the common value being

$$\frac{1}{v-1} \text{ trace } C_0 = \frac{vr}{v-1}\left(1 - \frac{1}{U} - \frac{1}{U'} + \frac{1}{UU'}\right),$$
$$= a, \text{ say}.$$

It therefore follows that for designs in which heterogeneity is eliminated in two directions, the efficiency factor is maximum if

$$\frac{1}{U'} LL' + \frac{1}{U} MM' \text{ is of the form}$$

$$\begin{bmatrix} p & q & q & \cdots & q \\ q & p & q & \cdots & q \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ q & q & q & \cdots & p \end{bmatrix}.$$

It should be observed that, for a Youden Square (where the rows are complete blocks and columns form a symmetrical balanced incomplete block design),

$$U = r, \qquad U' = v$$

and

$$L = E_{vv}$$

and

$$MM' = \begin{bmatrix} r & \lambda & \lambda & \cdots & \lambda \\ \lambda & r & \lambda & \cdots & \lambda \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \lambda & \lambda & \lambda & \cdots & r \end{bmatrix}.$$

and $LL'/U' + MM'/U$ is of the required form. Consequently, among designs in which heterogeneity is eliminated in two directions, a Youden Square, if it exists, has maximum efficiency.

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ON A MINIMAX PROPERTY OF A BALANCED INCOMPLETE BLOCK DESIGN

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Summary. It is shown that for a given set of parameters (b blocks, k plots per block and v treatments), among the class of connected incomplete block designs,

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a balanced incomplete block design (if it exists) is the design which maximizes the minimum efficiency, efficiency being defined as

$$\frac{\text{Variance of an estimated treatment contrast in a randomized block}}{\text{Variance of the estimated treatment contrast in the incomplete block}}.$$

The proof will be preceded by a lemma.

Notation. Capital letters will be used to denote matrices and boldface small letters to denote vectors. At times a matrix of m rows and n columns will be denoted by $A(m \times n)$.

LEMMA. *If $B(p \times p)$ is real symmetric and at least positive semidefinite of rank $r(\leqq p)$, then:*

(i) *The stationary values of*

$$\frac{\mathbf{a}'(1 \times p)B(p \times p)\mathbf{a}(p \times 1)}{\mathbf{a}'\mathbf{a}}$$

under the variation of $\mathbf{a}$ *(over all non-null* $\mathbf{a}$ *excepting the solutions of* $B\mathbf{a} = 0$*) are the characteristic roots of* B.

(ii) *In particular the largest and the smallest values of* $\mathbf{a}'B\mathbf{a}/\mathbf{a}'\mathbf{a}$ *(under the variation of all non-null* $\mathbf{a}$ *excepting the solutions of* $B\mathbf{a} = 0$*), are the largest and the smallest non-zero characteristic roots of* B.

(iii) $\mathbf{a}'B\mathbf{a}/\mathbf{a}'\mathbf{a}$ *attains its maximum (or minimum) value if and only if* $\mathbf{a}$ *is a latent vector corresponding to the maximum (or minimum) latent roots of* B.

For a proof of this lemma we refer to S. N. Roy [3] and H. W. Turnbull and A. C. Aitken [4].

Let us adopt the following notation:

$\lambda_{i\alpha}$ = number of blocks in which the ith and the αth treatments appear together.

r_i = number of blocks in which the ith treatment appears.

$$c_{i\alpha} = \begin{cases} \dfrac{-\lambda i\alpha}{k}\ i \neq \alpha;\ i = 1, 2, \cdots\ v;\ \alpha = 1, 2, \cdots, v. \\ r_i\left(1 - \dfrac{1}{k}\right), i = \alpha. \end{cases}$$

T_i = total yield of the ith treatment.

B_j = total yield of the jth block.

$$n_{ij} = \begin{cases} 1 \text{ if the } i\text{th treatment appears in the } j\text{th block,} \\ 0 \text{ otherwise.} \end{cases}$$

$$Q_i = T_i - \frac{1}{k}\sum_{j=1}^{b} n_{ij} B_j .$$

Finally let

$$\mathbf{Q}'(1 \times v) = (Q_1 Q_2 \cdots Q_v).$$

In any connected incomplete block design the adjusted normal equations are given by

$$Ct = Q$$

where

$$C = (c_{i\alpha}) \qquad i = 1, 2, \cdots, v, \quad \alpha = 1, 2, \cdots, v.$$

It is well known that C is symmetric positive semidefinite of rank $v - 1$ and that the only independent non-trivial solution of the equations $C\mathbf{x} = 0$ is

$$\mathbf{x}'(1 \times v) = (1, 1, \cdots, 1).$$

Let $\mathbf{m}'(1 \times v) = (m_1 m_2, \cdots, m_v)$ be a non-null vector such that $\sum_{i=1}^{v} m_i = 0$.

It is well known (e.g., see R. C. Bose and S. Ehrenfeld) that the variance of the "best estimate" of $\mathbf{m}'\mathbf{t}$ is given by $\boldsymbol{\varrho}'C\boldsymbol{\varrho}\sigma^2$ where $\boldsymbol{\varrho}$ is a solution of $C\boldsymbol{\theta} = \mathbf{m}$.

We shall now show that

$$\sup_{\mathbf{m}\epsilon M} \frac{\boldsymbol{\varrho}'C\boldsymbol{\varrho}}{\mathbf{m}'\mathbf{m}} = \frac{1}{\lambda_{\min}}$$

where M is the class of all non-null vectors $\mathbf{m}'(1 \times v) = (m_1, m_2, \cdots, m_v)$ such that $\sum_i m_i = 0$ and $\lambda_{\min}$ is the smallest of the $v - 1$ non-zero characteristic roots of C.

Since C is real symmetric, it follows that there exists an orthogonal matrix $P(v \times v)$ such that

$$P'CP = \begin{bmatrix} D_{\lambda_i}\,[(v-1) \times (v-1)] & 0\,[(v-1) \times 1] \\ 0\,[1 \times (v-1)] & 0 \end{bmatrix}$$

where D_{λ_i} is a diagonal matrix; the diagonal elements being $\lambda_1, \lambda_2, \cdots, \lambda_{v-1}$, the non-zero latent roots of C. Let

$$P = [P_1[v \times (v-1)] \qquad \mathbf{q}\,(v \times 1)].$$

Then $C = P_1 D_{\lambda_i} P_1'$.

It can be easily shown that

$$P_1 P_1' + \mathbf{q}\mathbf{q}' = I, \tag{2}$$

$$P_1' P_1 = I, \tag{3}$$

and that the rank of P_1 is $v - 1$ and

$$\mathbf{q}'(1 \times v) = \frac{1}{\sqrt{v}}(1, 1, \cdots, 1). \tag{4}$$

It can be seen that

$$\boldsymbol{\varrho} = [P_1 D_{\lambda_i}^{-1} P_1']\mathbf{m}$$

is a solution of $C\boldsymbol{\theta} = \mathbf{m}$, and

$$\frac{\boldsymbol{\varrho}'C\boldsymbol{\varrho}}{\mathbf{m}'\mathbf{m}} = \frac{\mathbf{m}'P_1 D_{\lambda_i}^{-1} P_1'\mathbf{m}}{\mathbf{m}'\mathbf{m}}.$$

Hence by virtue of the lemma stated earlier we have

$$\operatorname*{Sup}_{\mathbf{m}\varepsilon M} \frac{\mathbf{m}'(P_1 D_{\lambda_i}^{-1} P_1')\mathbf{m}}{\mathbf{m}'\mathbf{m}} = \frac{1}{\lambda_{\min}}.$$

The variance of the "best estimate" of $\mathbf{m}'\mathbf{t}$ in a randomized block is

$$(1/b)\mathbf{m}'\mathbf{m}\sigma^2.$$

Hence,

$$\text{efficiency} = \left(\frac{1}{b}\right) \frac{\mathbf{m}'\mathbf{m}}{\boldsymbol{\varrho}'C\boldsymbol{\varrho}}$$

where $\boldsymbol{\varrho}$ is a solution of $C\boldsymbol{\theta} = \mathbf{m}$. Now

$$\inf_{m \varepsilon M} \left[\frac{\mathbf{m}'\mathbf{m}}{\boldsymbol{\varrho}'C\boldsymbol{\varrho}}\right] = \left[\frac{1}{\operatorname*{Sup}_{m\varepsilon M} \frac{\boldsymbol{\varrho}'C\boldsymbol{\varrho}}{\mathbf{m}'\mathbf{m}}}\right] = \lambda_{\min}.$$

Hence, minimum efficiency $= \lambda_{\min}/b$. It can be shown that for any connected design $\lambda_{\min} \leqq \lambda v/k$, where

$$\lambda = \frac{bk(k-1)}{v(v-1)}.$$

Now if we can show that, $\lambda_{\min} = \lambda v/k$ if and only if the design is a balanced incomplete block design, then our problem is solved. If the design is a balanced incomplete block design, then, $\lambda_{\min} = \lambda v/k$, since $\lambda v/k$ is a latent root of multiplicity $v - 1$ for the C corresponding to the given design. The next thing we have to show is that if $\lambda_{\min} = \lambda v/k$, then the design is a balanced incomplete block design. Since $\lambda_{\min} = \lambda v/k$, it follows that all of the remaining $v - 2$ roots must be exactly $\lambda v/k$. Hence

$$C = P_1 D_{\lambda_i} P_1' = \frac{\lambda v}{k} P_1 P_1'.$$

By virtue of equations (2) and (4) we have

$$P_1 P_1' = I - \frac{1}{v} J$$

where J is a matrix of dimensions $v \times v$ in which every element is unity. Hence

$$C = \frac{\lambda v}{k}\left[I - \frac{1}{v} J\right].$$

Thus $\lambda_{i\alpha} = \lambda$ for all $i \neq \alpha$ hence, the result.

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A CHARACTERIZATION OF THE NORMAL DISTRIBUTION[1]

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1. Introduction. Using characteristic functions Lukacs [3] has shown that a necessary and sufficient condition for the independence of the sample mean and variance is that the parent population be normal. Geisser [2] has derived a similar theorem concerning the sample mean and the first order mean square successive difference. In section 2 of this note a general theorem of which Lukacs' and Geisser's results are particular cases has been proved.

Lukacs [3] has extended his theorem to the multivariate case, namely, that a necessary and sufficient condition that the sample mean vector is distributed independently of the variance-covariance matrix is that the parent population be multivariate normal. In section 3, the general theorem of section 2 is extended to the multivariate population of which Lukacs' theorem for the multivariate population is a particular case. To prove the necessity of this theorem, we extend, to the multivariate case, Daly's [1] result that if $f(x)$ is the normal density, then the sample mean and $g(x_1 \cdots x_n)$ are independently distributed where $g(x_1 \cdots x_n) = g(x_1 + a, \cdots, x_n + a)$.

2. Univariate case. Let $x_1, \cdots, x_n$ be independent and identically distributed with density function $f(x)$ and mean μ and variance σ^2.

Let,

$$\bar{x} = n^{-1} \sum_{j=1}^{n} x_j \cdots \tag{2.1}$$

and

$$\delta^2 = \left(\sum_{t=1}^{m} \sum_{j=1}^{n} l_{tj}^2\right)^{-1} \sum_{t=1}^{m} (l_{t1} x_1 + \cdots + l_{tn} x_n)^2, \qquad m \geqq 1 \tag{2.2}$$

where

$$\sum_{j=1}^{n} l_{tj} = 0 \quad \text{for} \quad t = 1, \cdots, m.$$

The following theorem is proved.

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