mu} + \bar{\beta}^k_. + \bar{\alpha}^k_j, \qquad j = N^{k-1} + 1, N^{k-1} + 2, \cdots, N^k$$

where

$$\bar{\beta}^t_. = \frac{\sum_{i=1}^{M^t} \bar{\beta}_i}{M^t}.$$

Imposing the linear restriction $\bar{\alpha}_. = 0$, we find from (3.1.k) that

$$\bar{\mu} + \bar{\beta}_i = Y_{i.}, \qquad i = 1, 2, \cdots, M^k.$$

Substituting this into (3.2.k) we have

$$\bar{\alpha}^k_j = Y^k_{.j} - \frac{N^{k-1} Y'^{k-1}_{..} + N_k Y^k_{..}}{N}, \qquad j = N^{k-1} + 1, N^{k-1} + 2, \cdots, N^k$$

Now since

$$\sum_{j=1}^{N^{k-1}} \bar{\alpha}_j = - \sum_{j=N^{k-1}+1}^{N} \bar{\alpha}_j$$

under the restriction, $\bar{\alpha}_. = 0$, we may now substitute back and solve (3.1.$k - 1$) obtaining

$$\bar{\mu} + \bar{\beta}_i = Y_{i.} + \frac{N_k}{N^{k-1}} \cdot \frac{N^{k-1}}{N} (Y^k_{..} - Y'^{k-1}_{..})$$

$$= Y_{i.} + \frac{N_k}{N} (Y^k_{..} - Y'^{k-1}_{..}), \qquad i = M^k + 1, M^k + 2, \cdots, M^{k-1}$$

Substituting back into (3.2.$k - 1$) we get

$$\bar{\alpha}^{k-1}_j = Y^{k-1}_{.j} - \frac{M^k}{M^{k-1}} \left[\frac{N^{k-1} Y'^{k-1}_{..} + N_k Y^k_{..}}{N} \right] - \frac{\sum_{i=M^k+1}^{M^{k-1}} Y_{i.}}{M^{k-1}}$$

$$- \frac{M^{k-1} - M^k}{M^{k-1}} \cdot \frac{N_k}{N} (Y^k_{..} - Y'^{k-1}_{..})$$

$$= Y^{k-1}_{.j} - \frac{M^k}{M^{k-1}} Y'^{k-1}_{..} - \frac{\sum_{i=M^k+1}^{M^{k-1}} Y_{i.}}{M^{k-1}} - \frac{N_k}{N} (Y^k_{..} = Y'^{k-1}_{..})$$

$$= Y^{k-1}_{.j} - \frac{N^{k-2} Y'^{k-2}_{..} + N_{k-1} Y^{k-1}_{..}}{N^{k-1}} - \frac{N_k}{N} (Y^k_{..} - Y'^{k-1}_{..}),$$

$$j = N^{k-2} + 1, N^{k-2} + 2, \cdots, N^{k-1}.$$

Finishing the solution in this manner, we obtain

$$\bar{\alpha}^p_j = Y^p_{.j} - \frac{N^{p-1} Y'^{p-1}_{..} + N_p Y^p_{..}}{N^p} - \sum_{t=p+1}^{k} \frac{N_t}{N_t} (Y^t_{..} - Y'^{t-1}_{..}),$$

$$j = N^{p-1} + 1, N^{p-1} + 2, \cdots, N^p, p = 1, 2, \cdots, k$$

which may be written as

$$\tilde{\alpha}_j^p = Y_{.j}^p - Y_{..}^p - \frac{N^{p-1}}{N^p}(Y_{..}'^{p-1} - Y_{..}^p) - \sum_{t=p+1}^{k} \frac{N_t}{N_t}(Y_{..}^t - Y_{..}'^{t-1}), \tag{3.3}$$

$$j = N^{p-1}+1, N^{p-1}+2, \cdots, N^p, p = 1, 2, \cdots, k$$

Now

$$Q_j^p = M^p Y_{.j}^p - \sum_{i=1}^{M^p} Y_{i.},$$

$$j = N^{p-1}+1, N^{p-1}+2, \cdots, N^p, p = 1, 2, \cdots, k$$

But

$$\frac{\sum_{i=1}^{M^p} Y_{i.}}{M^p} = \frac{M^k}{M^p} \cdot \frac{N^{k-1} Y_{..}'^{k-1} + N_k Y_{..}^k}{N^k} + \frac{\sum_{i=M^k+1}^{M^p} Y_{i.}}{M^p},$$

Hence

$$Q_j^p = M^p \Big[Y_{.j}^p - Y_{..}^p - \frac{N^{p-1}}{N^p}(Y_{..}'^{p-1} - Y_{..}^p) - \sum_{t=p+1}^{k} \frac{M^t N_t}{M^p N^t}(Y_{..}^t - Y_{..}'^{t-1}) \Big], \quad p = 1, 2, \cdots, k.$$

Therefore

$$\sum_{j=1}^{N} Q_j \tilde{\alpha}_j = \sum_{p=1}^{k} \sum_{j=N^{p-1}+1}^{N^p} Q_j^p \tilde{\alpha}_j^p = \sum_{p=1}^{k} \Big[M^p \sum_{j=N^{p-1}+1}^{N^p} (Y_{.j}^p - Y_{..}^p)^2 \Big]$$
$$+ \sum_{p=1}^{k} M^p N_p \Big[\frac{N^{p-1}}{N^p}(Y_{..}'^{p-1} - Y_{..}^p) + \sum_{t=p+1}^{k} \frac{M^t N_t}{M^p N^t}(Y_{..}^t - Y_{..}'^{t-1}) \Big] \cdot \Big[\frac{N^{p-1}}{N^p}(Y_{..}'^{p-1} - Y_{..}^p) + \sum_{t=p+1}^{k} \frac{N_t}{N^t}(Y_{..}^t - Y_{..}'^{t-1}) \Big].$$

Collecting coefficients of $(Y_{..}'^r - Y_{..}^{r+1})^2$, we have

$$\frac{M^{r+1} N_{r+1} (N^r)^2}{(N^{r+1})^2} + \frac{M^r N_r M^{r+1} (N_{r+1})^2}{M^r (N^{r+1})^2} + \cdots + \frac{M^1 N_1 M^{r+1} (N_{r+1})^2}{M^1 (N^{r+1})^2}.$$

Combining the last r terms this becomes

$$\frac{M^{r+1} N_{r+1} (N^r)^2}{(N^{r+1})^2} + \frac{N^r M^{r+1} (N_{r+1})^2}{(N^{r+1})^2} = \frac{M^{r+1} N_{r+1} N^r (N^r + N_{r+1})}{(N^{r+1})^2}$$
$$= \frac{M^{r+1} N_{r+1} N^r}{N^{r+1}}$$

Collecting coefficients of $(Y_{..}^{\prime r} - Y_{..}^{r+1})(Y_{..}^{\prime s} - Y_{..}^{s+1})$, $r < s$, we have

$$- \frac{M^{r+1} N_{r+1} N^r N_{s+1}}{N^{r+1} N^{s+1}} + \frac{M^r N_r M^{r+1} N_{r+1} N_{s+1}}{M^r N^{r+1} N^{s+1}} + \cdots + \frac{M^1 N_1 M^{r+1} N_{r+1} N_{s+1}}{M^1 N^{r+1} N^{s+1}}$$

$$- \frac{M^{r+1} N_{r+1} N^r M^{s+1} N_{s+1}}{N^{r+1} M^{r+1} N^{s+1}} + \frac{M^r N_r N_{r+1} M^{s+1} N_{s+1}}{N^{r+1} M^r N^{s+1}} + \cdots$$

$$+ \frac{M^1 N_1 N_{r+1} M^{s+1} N_{s+1}}{N^{r+1} M^1 N^{s+1}}.$$

Combining all but the first term of each part gives

$$-\frac{M^{r+1} N_{r+1} N^r N_{s+1}}{N^{r+1} N^{s+1}} + \frac{N^r M^{r+1} N_{r+1} N_{s+1}}{N^{r+1} N^{s+1}} - \frac{N_{r+1} N^r M^{s+1} N_{s+1}}{N^{r+1} N^{s+1}}$$

$$+ \frac{N^r N_{r+1} M^{s+1} N_{s+1}}{N^{r+1} N^{s+1}} = 0.$$

Now since these two general terms are the only possible ones involved in the second summation of the expression for

$$\sum_{j=1}^{N} Q_j \tilde{\alpha}_j ,$$

we have

$$\sum_{j=1}^{N} Q_j \tilde{\alpha}_j = \sum_{p=1}^{k} \left[M^p \sum_{j=N^{p-1}+1}^{N^p} (Y_{.j}^p - Y_{..}^p)^2 \right]$$

$$+ \sum_{p=1}^{k-1} \frac{M^{p+1} N_{p+1} N^p}{N^{p+1}} (Y_{..}^{\prime p} - Y_{..}^{p+1})^2$$

$$= \sum_{t=1}^{k} q_t^3 + \sum_{t=1}^{k-1} q_t^4$$

since $N^0 = 0$.

Now by subtraction the Error S.S. must be

$$\sum_{t=1}^{k} q_t^1 + \sum_{t=1}^{k-1} q_t^2 .$$

Also since the degrees of freedom for error and treatments eliminating blocks in Table 3.1 are the same as q and p of (2.2), then we have $W = v$. Thus we have shown that the test function given in section 2 is that given by the method of least squares.

4. Means and standard errors. We will now derive the best, linear, unbiased estimates of $\alpha_s - \alpha_t$ and the standard errors of these estimates.

THEOREM II.

$$\tilde{\alpha}_s^p = Y_{.s}^p - Y_{..}^p - \frac{N^{p-1}}{N^p} (Y_{..}^{\prime p-1} - Y_{..}^p) - \sum_{t=p+1}^{k} \frac{N_t}{N^t} (Y_{..}^t - Y_{..}^{\prime t-1}),$$

$$s = N^{p-1} + 1, N^{p-1} + 2, \cdots, N^p$$
$$p = 1, 2, \cdots, k$$

is the best, linear, unbiased estimate of $\alpha_s - \alpha_.$, *and therefore* $\tilde{\alpha}_s - \tilde{\alpha}_u$ *is the best, linear, unbiased estimate of* $\alpha_s - \alpha_u$.

PROOF. Since $\tilde{\alpha}_s$ was found by the method of least squares (3.3) using the linear restriction $\tilde{\alpha}_. = 0$, this result is a consequence of the Markoff Theorem [3].

THEOREM III. *The variance of the estimate of* $\alpha_s^p - \alpha_u^p$ *is* $2\sigma^2/M^p$ *if* $s \neq u$, *and* $s, u = N^{p-1} + 1, N^{p-1} + 2, \cdots, N^p$ *for* $p = 1, 2, \cdots, k$. *The variance of the estimate of* $\alpha_s^p - \alpha_u^r$ *is*

$$\sigma^2 \left[\frac{N^p - 1}{M^p N^p} + \frac{N^{r-1} + 1}{M^r N^{r-1}} + \sum_{t=p+1}^{r-1} \frac{N_t}{M^t N^t N^{t-1}} \right.$$

for $s = N^{p-1} + 1, N^{p-1} + 2, \cdots, N^p$, $u = N^{r-1} + 1, N^{r-1} + 2, \cdots, N^r$, $p = 1, 2, \cdots, k - 1$, $r = p + 1, p + 2, \cdots, k$.

PROOF.

$$\tilde{\alpha}_s^p - \tilde{\alpha}_u^p = Y_{.s}^p - Y_{.u}^p,$$

$s \neq t$ and $s, u = N^{p-1} + 1, N^{p-1} + 2, \cdots, N^p$, $p = 1, 2, \cdots, k$.

And

$$\text{Var}\,(\tilde{\alpha}_s^p - \tilde{\alpha}_u^p) = E\left[\frac{\sum_{i=1}^{M^p} e_{is}}{M^p} \frac{\sum_{i=1}^{M^p} e_{iu}}{M^p}\right]$$
$$= \frac{\sigma^2}{M^p} + \frac{\sigma^2}{M^p} = \frac{2\sigma^2}{M^p}, \qquad p = 1, 2, \cdots, k.$$

Now

$$\tilde{\alpha}_s^p - \tilde{\alpha}_u^r = Y_{.s}^p - \frac{N^{p-1} Y_{..}'^{p-1} + N_p Y_{..}^p}{N^p} - \sum_{t=p+1}^{r-1} \frac{N_t}{N^t} (Y_{..}^t - Y_{..}'^{t-1}) - (Y_{.u}^r + Y_{..}'^{r-1})$$

and by straightforward application of expected values we arrive at the result.

From the theory of least squares it follows that the error mean square

$$\frac{1}{q} \left| \sum_{t=1}^{k} q_t^1 + \sum_{t=1}^{k-1} q_t^2 \right|$$

(where q is defined in 2.2) is an unbiased estimate of σ^2 and is independent of $\tilde{\alpha}_i$. Therefore, these quantities may be used to set confidence limits about the difference between treatment means or any linear contrast of treatment means. Therefore, by using equation (2.1), the analysis of variance for the Staircase Design is easily computed. By using the formulas in Theorem II and Theorem III, the means and standard errors can be easily computed even if the number of steps is large. In another paper we will give detailed computing instructions with a numerical example of the Staircase Design.

We have assumed throughout that the variance of e_{ij} is a constant independent of i and j. It may be that if the variance of e_{ij} is in some way dependent on the number of experimental units in a block, and if the number of units differ widely, then this may somewhat invalidate the exact distributions of the test function. The variances will probably have to be quite different before the distributions are disturbed appreciably. On the other hand it may be that an experimenter divides his material into homogeneous groups with constant variances and finds he ends up with different number of plots in a block. This would suggest using the staircase design.

Also this design may be useful in case an experimenter desires to conduct an experiment on two sets of treatments and is satisfied with different precisions on the two sets.

REFERENCES

[1] R. C. Bose and K. R. Nair, "Partially balanced incomplete block designs", *Sankhya*, Vol. 4 (1939), pp. 337–372.
[2] R. A. Fisher, *The design of experiments.* Oliver and Goyd, Edinburgh 4th edition, 1947.
[3] F. N. David and J. Neyman, "Extension of the Markoff theorem on least squares", *Stat. Res. Memoirs*, Vol. 2 (1938), pp. 105–116.
[4] O. Kempthorne, "*The design and analysis of experiments*", John Wiley and Sons, Inc., New York 1952.
[5] W. G. Madow, "The distribution of quadratic forms in non-central normal random variables", *Ann. Math. Stat.* Vol. 11 (1940), pp. 100–103.
[6] C. R. Rao, "General methods of analysis for incomplete block designs", *J. Amer. Stät. Assn.*, Vol. 42 (1947), pp. 541–561.
[7] P. C. Tang, "The power function of the analysis of variance tests with tables and illustrations of their use", *Stat. Res. Memoirs*, Vol. 20 (1938), pp. 126–149.
[8] F. Yates, "Incomplete randomized blocks", *Ann. Eug.*, Vol. 7 (1936), pp. 121–140.
[9] W. J. Youden and W. S. Connor, "The chain block design," *Biometrics* Vol. 9 (1953) pp. 127–140.
[10] K. Kishen, "Symmetrical unequal block arrangements", *Sankhya*, Vol. 5 (1941) pp. 329–344.