ON A PROBABILITY PROBLEM IN THE THEORY OF COUNTERS

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1. Introduction. Let us suppose that particles arrive at a counter in the time interval $(0, \infty)$ according to a Poisson-process of density λ. Each particle arriving in the time interval $(0, \infty)$ independently of the others gives rise to an impulse with probability p or 1 according to whether at this instant there is an impulse present or there is no impulse present. The time durations of the impulses are identically distributed independent positive random variables with distribution function $H(x)$ and these random variables are independent of the instants of the arrivals and of the events of the realizations of the impulses. We define as "registered particles" those particles which occur at an instant when there is no impulse present. Denote by ν_t the number of the registered particles in the time interval $(0, t)$. The problem is to determine the distribution law of ν_t and its asymptotic behaviour as $t \to \infty$.

The particular case of the above problem, when the time durations of the impulses are constant, was investigated earlier by G. E. Albert and L. Nelson [1].

2. The structure of the process. Denote by $\{\tau_n\}$ the sequence of instants at which particles are registered. We say that the system at any instant t is in state A when no impulse covers the instant t and in state B otherwise. Then the system assumes the states $A, B, A, B, \cdots$ alternatingly. Let us denote by $\xi_1, \eta_1, \xi_2, \eta_2, \cdots$ the times spent in states A and B respectively. If the system at the instant t is in state A, then t is evidently a regeneration point of the process. Consequently $\{\xi_n\}$ and $\{\eta_n\}$ are independent sequences of identically distributed positive random variables. Clearly $\mathbf{P}\{\xi_n \leqq x\} = F(x) = 1 - e^{-\lambda x}$ if $x \geqq 0$. Write $\mathbf{P}\{\eta_n \leqq x\} = U(x)$, where $U(x)$ is still unknown. (We use $\mathbf{P}$ for the symbol of probability and $\mathbf{E}$ for expectation.) It can easily be seen that the instants of the transitions $A \to B$ coincide with the instants τ_n $(n = 1, 2, \cdots)$. Consequently the time differences $\tau_{n+1} - \tau_n$ $(n = 1, 2, \cdots)$ are identically distributed independent random variables with distribution function $G(x) = F(x) * U(x)$ i.e.

$$G(x) = \int_0^x U(x - y) e^{-\lambda y} \lambda \, dy, \tag{1}$$

while $\mathbf{P}\{\tau_1 \leqq x\} = F(x)$.

3. Notations. Let us introduce the following Laplace-Stieltjes transforms:

$$\gamma(s) = \int_0^\infty e^{-sx} \, dG(x) \tag{2}$$

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and

$$\omega(s) = \int_0^\infty e^{-sx}\, dU(x). \tag{3}$$

By (1) we have

$$\gamma(s) = \frac{\lambda}{\lambda + s}\,\omega(s). \tag{4}$$

Further put

$$\alpha = \int_0^\infty x\, dH(x), \qquad \beta^2 = \int_0^\infty (x - \alpha)^2\, dH(x), \tag{5}$$

$$\tau = \int_0^\infty x\, dU(x), \qquad \rho^2 = \int_0^\infty (x - \tau)^2\, dU(x), \tag{6}$$

$$A = \int_0^\infty x\, dG(x), \qquad B^2 = \int_0^\infty (x - A)^2\, dG(x). \tag{7}$$

By (1) we clearly have that $A = \tau + (1/\lambda)$ and $B^2 = \rho^2 + (1/\lambda^2)$.

Denote by $P(t)$ the probability that at the instant t the system is in state A, and put

$$\pi(s) = \int_0^\infty e^{-st} P(t)\, dt. \tag{8}$$

4. Theorems concerning ν_t. In what follows we shall give some general theorems for ν_t.

1. We have

$$\mathbf{P}\{\nu_t \leqq n\} = 1 - F(t) * G_n(t), \tag{9}$$

where $G_n(x)$ denotes the n-fold convolution of $G(x)$ with itself. ($G_0(x) = 1$ if $x \geqq 0$ and $G_0(x) = 0$ if $x < 0$). For

$$\mathbf{P}\{\nu_t \leqq n\} = \mathbf{P}\{t < \tau_{n+1}\} = 1 - \mathbf{P}\{\tau_{n+1} \leqq t\},$$

and $\tau_{n+1} = \tau_1 + (\tau_2 - \tau_1) + \cdots + (\tau_{n+1} - \tau_n)$ is a sum of independent random variables.

2. If $A < \infty$, then we have

$$\lim_{T\to\infty} \mathbf{P}\{\nu_{T+t} - \nu_T \leqq n\} = 1 - G^*(t) * G_n(t), \tag{10}$$

where

$$G^*(t) = \begin{cases} \dfrac{1}{A}\displaystyle\int_0^t [1 - G(u)]\, du & \text{if } t \geqq 0 \\ 0 & \text{if } t < 0. \end{cases}$$

The proof is similar to that of (9), only we must use the result

$$\lim_{T\to\infty} \mathbf{P}\{\nu_{T+t} - \nu_T \geqq 1\} = G^*(t),$$

which was proved by J. L. Doob [2].

3. If $B^2 < \infty$, then we have

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\nu_t - \frac{t}{A}}{\sqrt{\frac{B^2 t}{A^3}}} \leqq x\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du. \tag{11}$$

This can be proved by the aid of the method of W. Feller [3]. (Cf. [5]).

4. If $B^2 < \infty$, then we have

$$\mathbf{P}\left\{\limsup_{t\to\infty} \frac{\nu_t - \frac{t}{A}}{\sqrt{\frac{2B^2}{A^3}\, t \log\log t}} = 1\right\} = \mathbf{P}\left\{\liminf_{t\to\infty} \frac{\nu_t - \frac{t}{A}}{\sqrt{\frac{2B^2}{A^3}\, t \log\log t}} = -1\right\} = 1. \tag{12}$$

This can be proved by the aid of the law of the iterated logarithm stated by P. Hartman and A. Wintner [4].

5. Applying the strong law of large numbers we obtain

$$\mathbf{P}\left\{\lim_{t\to\infty} \frac{\nu_t}{t} = \frac{1}{A}\right\} = 1, \tag{13}$$

(cf. J. L. Doob [2]).

It is easy to see that $\mathbf{E}\{\nu_t\} = M(t)$ can be expressed as follows:

$$M(t) = \sum_{n=1}^{\infty} \mathbf{P}\{\tau_n \leqq t\}. \tag{14}$$

6. If $A < \infty$, then for any $h > 0$ we have

$$\lim_{t\to\infty} \frac{M(t+h) - M(t)}{h} = \frac{1}{A}, \tag{15}$$

by the theorem of J. L. Doob [2].

7. If $B^2 < \infty$, then we have

$$\int_0^\infty e^{-st}\, dM(t) = \frac{1}{As} + \frac{B^2 + A^2}{2A^2} - \frac{1}{\lambda A} + o(s) \tag{16}$$

if $s \to 0$. For by (9) and (14) we have

$$\int_0^\infty e^{-st}\, dM(t) = \frac{\lambda}{(\lambda + s)[1 - \gamma(s)]} \tag{17}$$

and

$$\gamma(s) = 1 - sA + \frac{s^2}{2}(B^2 + A^2) + o(s^2)$$

if $s \to 0$.

8. For the Laplace-transform of $P(t)$ we have

$$\pi(s) = \int_0^\infty e^{-st} P(t)\, dt = \frac{1}{(\lambda + s)[1 - \gamma(s)]}, \tag{18}$$

and

$$P = \lim_{t\to\infty} P(t) = \frac{1}{\lambda A}. \tag{19}$$

PROOF. As $M(t + \Delta t) = M(t) + P(t)\lambda\Delta t + o(\Delta t)$, we have $M'(t) = \lambda P(t)$, and thus (18) follows from (17). Now

$$P(t) = 1 - \int_0^t [1 - U(t - x)]\, dM(x), \tag{20}$$

for by the theorem of total probability we have

$$1 - P(t) = \sum_{n=1}^\infty \int_0^t [1 - U(t - x)]\, d\mathbf{P}\{\tau_n \leqq x\} = \int_0^t [1 - U(t - x)]\, dM(x),$$

which agrees with (20). By virtue of (15) we obtain from (20)

$$\lim_{t\to\infty} P(t) = 1 - \frac{1}{A}\int_0^\infty [1 - U(x)]\, dx = 1 - \frac{\tau}{A}.$$

Since $\tau = A - (1/\lambda)$, equation (19) follows.

REMARK. Taking into consideration that $M'(t) = \lambda P(t)$, we obtain from (20) the following integral equation for $P(t)$:

$$P(t) = 1 - \lambda \int_0^t [1 - U(t - x)] P(x)\, dx. \tag{21}$$

From (18) or from (21) we obtain that

$$\omega(s) = \int_0^\infty e^{-sx}\, dU(x) = \frac{\lambda + s}{\lambda}\left[1 - \frac{1}{(\lambda + s)\pi(s)}\right]. \tag{22}$$

To apply the above theorems it remains only to determine $G(x)$, A, and B^2.

5. The determination of $G(x)$, A, and B^2.

THEOREM. *If $0 < p \leqq 1$, then we have*

$$\gamma(s) = \int_0^\infty e^{-sx}\, dG(x) = \frac{\lambda p + s}{p(\lambda + s)} - \frac{1}{p(\lambda + s)}\left\{\int_0^\infty \exp\left[-st - \lambda p \int_0^t (1 - H(x))\, dx\right] dt\right\}^{-1}; \tag{23}$$

if $\alpha < \infty$, then

$$A = \frac{e^{\lambda p\alpha} + p - 1}{\lambda p}, \tag{24}$$

and if $\beta^2 < \infty$, then

$$B^2 = \frac{2e^{\lambda p\alpha}}{\lambda p}\int_0^\infty \left\{\exp\left[\lambda p \int_t^\infty (1 - H(x))\, dx\right] - 1\right\} dt + \frac{2e^{\lambda p\alpha} - e^{2\lambda p\alpha} + p^2 - 1}{(\lambda p)^2}. \tag{25}$$

If $p = 0$ then $U(x) = H(x)$ and consequently

$$G(x) = \int_0^x H(x - y)e^{-\lambda y}\lambda\, dy, \tag{26}$$

$$A = \frac{1 + \lambda\alpha}{\lambda}, \tag{27}$$

and

$$B^2 = \frac{1 + \lambda^2\beta^2}{\lambda^2}. \tag{28}$$

PROOF. Let us consider a new process which is a particular case of the process defined in the Introduction. Suppose that the density of the underlying Poisson-process is λ^* and each particle gives rise to an impulse (with probability $p^* = 1$). Let $H^*(x) = H(x)$ be the distribution function of the duration of the impulses. This is the case of Type II counter. Denote by $\{\xi_n^*\}$ and $\{\eta_n^*\}$ the sequences of the times spent in state A and B respectively. Clearly $\mathbf{P}\{\xi_n^* \leqq x\} = 1 - e^{-\lambda^* x}$ if $x \geqq 0$. Write $\mathbf{P}\{\eta_n^* \leqq x\} = U^*(x)$. Denote by $P^*(t)$ the probability that at the instant t there is no impulse present. We have showed in [5] that

$$P^*(t) = \exp\left[-\lambda^* \int_0^t [1 - H(x)]\, dx\right]. \tag{29}$$

Applying (22) it follows that

$$\omega^*(s) = \int_0^\infty e^{-sx}\, dU^*(x) = \frac{\lambda^* + s}{\lambda^*} - \frac{1}{\lambda^* \pi^*(s)}, \tag{30}$$

where

$$\pi^*(s) = \int_0^\infty e^{-st} P^*(t)\, dt = \int_0^\infty \exp\left\{-st - \lambda^* \int_0^t [1 - H(x)]\, dx\right\} dt. \tag{31}$$

Now we observe that, if in this new process $\lambda^* = \lambda p$, then we have

$$U^*(x) = U(x), \tag{32}$$

where $U(x)$ is related to the general process. The equality (32) can easily be seen if we take into consideration that the arrivals of those particles in the general process, which arrive during a dead time and which give rise to an impulse, form a Poisson process with density λp. Accordingly by (30) and (31) we have

$$\begin{aligned} \omega(s) = \int_0^\infty e^{-sx}\, dU(x) &= \frac{\lambda p + s}{\lambda p} \\ &- \left\{\lambda p \int_0^\infty \exp\left[-st - \lambda p \int_0^t (1 - H(x))\, dx\right] dt\right\}^{-1}, \end{aligned} \tag{33}$$

and by (4) we obtain (23), which was to be proved.

If we introduce for the new process the analogous quantities $M^*(t)$, A^* and B^{*2} corresponding to (7) and (14), then by (16) we obtain that

$$\int_0^\infty e^{-st}\, dM^*(t) = \lambda^* \int_0^\infty e^{-st}\, P^*(t)\, dt = \frac{1}{A^* s} + \frac{B^{*2} + A^{*2}}{2A^{*2}} - \frac{1}{\lambda^* A^*} + o(s) \tag{34}$$

if $s \to 0$. Since $P^* = \lim_{t\to\infty} P^*(t) = e^{-\lambda^* \alpha}$, we obtain from (34) that

$$A^* = e^{\lambda^* \alpha} / \lambda^* \tag{35}$$

and, further,

$$B^{*2} = 2\lambda^* A^{*2} \int_0^\infty [P^*(t) - P^*]\, dt - A^{*2} + 2A^*/\lambda^*. \tag{36}$$

If in particular $\lambda^* = \lambda p$, then clearly

$$A - \frac{1}{\lambda} = A^* - \frac{1}{\lambda^*}$$

and

$$B^2 - \frac{1}{\lambda^2} = B^{*2} - \frac{1}{\lambda^{*2}},$$

and thus (24) and (26) are proved. The case $p = 0$ is evident.

Finally we remark that the more general case when the arrivals of the particles to the counter form a recurrent process was dealt by the author [6], [7], [8], but explicit solution was given only for a particular distribution $H(x)$.

REFERENCES

[1] G. E. Albert and L. Nelson, "Contributions to the statistical theory of counters data, *Ann. Math. Stat.*, 24 (1953), 9–22.

[2] J. L. Doob, "Renewal theory from the point of view of the theory of probability," *Trans. Amer. Math. Soc.*, 63 (1948), pp. 422–438.

[3] W. Feller, "Fluctuation theory of recurrent events," *Trans. Amer. Math. Soc.*, 67 (1949), pp. 98–119.

[4] P. Hartman and W. Wintner, "On the law of the iterated logarithm," *Amer. J. Math.*, 63 (1941), pp. 169–176.

[5] L. Takács, "On a probability problem arising in the theory of counters," *Proc. Cambr. Phil. Soc.*, 52 (1956), pp. 488–498.

[6] L. Takács, "On stochastic processes in the theory of counters," (Hungarian), *Magy. Tud. Akad.* III. *Oszt. Közl.*, 6 (1956), pp. 369–421.

[7] L. Takács, "On the sequence of events, selected by a counter from a recurrent process of events," *The Theory of Probability and Its Applications* (Moscow) 1 (1956), pp. 90–102.

[8] L. Takács, "On some probability problems concerning the theory of counters," *Acta Math. Acad. Sci. Hung.*, 8 (1957), pp. 127–138.