with the two branches of this hyperbola. It is readily seen that the condition that such an x-interval exists and be of finite length is equivalent to the condition that the two asymptotes of the hyperbola have slopes of equal sign. Since these slopes are $b_1 - K/[\sum(x - \bar{x})^2]^{\frac{1}{2}}$ and $b_1 + K/[\sum(x - \bar{x})^2]^{\frac{1}{2}}$, the condition in question is $b_1^2 - K^2/(\sum(x - \bar{x})^2) > 0$. This is condition (10) obtained previously by a different line of reasoning.

It may be observed, finally, that the inverse problem, viz, to determine uncertainty intervals for observed y values corresponding to given x values [2] is not a classical case of interval estimation, since it is concerned with bracketing a random variable, not a population parameter, by means of two statistics. Intervals of this type are discussed by Weiss [7].

Applications of the procedure outlined in this note to a problem in chemistry are discussed elsewhere [5].

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A NOTE ON INCOMPLETE BLOCK DESIGNS

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1. Introduction. Kempthorne [1] has shown the efficiency factor of an incomplete block design to be a quantity proportional to the harmonic mean of the nonzero latent roots of the matrix of coefficients of the reduced normal equations for the intra-block estimates of treatment effects. He has further stated that the geometric mean in a certain sense corresponds to the generalized variance but has not explicitly explained it. The present note is intended to clear this point and to prove that the design with highest efficiency factor (in any case, whether the harmonic mean or the geometric mean is taken as a measure of efficiency) is

(a) a balanced incomplete block design, if such a design exists; and

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(b) a Youden Square, if it exists, among designs in which heterogeneity is eliminated in two directions.

There is some overlap between this paper and the ones by Kiefer and by Mote in this issue.

2. Incomplete block design. Let there be v treatments and b blocks of k plots each. Let r be the number of replications of each treatment and let N be the incidence matrix of the design (rows refer to the treatments and columns to blocks). Each element of N, for an incomplete block design, is either 0 or 1, Then the matrix of coefficients of the reduced normal equations for the intra-block estimates t_i of the treatment effects is

$$C = rI_v - \frac{1}{k} NN',$$

where I_v denotes the identity matrix of order v. For any design, C has one zero latent root, the corresponding latent vector having all the elements equal. Let the non-zero roots of C be $\lambda_1, \lambda_2, \cdots, \lambda_{v-1}$ and let $m_1, m_2, \cdots, m_{v-1}$ be the corresponding orthogonal normalized latent vectors (column). Kempthorne [1] chooses the average variance of elementary treatment contrasts like $t_i - t_j$ to arrive at the harmonic mean of the λ's as a definition of efficiency factor of a design. The author, however, feels that, instead, a complete set of $v - 1$ orthogonal normalized treatment contrasts be chosen because,

(1) their average variance leads to the harmonic mean of the λ's; and

(2) their generalized variance leads to the geometric mean of the λ's, as a criterion to measure the efficiency of a design. Let $l_i\,(i = 1, 2, \cdots, v - 1)$ be orthogonal normalized column vectors so that $l_i't$ where

$$t = \begin{bmatrix} t_1 \\ t_2 \\ \cdots \\ t_v \end{bmatrix}$$

form a complete set of orthogonal normalized treatment contrasts. Then, if we observe that

$$\sum_{i=1}^{v-1} (l_i' t)^2 = \sum_{i=1}^{v} (t_i - \bar{t})^2,$$

$$= \frac{1}{2v} \sum_{\substack{i,j=1 \\ i \neq j}}^{v} (t_i - t_j)^2$$

and use Kempthorne's [1] result about average variance of $t_i - t_j$, it follows readily that the average variance of a full set of orthogonal normalized treatment contrasts is proportional to the hormonic mean of $\lambda_1, \lambda_2, \cdots, \lambda_{v-1}$.

However, if we consider the generalized variance of $l_i't\,(i = 1, 2, \cdots, v - 1)$, it can be shown that it is proportional to $(\lambda_1\lambda_2 \cdots \lambda_{v-1})^{-1}$. This can be proved by using the fact that the transformation from $l_i't\,(i = 1, 2, \cdots, v - 1)$

to $m_i' t$ $(i = 1, 2, \cdots, v - 1)$ is orthogonal and that

$$V(m_i' t) = \frac{\sigma^2}{\lambda_i}; \qquad i = 1, 2, \cdots, v - 1$$

and

$$\text{Cov}\,(m_i' t, m_j' t) = 0, \qquad i \neq j,$$

where σ^2 is the variance of the yield of a plot.

Thus, either $\sum_1^{v-1} 1/\lambda_i$ or $(\lambda_1 \lambda_2 \cdots \lambda_{v-1})^{-1}$ can be taken as a measure of efficiency of the design. It should be noted that

$$\sum_{i=1}^{v-1} \lambda_i = \text{trace } C = vr\left(1 - \frac{1}{k}\right).$$

Hence to obtain a design with highest efficiency we have to minimise either $\sum_{i=1}^{v-1} 1/\lambda_i$ or $(\lambda_1 \lambda_2 \cdots \lambda_{v-1})^{-1}$ subject to the condition that

$$\sum_{i=1}^{v-1} \lambda_i = \text{constant}.$$

This immediately leads to

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{v-1} = \frac{vr}{v - 1}\left(1 - \frac{1}{k}\right)$$

and consequently,

$$C = \frac{vr}{v - 1}\left(1 - \frac{1}{k}\right)\left(I_v - \frac{1}{v} E_{vv}\right)$$

where E_{pq} denotes a $p \times q$ matrix, all the elements of which are unity. This proves, therefore, that the design with the highest efficiency is a balanced incomplete block design, if such a design exists.

3. Designs in which heterogeneity is eliminated in two directions. Let there be UU' plots arranged in U rows and U' columns, and let v treatments be assigned to these plots in such a way that every treatment is replicated r times and the ith treatment occurs l_{ij} times in the jth row and m_{ik} times in the kth column $(i = 1, 2, \cdots, v;\ j = 1, 2, \cdots, U;\ k = 1, 2, \cdots, U')$ where l_{ij} and m_{ik} are either 0 or 1. Let $L = [l_{ij}]$ and $M = [m_{ik}]$. Then the matrix of coefficients of reduced normal equations for treatments effects after eliminating row and column effects is

$$C_0 = rI_v - \frac{1}{U'} LL' - \frac{1}{U} MM' + \frac{r^2}{UU'} E_{vv}.$$

This matrix C_0 plays the same role as C in section 2. Hence for a design of this type, the efficiency is maximum if all the non-zero latent roots of C_0 are equal, the common value being

$$\frac{1}{v-1} \text{trace } C_0 = \frac{vr}{v-1}\left(1 - \frac{1}{U} - \frac{1}{U'} + \frac{1}{UU'}\right),$$
$$= a, \text{ say.}$$

It therefore follows that for designs in which heterogeneity is eliminated in two directions, the efficiency factor is maximum if

$$\frac{1}{U'} LL' + \frac{1}{U} MM' \text{ is of the form}$$

$$\begin{bmatrix} p & q & q & \cdots & q \\ q & p & q & \cdots & q \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ q & q & q & \cdots & p \end{bmatrix}.$$

It should be observed that, for a Youden Square (where the rows are complete blocks and columns form a symmetrical balanced incomplete block design),

$$U = r, \qquad U' = v$$

and

$$L = E_{vv}$$

and

$$MM' = \begin{bmatrix} r & \lambda & \lambda & \cdots & \lambda \\ \lambda & r & \lambda & \cdots & \lambda \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \lambda & \lambda & \lambda & \cdots & r \end{bmatrix}.$$

and $LL'/U' + MM'/U$ is of the required form. Consequently, among designs in which heterogeneity is eliminated in two directions, a Youden Square, if it exists, has maximum efficiency.

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ON A MINIMAX PROPERTY OF A BALANCED INCOMPLETE BLOCK DESIGN

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Summary. It is shown that for a given set of parameters (b blocks, k plots per block and v treatments), among the class of connected incomplete block designs,

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