Using 1.6 and the linear independence of the B's, 2.1 yields

$$(2.2)\quad (c_0 I, c_1 I, \cdots, c_m I)\begin{bmatrix} (d_{00}-e)I & d_{01} I & \cdots & d_{0m} I \\ d_{10} I & (d_{11}-e)I & & d_{1m} I \\ \vdots & \vdots & & \vdots \\ d_{m0} I & d_{m1} I & \cdots & (d_{mm}-e)I \end{bmatrix} = 0.$$

Therefore

$$(2.3)\quad (c_0, c_1, \cdots, c_m)(D - eI) = 0.$$

If C has m^* distinct non-zero characteristic roots, $e_1, e_2, \cdots, e_{m^*}$, then we may write

$$C = e_1E(e_1) + e_2E(e_2) + \cdots + e_{m^*}E(e_{m^*}).$$

Now using Theorem 2 we have

THEOREM 3. *The C matrix of* a P.B.I.B. (m) *may be expressed as a linear function of the $m + 1$ commutative and linearly independent matrices $B_0, B_1, \cdots, B_m$.*

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ON A FACTORIZATION THEOREM IN THE THEORY OF ANALYTIC CHARACTERISTIC FUNCTIONS[1]

Dedicated to Paul Lévy on the occasion of his seventieth birthday

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1. Introduction. Let $F(x)$ be a distribution function, that is, a non-decreasing right-continuous function such that $F(-\infty) = 0$ and $F(+\infty) = 1$. The characteristic function

$$(1.1)\quad \phi(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x)$$

of the distribution function $F(x)$ is defined for all real t. A characteristic function is said to be an *analytic characteristic function* if it coincides with a regular analytic function $\phi(z)$ in some neighborhood of the origin in the complex z-plane.

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Then it follows from a theorem due to Boas [1] that the analytic characteristic function $\phi(z)$ is also regular in a horizontal strip $-\alpha < Im\ z < +\beta$ of the complex z-plane containing the real axis. It is also well known that the analyticity of the characteristic function $\phi(z)$ in the horizontal strip $| Im\ z | \leqq R (R > 0)$ is equivalent to the condition that (a) the corresponding distribution function $F(x)$ has moments μ_k of all orders k and further (b) $\lim \sup_{k\to\infty}[\mu_k/k!]^{1/k}$ is finite and equal to $1/R$. In other words, the analytic characteristic function $\phi(z)$ has the power series expansion

$$\phi(z) = \sum_{k=0}^{\infty} \frac{i^k \mu_k}{k!} z^k \tag{1.2}$$

about the origin $z = 0$ in the circle $| z | \leqq R$ (z complex) where $R > 0$ is the radius of convergence of the series. The characteristic function $\phi(z)$ is said to be an *entire characteristic function* if its strip of regularity comprises the whole complex z-plane. A summary of most of the important properties of analytic characteristic functions is given in a recent paper by Lukacs [6].

In the present paper we shall discuss some results concerning the decomposition properties of analytic characteristic functions. In this direction a very interesting theorem has been recently obtained by Linnik [5], [7] which may be considered as an analytical extension of Cramér's theorem [2] on the normal law. The theorem is as follows:

THEOREM OF LINNIK. *Let $\phi_1(t), \phi_2(t), \cdots, \phi_n(t)$ denote the characteristic functions of some non-degenerate distributions and let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be some positive numbers. Let the functions $\phi_j(t)$ satisfy the equation*

$$\prod_{j=1}^{n} \{\phi_j(t)\}^{\alpha_j} = \exp \{i\mu t - \tfrac{1}{2}\sigma^2 t^2\} \tag{1.3}$$

for all real t in a certain neighborhood $| t | < \delta (\delta > 0)$ of the origin, where $\sigma^2 > 0$ and μ are real numbers. Then each factor $\phi_j(t)$ is the characteristic function of a normal distribution.

In the following section we shall deal with some related factorization theorems (Theorems 2.1 and 2.2) for analytic characteristic functions. These theorems may be considered as generalizations of the theorem of Linnik stated above.

2. The Theorems. We now consider the following theorems:

THEOREM 2.1. *Let $\phi_1(t), \phi_2(t), \cdots, \phi_n(t)$ denote the characteristic functions of some non-degenerate distributions. Let further $\phi(z)$ denote an analytic characteristic function and $\alpha_1, \alpha_2, \cdots, \alpha_n$ be some positive numbers. Let the functions $\phi_j(t)$ satisfy the equation*

$$\prod_{j=1}^{n} \{\phi_j(t)\}^{\alpha_j} = \phi(t) \tag{2.1}$$

for all real t in a certain neighborhood $|t| < \delta (\delta > 0)$ of the origin. Then each of the factors $\phi_j(z)$ is also an analytic characteristic function which is regular at least in the strip of regularity of $\phi(z)$.

This theorem has already been obtained by Dugué and stated without proof in [3]. The author has also independently obtained a proof of this theorem, following a method closely similar to that used by Linnik in [7]. Proceeding along the same lines as the proof of the theorem of Linnik [7], we can show that each of the corresponding distribution functions has finite moments of all orders and then finally each $\phi_j(z)$ is an analytic characteristic function having a power series expansion about $z = 0$ with a positive radius of convergence. Since Linnik's method of proof has been already presented by the author in [4], the proof of Theorem 2.1 is omitted. It is understood that the reader may easily construct a proof of Theorem 2.1, following the procedure indicated in [4]. We shall next prove a related theorem on the entire characteristic function.

THEOREM 2.2. *Under the same conditions as in Theorem* 2.1, *let $\phi(z)$ be an entire characteristic function of some finite order ρ. Then each of the factors $\phi_j(z)$ is also an entire characteristic function of finite order not exceeding ρ.*

PROOF. First of all, we give a precise definition of the *order* of an entire characteristic function. Let $f(z)$ be an entire characteristic function of some finite order ρ. We denote by

$$M(r, f) = \max_{|z| \leqq r} |f(z)| \tag{2.2}$$

the maximum modulus of the function $f(z)$ in the circle $|z| \leqq r$ (z complex). This value is evidently assumed on the perimeter of this circle. Then using the well known property of the positive definite functions

$$\max_{-\infty \leqq t \leqq +\infty} |f(t + iv)| \leqq f(iv) \qquad (t \text{ and } v \text{ real}) \tag{2.3}$$

we can easily deduce from (2.2) that

$$M(r, f) = \max [f(ir), f(-ir)]. \tag{2.4}$$

The order ρ of an entire characteristic function $f(z)$ is then defined as

$$\rho = \limsup_{r \to \infty} \frac{\ln \ln M(r, f)}{\ln r}. \tag{2.5}$$

We now turn to the proof of Theorem 2.2. Without any loss of generality in the proof, we introduce the symmetrized characteristic functions

$$\begin{cases} \theta_j(t) = \phi_j(t)\phi_j(-t), \\ \theta(t) = \phi(t)\phi(-t). \end{cases} \qquad j = 1, 2, \cdots . n, \tag{2.6}$$

Then it is easy to verify from (2.1) that the characteristic functions $\theta_j(t)$ satisfy the equation

$$\prod_{j=1}^{n} \{\theta_j(t)\}^{\alpha_j} = \theta(t) \tag{2.7}$$

for all real t in a certain neighborhood of the origin. We can see easily that under the conditions of the theorem the symmetric characteristic function $\theta(z) = \phi(z)\phi(-z)$ is also an entire characteristic function of the same order ρ as

$\phi(z)$. It then follows at once from Theorem 2.1 that each of the factors $\theta_j(z)$ in Eq. (2.7) is also an entire characteristic function when $\theta(z)$ is an entire function. Thus we have the equation

$$\prod_{j=1}^{n} \{\theta_j(z)\}^{\alpha_j} = \theta(z) \tag{2.8}$$

holding for all complex z.

We now consider the behavior of each of the functions $\theta_j(z)$ for purely imaginary values of z. For this purpose, we substitute $z = iv$ (v real) in Eq. (2.8), thus obtaining

$$\prod_{j=1}^{n} \{\theta_j(iv)\}^{\alpha_j} = \theta(iv). \tag{2.9}$$

Now we note that the distribution function corresponding to each $\theta_j(z)$ is symmetric about the origin and hence has all moments of odd order equal to zero. Let us denote by $\mu_{2k,j}$ the moment of even order $2k$ of the distribution function corresponding to the characteristic function $\theta_j(z)$, $j = 1, 2, \cdots, n$. Then we have

$$\theta_j(iv) = \sum_{k=0}^{\infty} \frac{\mu_{2k,j}}{(2k)!} v^{2k} \geqq 1, \qquad j = 1, 2, \cdots, n. \tag{2.10}$$

Using (2.10) in Eq. (2.9), we have, for every j, the inequality

$$\{\theta_j(iv)\}^{\alpha_j} \leqq \theta(iv). \tag{2.11}$$

We denote by $M(r, \theta_j)$ and $M(r, \theta)$ the maximum moduli of the characteristic functions $\theta_j(z)$ and $\theta(z)$ respectively in the circle $|z| \leqq r$ (z complex) as in (2.4). Then noting the consequence of symmetrization of $\theta_j(z)$ and $\theta(z)$, we can easily verify

$$\begin{cases} M(r, \theta_j) = \theta_j(ir) = \theta_j(-ir), \\ M(r, \theta) = \theta(ir) = \theta(-ir). \end{cases} \qquad j = 1, 2, \cdots, n, \tag{2.12}$$

Then substituting the relations obtained in (2.12) in the inequality (2.11), we get for every j

$$\{M(r, \theta_j)\}^{\alpha_j} \leqq M(r, \theta), \qquad j = 1, 2, \cdots, n. \tag{2.13}$$

Then using the definition of the order of an entire characteristic function as given in (2.5), it follows easily from (2.13) that each of the factors $\theta_j(z)$ is an entire function of order not exceeding ρ. This at once establishes that each of the factors $\phi_j(z)$ is also an entire characteristic function of order not exceeding ρ, thus completing the theorem.

3. Applications. We now apply the theorems in the preceding section to give a simple proof of the theorem of Linnik.

In this case it is given that $\phi(t) = e^{Q(t)}$, where $Q(t)$ is a quadratic polynomial in t. Thus it is known that $\phi(z)$ is an entire characteristic function of order two and without any zeros. Hence applying Theorems 2.1 and 2.2, it follows at once that each of the factors $\phi_j(z)$ is also an entire characteristic function of order not exceeding two and without any zeros in the complex plane. Then the proof follows at once, using the factorization theorem of Hadamard to each of the factors $\phi_j(z)$.

In conclusion the author wishes to express his thanks to Professor Eugene Lukacs for calling his attention to the paper by Dugué [3].

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BOUNDS FOR MILLS' RATIO FOR THE TYPE III POPULATION

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1. Introduction and summary. Cohen [1] and Des Raj [2] have shown that in estimating the parameters of truncated type III populations, it is necessary to calculate for several values of x the Mills ratio of the ordinate of the standardized type III curve at x to the area under the curve from x to ∞. Des Raj [3] has also noted that for large values of x the existing tables of Salvosa [4] are inadequate for this purpose and he has found lower and upper bounds for the ratio. The object of this note is to improve these bounds, by obtaining monotonic sequences of lower and upper bounds through the use of continued fractions.

2. Approximations to the ratio. Taking the type III population in the standardized form

$$C\, f(x)\, dx, \qquad -2/\alpha \leqq x \leqq \infty, \qquad 0 \leqq \alpha \leqq 2,$$

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