ON THE KOLMOGOROV AND SMIRNOV LIMIT THEOREMS FOR DISCONTINUOUS DISTRIBUTION FUNCTIONS

By Paul Schmid

Swiss Forest Research Institute and Federal Institute of Technology

1. Introduction. Let $X_1, X_2, \ldots, X_N$ be N independent random variables with the same distribution function $F(x)$. $S_N(x)$ is the empirical distribution function, i.e., $S_N(x) = k/N$ if exactly k of the N values X_i are less than or equal to x. It is of theoretical and practical interest to analyze the behavior of the statistics

$$\sup_{-\infty<x<\infty} |S_N(x) - F(x)| \cdot N^{\frac{1}{2}}$$

and

$$\sup_{-\infty<x<\infty} (S_N(x) - F(x)) \cdot N^{\frac{1}{2}}.$$

Kolmogorov [12] proved in a famous paper in 1933 that for $\lambda > 0$

$$\text{I} \qquad \lim_{N\to\infty} P[\sup_{-\infty<x<\infty} |S_N(x) - F(x)| \cdot N^{\frac{1}{2}} < \lambda] = \sum_{k=-\infty}^{+\infty} (-1)^k e^{-2\lambda^2 k^2}$$

if $F(x)$ is a continuous distribution function. Smirnov [21] obtained a similar result in 1939, when he showed that

$$\text{II} \qquad \lim_{N\to\infty} P[\sup_{-\infty<x<\infty} (S_N(x) - F(x)) \cdot N^{\frac{1}{2}} < \lambda] = 1 - e^{-2\lambda^2}$$

holds for continuous distribution functions $F(x)$.

Kolmogorov converts in his proof to a generalization of the Central Limit theorem, whereas Smirnov's theorem was a corollary to a more intricate theorem. But the two formulae can be proved by reciprocal methods. They have also been proved by Feller [11] and by Doob [10] and Donsker [9]. Feller made use of characteristic functions and Doob employed stochastic processes. Smirnov [22] found in 1944 the first terms of the asymptotic expansion for the probability in II and an exact formula for finite N. Chung [7] and Blackman [5], [6] were successful in finding the asymptotic expansion for the probability in I.

A somewhat more general form of the statistics, namely

$$\sup_{-\infty<x<\infty} |S_N(x) - F(x)| N^{\frac{1}{2}} \cdot \varphi(F(x)),$$

where $\varphi(y)$ is a positive definite weight function, was discussed by Anderson and Darling [1]. They found the limit distributions for some special weight functions,

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by means of stochastic processes. Similar results were obtained by Maniya [16] and Malmquist [15]. Rényi [19], in 1953, established the relations

$$\text{III} \quad \lim_{N\to\infty} P\left[\sup_{a\leqq F(x)} \left|\frac{S_N(x) - F(x)}{F(x)}\right| \cdot N^{\frac{1}{2}} < \lambda\right] = \frac{4}{\pi}\sum_{k=0}^{\infty}(-1)^k \frac{\exp\left[-\frac{(2k+1)^2\pi^2}{8}\frac{1-a}{a\lambda^2}\right]}{2k+1}$$

and

$$\text{IV} \quad \lim_{N\to\infty} P\left[\sup_{a\leqq F(x)} \left(\frac{S_N(x) - F(x)}{F(x)}\right) N^{\frac{1}{2}} < \lambda\right] = \sqrt{\frac{2}{\pi}}\int_0^{\lambda[a/(1-a)]^{\frac{1}{2}}} e^{-t^2/2}\, dt,$$

where $F(x)$ is a continuous distribution function, $a > 0$, $\lambda > 0$.

The statistics treated here are well suited to test if a sample comes from a population with the distribution function $F(x)$. These test functions have the great advantage in that their distributions are independent of the distribution $F(x)$ of the population. Massey [17], Birnbaum [2], and Malmquist [15] investigated the power of the statistics of Kolmogorov and Smirnov. The limit distributions of these statistics have been tabulated by Smirnov [23], and the distribution for finite N by Massey [18], Birnbaum and Tingey [3], [4]. Rényi tabulated his own limit distributions. Hence, today it is practicable to use these statistics.

In this paper Theorems I through IV are extended for the case of discontinuous distribution functions $F(x)$. The probabilities in question converge also in this case, but the limit distributions are no longer independent of $F(x)$. They depend on the values of $F(x)$ at the discontinuity points, but not on the form of the function between the points of discontinuity. Theorems 1 and 2 are proved by a generalization of the method of Kolmogorov. They can also be proved with the help of stochastic processes, as Doob did it for the case of continuous $F(x)$. We bypass representation of this method since it involves techniques similar to those of Anderson and Darling. The proofs of Theorems 3 and 4 follow in part the methods applied by Rényi, but also make use of the generalization by Kolmogorov of the Central Limit theorem. A part of these results has already been published [20].

I should like to thank W. Saxer for suggesting this topic.

2. Extension of the limit theorems of Kolmogorov and Smirnov. Let $F(x)$ be a distribution function continuous for $x \neq x_\nu$, where $F(x_\nu - 0) = f_{2\nu-1}$, $F(x_\nu) = f_{2\nu}$, for $\nu = 1, 2, \cdots, n$, and $f_{2n+1} = 1$. Denote the corresponding empirical distribution function by $S_N(x)$.

THEOREM 1. *If* $\lambda > 0$, *then*

$$(1) \qquad \lim_{N\to\infty} P\left[\sup_{-\infty<x<\infty} |S_N(x) - F(x)| < \lambda N^{-\frac{1}{2}}\right] = \Phi(\lambda),$$

$$(2) \qquad \Phi(\lambda) = \sum_{k=-\infty}^{+\infty}(-1)^k e^{-2\lambda^2k^2}\, c\int\cdots\int_{G_k} \exp\left[-\frac{1}{2}\sum_{i,j=1}^{2n}\Lambda_{ij}\, x_i\, x_j\right] dx_1 \cdots dx_{2n},$$

where

$$\Lambda_{jj} = \frac{f_{j+1} - f_{j-1}}{(f_{j+1} - f_j)(f_j - f_{j-1})}, \qquad \Lambda_{jj-1} = \Lambda_{j-1j} = \frac{-1}{f_j - f_{j-1}},$$

$$\Lambda_{ij} = 0, \qquad \text{for } i < j - 1 \text{ or } i > j + 1,$$

$$c = (2\pi)^{-n} \prod_{j=1}^{2n+1} (f_j - f_{j-1})^{-\frac{1}{2}}$$

and

$$G_k = \bigcup_{p_1, p_2, \cdots, p_n = -\infty}^{+\infty} \{-\lambda < x_{2\nu-1} + 2\lambda(p_\nu + kf_{2\nu-1}) < \lambda,$$
$$-\lambda < x_{2\nu} + 2\lambda(p_\nu + kf_{2\nu}) < \lambda, \qquad \nu = 1, \cdots, n\}.$$

THEOREM 2. *If* $\lambda > 0$, *then*

$$\lim_{N\to\infty} P[\sup_{-\infty<x<\infty} (S_N(x) - F(x)) < \lambda N^{-\frac{1}{2}}] = \Phi^+(\lambda), \tag{3}$$

$$\lim_{N\to\infty} P[\sup_{-\infty<x<\infty} (F(x) - S_N(x)) < \lambda N^{-\frac{1}{2}}] = \Phi^+(\lambda), \tag{4}$$

$$\Phi^+(\lambda) = \sum_{k=0}^{1} (-1)^k e^{-2\lambda^2 k^2} c \int \cdots \int_{G_k^+} \exp\left[-\frac{1}{2} \sum_{i,j=1}^{2n} \Lambda_{ij}\, x_i\, x_j\right] dx_1 \cdots dx_{2n}, \tag{5}$$

where

$$G_k^+ = \bigcup_{p_1, \cdots, p_n = 0}^{1} \{-\infty < (-1)^{p_\nu}(x_{2\nu-1} + 2\lambda k \cdot f_{2\nu-1}) + 2\lambda p_\nu < \lambda,$$
$$-\infty < (-1)^{p_\nu}(x_{2\nu} + 2\lambda k f_{2\nu}) + 2\lambda p_\nu < \lambda, \qquad \nu = 1, \cdots, n\}.$$

For $\lambda \leqq 0$ all limits are 0. The convergence is uniform in λ in all cases.

If the number of jumps of $F(x)$ is countably infinite, a further limit process has to be made in which at first only the highest jumps of $F(x)$ are taken into account. The two limit processes can be interchanged, because $\Phi(\lambda)$ and $\Phi^+(\lambda)$ are continuous functions of the values of $F(x)$ at the points of discontinuity. Hence further difficulties do not arise in this, the most general case. We will prove Theorem 1 for the case of a distribution function for which the inequalities

$$f_{2\nu+1} > f_{2\nu}, \qquad \nu = 0, 1, \cdots, n,$$

are valid. The results must then hold for any distribution function with n jumps, because both sides of (1) depend continuously on the f's.

If the random variable X has the distribution function $F(x)$, then $Y = F(X)$ is also a random variable, the distribution of which has to fulfill

$$P[F(X) \leqq 0] = 0, \qquad P[F(X) \geqq 1] = 0, \qquad P[f_{2\nu-1} \leqq F(X) < f_{2\nu}] = 0$$

and, for $f_{2\nu} \leqq y \leqq f_{2\nu+1}$,

$$P[F(X) \leqq y] = P[X \leqq F^{-1}(y)] = F(F^{-1}(y)) = y.$$

Furthermore, since

$$P[F(X) = f_{2\nu}] = P[X = x_\nu] = f_{2\nu} - f_{2\nu-1},$$

Y will have the distribution function

$$F^0(y) = \begin{cases} 0, & \text{for } y \leqq 0, \\ y, & \text{for } f_{2\nu} \leqq y \leqq f_{2\nu+1}, \quad \nu = 0, 1, \cdots, n, \\ f_{2\nu-1}, & \text{for } f_{2\nu-1} \leqq y < f_{2\nu}, \quad \nu = 1, 2, \cdots, n, \\ 1, & \text{for } y \geqq 1. \end{cases} \tag{6}$$

Let $S_N^0(y)$ be the empirical distribution function corresponding to $F^0(y)$. Then we have, for $f_{2\nu} \leqq F(x) \leqq f_{2\nu+1}$,

$$\begin{aligned} S_N^0(F(x)) &= \frac{1}{N} \text{ (Number of } F(X_i), F(X_i) \leqq F(x)) \\ &= \frac{1}{N} \text{ (Number of } X_i, X_i \leqq x) = S_N(x) \end{aligned}$$

and $F^0(F(x)) = F(x)$. Hence

$$\sup_{-\infty < x < \infty} |S_N(x) - F(x)| = \sup_{-\infty < x < \infty} |S_N^0(x) - F^0(x)|,$$

because the other values of $F(x)$ cannot be attained. If we denote by I the union of the closed intervals $[f_{2\nu}, f_{2\nu+1}]$, $\nu = 0, 1, \cdots, n$, we obtain

$$P[\sup_{-\infty < x < \infty} |S_N(x) - F(x)| < \lambda N^{-\frac{1}{2}}] = P[\sup_{x \varepsilon I} | S_N^0(x) - x | < \lambda N^{-\frac{1}{2}}]. \tag{7}$$

Denote by M_N the set of integers j such that $j/N \varepsilon I$,

$$M_N = \{k_0 = 0, 1, \cdots, k_1; k_2, k_2 + 1, \cdots, k_3; \cdots; k_{2n}, k_{2n} + 1, \cdots, k_{2n+1} = N\}. \tag{8}$$

The k_i are defined such that $k_i/N \to f_i$, as $N \to \infty$. We wish to analyze the behavior of

$$P\left[\max_{j \varepsilon M_N} \left| S_N^0\left(\frac{j}{N}\right) - \frac{j}{N} \right| < \lambda N^{-\frac{1}{2}}\right] \tag{9}$$

when $N \to \infty$.

The event $\mathcal{E}_{ik}$, $k \varepsilon M_N$, happens if simultaneously all inequalities

$$\left| S_N^0\left(\frac{j}{N}\right) - \frac{j}{N} \right| < \lambda N^{-\frac{1}{2}}, \qquad \text{for } j \leqq k, \ j \varepsilon M_N,$$

and the equality

$$S_N^0\left(\frac{k}{N}\right) - \frac{k}{N} = \frac{i}{N}$$

are fulfilled. P_{ik} is the probability of $\mathcal{E}_{ik}$. P_{0N} is equal to the probability in (9).

We can calculate the P_{ik} recursively by means of the initial conditions $P_{00} = 1$, $P_{i0} = 0$ for $i \neq 0$, and the equations

$$
\begin{aligned}
P_{i\,k+1} &= \sum_j P_{jk}\, P[\mathcal{E}_{i\,k+1} \mid \mathcal{E}_{jk}] \\
&= \sum_{|j|<\lambda N^{\frac{1}{2}}} P_{jk}\, P\left[S_N^0\left(\frac{k+1}{N}\right) - S_N^0\left(\frac{k}{N}\right) \right. \\
&\qquad \left. = \frac{i-j+1}{N} \,\middle|\, S_N^0\left(\frac{k}{N}\right) = \frac{k+j}{N}\right],
\end{aligned} \tag{10}
$$

for $k \,\varepsilon\, M_N$, $k + 1 \,\varepsilon\, M_N$, and

$$
\begin{aligned}
P_{i\,k_{2\nu}} &= \sum_j P_{j\,k_{2\nu-1}}\, P[\mathcal{E}_{i\,k_{2\nu}} \mid \mathcal{E}_{i\,k_{2\nu-1}}] \\
&= \sum_{|j|<\lambda N^{\frac{1}{2}}} P_{j\,k_{2\nu-1}}\, P\left[S_N^0\left(\frac{k_{2\nu}}{N}\right) - S_N^0\left(\frac{k_{2\nu-1}}{N}\right) \right. \\
&\qquad \left. = \frac{i-j+k_{2\nu}-k_{2\nu-1}}{N} \,\middle|\, S_N^0\left(\frac{k_{2\nu-1}}{N}\right) = \frac{j+k_{2\nu-1}}{N}\right],
\end{aligned} \tag{11}
$$

for $\nu = 1, \cdots, n$.

The occurring conditional probabilities give

$$
\begin{aligned}
&P\left[S_N^0\left(\frac{k+1}{N}\right) - S_N^0\left(\frac{k}{N}\right) = \frac{i-j+1}{N} \,\middle|\, S_N^0\left(\frac{k}{N}\right) = \frac{k+j}{N}\right] \\
&\qquad = \binom{N-k-j}{i-j+1}\left(\frac{1}{N-k}\right)^{i-j+1}\left(\frac{N-k-1}{N-k}\right)^{N-k-i-1},
\end{aligned} \tag{12}
$$

for $k \,\varepsilon\, M_N$, $k + 1 \,\varepsilon\, M_N$, and

$$
\begin{aligned}
&P\left[S_N^0\left(\frac{k_{2\nu}}{N}\right) - S_N^0\left(\frac{k_{2\nu-1}}{N}\right) = \frac{i-j+k_{2\nu}-k_{2\nu-1}}{N} \,\middle|\, S_N^0\left(\frac{k_{2\nu-1}}{N}\right) = \frac{k_{2\nu-1}+j}{N}\right] \\
&\quad = \binom{N-k_{2\nu-1}-j}{i-j+k_{2\nu}-k_{2\nu-1}}\left(\frac{k_{2\nu}-k_{2\nu-1}}{N-k_{2\nu-1}}\right)^{i-j+k_{2\nu}-k_{2\nu-1}}\left(\frac{N-k_{2\nu}}{N-k_{2\nu-1}}\right)^{N-k_{2\nu}-i},
\end{aligned} \tag{13}
$$

for $\nu = 1, \cdots, n$, according to the laws of the binomial distribution.

The recursion formulae can be simplified if we introduce the new terms

$$
Q_{ik} = \frac{N^N(N-k-i)!}{N!(N-k)^{N-k-i}e^k} P_{ik}. \tag{14}
$$

Now we have

$$
Q_{00} = 1; \quad Q_{i0} = 0, \quad \text{for } i \neq 0; \quad Q_{ik} = 0, \quad \text{for } |i| \geqq \lambda N^{\frac{1}{2}};
$$

$$
Q_{i\,k+1} = \sum_{|j|<\lambda N^{\frac{1}{2}}} Q_{jk} \frac{1}{(i-j+1)!} e^{-1},
$$

for $k \,\varepsilon\, M_N$ and $k + 1 \,\varepsilon\, M_N$; $|i| < \lambda N^{\frac{1}{2}}$,

$$
Q_{i\,k_{2\nu}} = \sum_{|j|<\lambda N^{\frac{1}{2}}} Q_{j\,k_{2\nu-1}} \frac{(k_{2\nu}-k_{2\nu-1})^{i-k+k_{2\nu}-k_{2\nu-1}}}{(i-j+k_{2\nu}-k_{2\nu-1})!\,e^{k_{2\nu}-k_{2\nu-1}}}, \tag{15}
$$

for $\nu = 1, \cdots, n$, and for the probability (9) we obtain

$$P_{0N} = \frac{N!e^N}{N^N} Q_{0N}. \tag{16}$$

For finite N, P_{0N} is evaluable, but as $N \to \infty$ the number of necessary recursion steps tends to infinity.

Let Y_j, $j \in M_N$, be independent random variables with the distributions

$$P\left[Y_j = \frac{i-1}{\lambda N^{\frac{1}{2}}}\right] = \frac{1}{i!e}, \qquad i = 0, 1, 2, \cdots; j \neq k_{2\nu}, \tag{17}$$

$$P\left[Y_{k_{2\nu}} = \frac{i - k_{2\nu} + k_{2\nu-1}}{\lambda N^{\frac{1}{2}}}\right] = \frac{(k_{2\nu} - k_{2\nu-1})^i}{i!\, e^{k_{2\nu}-k_{2\nu-1}}}, \qquad i = 0, 1, 2, \cdots. \tag{18}$$

Then

$$E(Y_j) = 0,$$

$$E(Y_j^2) = \frac{1}{\lambda^2 N}, \quad j \neq k_{2\nu}; \qquad E(Y_{k_{2\nu}}^2) = \frac{k_{2\nu} - k_{2\nu-1}}{\lambda^2 N},$$

$$E(|Y_j|^3) = \left(1 + \frac{2}{e}\right)\frac{1}{\lambda^3 N^{\frac{3}{2}}}, \quad j \neq k_{2\nu}; \qquad E(|Y_{k_{2\nu}}|^3) \sim \sqrt{\frac{8}{\pi}}\,\frac{(k_{2\nu} - k_{2\nu-1})^{\frac{3}{2}}}{\lambda^3 N^{\frac{3}{2}}}.$$

The event $\mathfrak{D}_{ik}$, $k \in M_N$, take place if the inequalities

$$\left|\sum_{l \leq j} Y_l\right| < 1$$

for all $j \leq k$ and the equality

$$\sum_{l \leq k} Y_l = \frac{i}{\lambda N^{\frac{1}{2}}}$$

are simultaneously fulfilled. The probability of $\mathfrak{D}_{ik}$ is R_{ik}, $R_{00} = 1$, $R_{i0} = 0$ for $i \neq 0$.

We can easily verify that the recursion formulae for the R_{ik} are the same as for the Q_{ik}. Therefore,

$$R_{ik} = Q_{ik}, \tag{19}$$

for all i and k. For the probability in (9) we obtain

$$P_{0N} = \frac{N!e^N}{N^N} R_{0N}. \tag{20}$$

We can evaluate R_{0N} in $2n + 1$ recursion steps from

$$R_{00} = 1, \qquad R_{i0} = 0, \qquad \text{for } i \neq 0,$$

$$\begin{aligned} R_{ik_{2\nu+1}} &= \sum_{|j| < \lambda N^{\frac{1}{2}}} R_{jk_{2\nu}} P[\mathfrak{D}_{ik_{2\nu+1}} \mid \mathfrak{D}_{jk_{2\nu}}], & \nu = 0, \cdots, n, \\ R_{ik_{2\nu}} &= \sum_{|j| < \lambda N^{\frac{1}{2}}} R_{jk_{2\nu-1}} \frac{(k_{2\nu} - k_{2\nu-1})^{i-j+k_{2\nu}-k_{2\nu-1}}}{(i - j + k_{2\nu} - k_{2\nu-j})!\, e^{k_{2\nu}-k_{2\nu-1}}}, & \nu = 1, \cdots, n. \end{aligned} \tag{21}$$

These conditional probabilities can be written in the form

$$P[\mathfrak{D}_{ik_{2\nu+1}} \mid \mathfrak{D}_{jk_{2\nu}}] = P\left[-1 - \frac{j}{\lambda N^{\frac{1}{2}}} < \sum_{r=k_{2\nu}+1}^{l} Y_r < 1 - \frac{j}{\lambda N^{\frac{1}{2}}}, \quad l = k_{2\nu}+1, \cdots, k_{2\nu+1}; \sum_{r=k_{2\nu}+1}^{k_{2\nu+1}} Y_r = \frac{i-j}{\lambda N^{\frac{1}{2}}}\right], \tag{22}$$

and their limits for $N \to \infty$ can be obtained by the following lemma of Kolmogorov.

LEMMA, [12]. *Let $Y_{M1}, \cdots, Y_{Mm_M}$ be, for each M, independent random variables, whose values are multiples of $\epsilon = \epsilon(M)$, with*

$$E(Y_{Mj}) = 0, \qquad E(Y_{Mj}^2) = 2b_{Mj}, \qquad E(|Y_{Mj}|^3) = d_{Mj}.$$

Let a and b be two numbers such that $a < 0$ and $b > 0$. Assume the existence of positive numbers $A, \cdots, E$, such that, for all M, the inequalities (i) *through* (iv) *are fulfilled:*

(i) $A < \sum_{j=1}^{m_M} b_{Mj} < B$,

(ii) $\dfrac{d_{Mj}}{b_{Mj}} < C\epsilon$, for all j,

(iii) $P[Y_{Mj} = l_{Mj}\epsilon] > D$ and $P[Y_{Mj} = (l_{Mj}+1)\epsilon] > D$ for all j and suitably chosen l_{Mj},

(iv) $a + E < i_M\epsilon < b - E$.

Then

$$P\left[a < \sum_{k=1}^{j} Y_{Mk} < b, j = 1, 2, \cdots, m_M; \sum_{k=1}^{m_M} Y_{Mk} = i_M\epsilon\right] = \epsilon\left(u\left(0, 0, i_M\epsilon, 2\sum_{k=1}^{m_M} b_{Mk}\right) + \Delta\right),$$

where $u(\sigma, \tau, s, t)$ is Green's function for the heat equation

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial s^2}$$

in the region G,

$$G = \{a < s < b, t > 0\}.$$

If $\epsilon(M) \to 0$, then $\Delta \to 0$.

This lemma can be applied to the random variables $Y_{k_{2\nu}+1}, Y_{k_{2\nu}+2}, \cdots, Y_{k_{2\nu+1}}$. It should be noticed that the variables $Y_{k_{2\nu}}$ do not fulfill condition (ii) and hence must be treated independently. Our recursion formulae are now

$$R_{00} = 1, \qquad R_{i0} = 0, \qquad i \neq 0,$$

$$R_{ik_{2\nu+1}} = \sum_{|j| < \lambda N^{\frac{1}{2}}} R_{jk_{2\nu}} \frac{1}{\lambda N^{\frac{1}{2}}}\left(u_j\left(0, 0; \frac{i-j}{\lambda N^{\frac{1}{2}}}, \frac{k_{2\nu+1} - k_{2\nu}}{2\lambda^2 N}\right) + \Delta\right), \qquad \nu = 0, \cdots, n, \tag{23}$$

$$R_{ik_{2\nu}} = \sum_{|j| < \lambda N^{\frac{1}{2}}} R_{jk_{2\nu-1}} \frac{(k_{2\nu} - k_{2\nu-1})^{i-j+k_{2\nu}-k_{2\nu-1}}}{(i - j + k_{2\nu} - k_{2\nu-1})!\, e^{k_{2\nu}-k_{2\nu-1}}}, \qquad \nu = 1, \cdots, n,$$

where $u_j(\sigma, \tau; s, t)$ is Green's function for the heat equation in the region G_j,

$$G_j = \left\{-1 - \frac{j}{\lambda N^{\frac{1}{2}}} < s < 1 - \frac{j}{\lambda N^{\frac{1}{2}}}, t > 0\right\},$$

or

$$u_j(\sigma, \tau; s, t) = \frac{1}{2\sqrt{\pi(t-\tau)}} \sum_{l=-\infty}^{+\infty} (-1)^l \cdot \exp\left[- \frac{\left(s + \frac{j}{\lambda N^{\frac{1}{2}}} - (-1)^l \left(\sigma + \frac{j}{\lambda N^{\frac{1}{2}}}\right) - 2l\right)^2}{4(t-\tau)}\right]. \tag{24}$$

If N tends to infinity, the Δ's disappear and the sums over j go over into integrals, with the exception of the sum in the first step which consists of only one summand,

$$R_{jk_1} = \frac{1}{\lambda N^{\frac{1}{2}}}\left(u\left(0, 0; \frac{j}{\lambda N^{\frac{1}{2}}}, \frac{k_1}{2\lambda^2 N}\right) + \Delta\right).$$

With this exception all sums tend to finite positive limits. The factor in (20), multiplied by $N^{-\frac{1}{2}}$ also tends to a finite limit, namely

$$N^{-\frac{1}{2}} \frac{N! e^N}{N^N} \sim \sqrt{2\pi}.$$

For $r \cdot N^{-\frac{1}{2}} \rightarrow x$, we obtain

$$\frac{N^{\frac{1}{2}}(k_{2\nu} - k_{2\nu-1})^{r+k_{2\nu}-k_{2\nu-1}}}{(r + k_{2\nu} - k_{2\nu-1})! \, e^{k_{2\nu}-k_{2\nu-1}}} \sim \frac{1}{\sqrt{2\pi(f_{2\nu} - f_{2\nu-1})}} \exp\left[-\frac{1}{2}\frac{x^2}{f_{2\nu} - f_{2\nu-1}}\right].$$

Finally we have

$$\begin{aligned} \lim_{N\to\infty} P\left[\max_{j\varepsilon M_N}\left|S_N^0\left(\frac{j}{N}\right) - \frac{j}{N}\right| < \lambda N^{-\frac{1}{2}}\right] &= \sum_{j_0, j_1, \cdots, j_n = -\infty}^{+\infty} (-1)^{\sum_{i=0}^{n} j_i}(2\pi)^{-n} \prod_{j=1}^{2n+1} (f_j - f_{j-1})^{-\frac{1}{2}} \\ &\quad \cdot \int_{-\lambda < x_j < \lambda} \cdots \int \exp\left[-\frac{1}{2}\sum_{\nu=1}^{n} \frac{(x_{2\nu} - x_{2\nu-1})^2}{f_{2\nu} - f_{2\nu-1}}\right. \\ &\quad \left. - \frac{1}{2}\sum_{\nu=0}^{n} \frac{(x_{2\nu+1} - (-1)^{j_\nu} x_{2\nu} - 2\lambda j_\nu)^2}{f_{2\nu+1} - f_{2\nu}}\right] dx_1 \cdots dx_{2n}, \end{aligned} \tag{25}$$

where x_0 and x_{2n+1} should be replaced by 0. This expression is $\Phi(\lambda)$.

Let us now prove that for those values of λ and sequences of N for which $\lambda N^{\frac{1}{2}}$ are integers,

$$\lim_{N\to\infty} P[\sup_{x\varepsilon I} | S_N^0(x) - x | < \lambda N^{-\frac{1}{2}}] = \lim_{N\to\infty} P\left[\max_{j\varepsilon M_N}\left|S_N^0\left(\frac{j}{N}\right) - \frac{j}{N}\right| < \lambda N^{-\frac{1}{2}}\right] \tag{26}$$

must be true.

To each $x \varepsilon I$ there exists a $j \varepsilon M_N$ such that either $x = j/N$ or $x = j/N + \epsilon$ with $0 < \epsilon < 1/N$. Set $S_N^0(x) = i/N$. From

$$S_N^0(x) - x = \frac{i-j}{N} - \epsilon \geqq \frac{\lambda N^{\frac{1}{2}}}{N}$$

follows, for $\epsilon > 0$,

$$S_N^0\left(\frac{j+1}{N}\right) - \frac{j+1}{N} \geqq S_N^0(x) - \frac{j+1}{N} = \frac{i-j-1}{N} \geqq \frac{\lambda N^{\frac{1}{2}}}{N},$$

because the value to the right is a multiple of $1/N$. From

$$S_N^0(x) - x = \frac{i-j}{N} - \epsilon \leqq -\frac{\lambda N^{\frac{1}{2}}}{N}$$

follows analogously

$$S_N^0\left(\frac{j}{N}\right) - \frac{j}{N} \leqq S_N^0(x) - \frac{j}{N} = \frac{i-j}{N} \leqq -\frac{\lambda N^{\frac{1}{2}}}{N}.$$

The second probability in (26) cannot be smaller than the first one and the limit of the second probability depends continuously on the endpoints of the intervals of I. Therefore the two limits have to be equal. The convergence must be uniform in λ, since $\Phi(\lambda)$ is a bounded and continuous function. Hence

$$P\,[\sup_{x \varepsilon I} | S_N^0(x) - x | < \lambda N^{-\frac{1}{2}}]$$

tends to $\Phi(\lambda)$ for all λ and all sequences of N. In view of (7), this proves Theorem 1.

Theorem 2 can be proved in a similar manner. We now disregard the absolute value signs in the definition of $\mathcal{E}_{ik}$ and $\mathfrak{D}_{ik}$. The summations in (10), (11), (15), (21) and (23) go from $-\infty$ to $\lambda N^{\frac{1}{2}}$ and the lower boundaries for the partial sums in (22) are omitted. Green's function for the heat equation in the region G_j^+,

$$G_j^+ = \left\{s < 1 - \frac{j}{\lambda N^{\frac{1}{2}}},\, t > 0\right\},$$

is now

$$u_j^+(\sigma, \tau; s, t) = \frac{1}{2\sqrt{\pi(t-\tau)}} \sum_{l=0}^{1} (-1)^l \cdot \exp\left[\frac{-\left(s + \frac{j}{\lambda N^{\frac{1}{2}}} - (-1)^l\left(\sigma + \frac{j}{\lambda N^{\frac{1}{2}}}\right) - 2l\right)^2}{4(t-\tau)}\right].$$

Hence

$$\lim_{N\to\infty} P\left[\max_{j\varepsilon M_N}\left(S_N^0\left(\frac{j}{N}\right) - \frac{j}{N}\right) < \lambda N^{-\frac{1}{2}}\right]$$
$$= (2\pi)^{-n} \prod_{l=1}^{2n+1} (f_l - f_{l-1})^{-\frac{1}{2}} \sum_{j_0, j_1, \cdots, j_n = 0}^{1} (-1)^{\sum_{i=0}^{n} j_i}$$

$$\cdot \int_{x_j<\lambda} \cdots \int \exp\left[-\frac{1}{2} \sum_{\nu=1}^{n} \frac{(x_{2\nu} - x_{2\nu-1})^2}{f_{2\nu} - f_{2\nu-1}} - \frac{1}{2} \sum_{\nu=0}^{n} \frac{(x_{2\nu+1} - (-1)^{j_\nu} x_{2\nu} - 2\lambda j_\nu)^2}{f_{2\nu+1} - f_{2\nu}} \right] dx_1 \cdots dx_{2n},$$

where again x_0 and x_{2n+1} are 0. This proves Theorem 2.

3. Extension of the limit theorems of Rényi. Let $F(x)$ be a continuous function for $x \neq x_\nu$, with $F(x_\nu - 0) = f_{2\nu-1}$ and $F(x_\nu) = f_{2\nu}$, for $\nu = 1, 2, \cdots, n$, and $f_{2n+1} = 1$. Let f_0 be a positive number such that $f_0 \leqq f_1$. If $f_0 > f_1$, then we get the same results except that only the $f_i \geqq f_0$ will appear. Denote the empirical distribution function by $S_N(x)$.

THEOREM 3. *If* $\lambda > 0$, *then*

$$\lim_{N\to\infty} P\left[\sup_{f_0 \leqq F(x)} \left| \frac{S_N(x) - F(x)}{F(x)} \right| < \lambda N^{-\frac{1}{2}} \right] = \Psi(\lambda), \tag{27}$$

$$\Psi(\lambda) = \sum_{k=-\infty}^{+\infty} (-1)^k d \int \cdots \int_{H_k} \exp\left[-\frac{1}{2} \sum_{i,j=0}^{2n} \Lambda_{ij} x_i x_j \right] dx_0 \cdots dx_{2n}, \tag{28}$$

where

$$\Lambda_{jj} = \frac{(f_{j+1} - f_{j-1}) f_j^2}{(f_{j+1} - f_j)(f_j - f_{j-1})}, \quad \Lambda_{j-1\,j} = \Lambda_{jj-1} = \frac{-f_j f_{j-1}}{(f_j - f_{j-1})},$$

$$\Lambda_{ij} = 0, \qquad \text{for } i < j - 1 \quad \text{or} \quad i > j + 1,$$

$$d = (2\pi)^{-n-\frac{1}{2}} \prod_{j=0}^{2n} (f_{j+1} - f_j)^{-\frac{1}{2}} (f_{j+1}^{\frac{1}{2}} f_j^{\frac{1}{2}}),$$

and

$$H_k = \bigcup_{p_1, \cdots, p_n = -\infty}^{+\infty} \{ -\lambda < (-1)^k x_0 + 2\lambda k < \lambda;\ -\lambda < (-1)^{p_\nu} x_{2\nu-1} + 2\lambda p_\nu < \lambda,$$
$$-\lambda < (-1)^{p_\nu} x_{2\nu} + 2\lambda p_\nu < \lambda, \nu = 1, \cdots, n \}.$$

THEOREM 4. *If* $\lambda > 0$, *then*

$$\lim_{N\to\infty} P\left[\sup_{f_0 \leqq F(x)} \frac{S_N(x) - F(x)}{F(x)} < \lambda N^{-\frac{1}{2}} \right] = \Psi^+(\lambda), \tag{29}$$

$$\lim_{N\to\infty} P\left[\sup_{f_0 \leqq F(x)} \frac{F(x) - S_N(x)}{F(x)} < \lambda N^{-\frac{1}{2}} \right] = \Psi^+(\lambda), \tag{30}$$

$$\Psi^+(\lambda) = \sum_{k=0}^{1} (-1)^k d \int_{H_k^+} \cdots \int \exp\left[-\frac{1}{2} \sum_{i,j} \Lambda_{ij} x_i x_j \right]_{dx_0 \cdots dx_{2n}}, \tag{31}$$

where

$$H_k^+ = \bigcup_{p_1, \cdots, p_n = 0}^{1} \{ -\infty < (-1)^k x_0 + 2\lambda k < \lambda;$$
$$-\infty < (-1)^{p_\nu} x_{2\nu-1} + 2\lambda p_\nu < \lambda, -\infty < (-1)^{p_\nu} x_{2\nu} + 2\lambda p_\nu < \lambda, \nu = 1, \cdots, n \}.$$

The convergence is in both theorems uniform in λ and for $\lambda \leqq 0$ all limits are 0.

These theorems can also be extended for distribution functions with infinitely many points of discontinuity.

We introduce again the random variable $Y = F(X)$ with the distribution function $F^0(x)$ and the set I as the union of the intervals $[f_{2\nu}, f_{2\nu+1}]$, $\nu = 0, 1, \cdots, n$. For any $F(x) \varepsilon I$ we have

$$\frac{S_N(x) - F(x)}{F(x)} = \frac{S_N^0(F(x)) - F^0(F(x))}{F^0(F(x))},$$

and therefore

$$\sup_{F(x) \geqq f_0} \left| \frac{S_N(x) - F(x)}{F(x)} \right| = \sup_{x \varepsilon I} \left| \frac{S_N^0(x) - x}{x} \right|.$$

Let $R_N(x)$ be the empirical distribution function of a sample $Z_1, Z_2, \cdots, Z_N$ from a population with the distribution

$$P[Z \leqq x] = x, \qquad 0 \leqq x \leqq 1, \tag{32}$$

then

$$P\left[\sup_{x \varepsilon I} \left| \frac{S_N^0(x) - x}{x} \right| < \lambda N^{-\frac{1}{2}}\right] = P\left[\sup_{x \varepsilon I} \left| \frac{R_N(x) - x}{x} \right| < \lambda N^{-\frac{1}{2}}\right], \tag{33}$$

since the distributions of the two populations coincide for $x \varepsilon I$. Thus,

$$P\left[\sup_{F(x) \geqq f_0} \left| \frac{S_N(x) - F(x)}{F(x)} \right| < \lambda N^{-\frac{1}{2}}\right] = P\left[\sup_{x \varepsilon I} \left| \frac{R_N(x) - x}{x} \right| < \lambda N^{-\frac{1}{2}}\right]. \tag{34}$$

The set I_ϵ is defined as the union of the intervals $[f_{2\nu} - \epsilon, f_{2\nu+1} + \epsilon]$, $\nu = 0, 1, \cdots, n$, for $\epsilon > 0$. If $|R_N(x) - x| \leqq \epsilon$, then

$$\sup_{R_N(x) \varepsilon I} \left| \frac{R_N(x) - x}{x} \right| \leqq \sup_{x \varepsilon I_\epsilon} \left| \frac{R_N(x) - x}{x} \right|,$$

since $R_N(x) \varepsilon I$ implies that $x \varepsilon I_\epsilon$. We see that

$$\begin{aligned} &P\left[\sup_{x \varepsilon I_\epsilon} \left| \frac{R_N(x) - x}{x} \right| < \lambda N^{-\frac{1}{2}}\right] \\ &\qquad \leqq P[| R_N(x) - x | > \epsilon] + P\left[\sup_{R_N(x) \varepsilon I} \left| \frac{R_N(x) - x}{x} \right| < \lambda N^{-\frac{1}{2}}\right]. \end{aligned} \tag{35}$$

By a similar procedure we have

$$\begin{aligned} &P\left[\sup_{R_N(x) \varepsilon I_{2\epsilon}} \left| \frac{R_N(x) - x}{x} \right| < \lambda N^{-\frac{1}{2}}\right] \\ &\qquad \leqq P[| R_N(x) - x | > \epsilon] + P\left[\sup_{x \varepsilon I_\epsilon} \left| \frac{R_N(x) - x}{x} \right| < \lambda N^{-\frac{1}{2}}\right]. \end{aligned} \tag{36}$$

It is sufficient to prove

$$\lim_{N \to \infty} P\left[\sup_{R_N(x) \varepsilon I} \left| \frac{R_N(x) - x}{x} \right| < \lambda N^{-\frac{1}{2}}\right] = \Psi(\lambda) \tag{37}$$

since the probability

$$P[\,|R_N(x) - x| > \epsilon]$$

tends to 0 as $N \to \infty$. The function $\Psi(\lambda)$ is continuously dependent on the boundaries of I. Therefore, from (35), (36) and (37) we get

$$\lim_{N\to\infty} P\left[\sup_{x\epsilon I}\left|\frac{R_N(x) - x}{x}\right| < \lambda N^{-\frac{1}{2}}\right] = \Psi(\lambda) \tag{38}$$

and from (34) follows the statement of Theorem 3.

We arrange the numbers $Z_1, \cdots, Z_N$ of the sample according to their values and denote by Z_k^* the one for which there are exactly $k - 1$ smaller numbers in the sample. The probability of ties is 0 since (33) is a continuous distribution function. $R_N(x)$ is equal to k/N in $Z_k^* \leqq x \leqq Z_{k+1}^*$. In this interval

$$\sup_{Z_k^*\leqq x<Z_{k+1}^*}\left|\frac{R_N(x) - x}{x}\right| = \max\left\{\left|\frac{k/N}{Z_k^*} - 1\right|, \left|\frac{k/N}{Z_{k+1}^*} - 1\right|\right\}.$$

For $Z_{k+1}^* \geqq k/N$

$$\left|\frac{k/N}{Z_{k+1}^*} - 1\right| \leqq \left|\frac{(k+1)/N}{Z_{k+1}^*} - 1\right| + \frac{1}{f_0 N}$$

since $k/N \geqq f_0$ implies that $Z_{k+1}^* \geqq f_0$. For $Z_{k+1}^* < k/N$

$$\left|\frac{k/N}{Z_{k+1}^*} - 1\right| \leqq \left|\frac{(k+1)/N}{Z_{k+1}^*} - 1\right|.$$

Therefore,

$$\max_{k/N\epsilon I_{1/N}}\left|\frac{k/N}{Z_k^*} - 1\right| \leqq \sup_{R_N(x)\epsilon I_{1/N}}\left|\frac{R_N(x) - x}{x}\right| \leqq \max_{k/N\epsilon I_{2/N}}\left|\frac{k/N}{Z_k^*} - 1\right| + \frac{1}{f_0 N}$$

and (37) is equivalent to

$$\lim_{N\to\infty} P\left[\max_{k/N\epsilon I_{1/N}}\left|\frac{k/N}{Z_k^*} - 1\right| < \lambda N^{-\frac{1}{2}}\right] = \Psi(\lambda). \tag{39}$$

We can write this equation as

$$\lim_{N\to\infty} P\left[\max_{k/N\epsilon I_{1/N}}\left|\ln\left(\frac{k/N}{Z_k^*}\right)\right| < \lambda N^{-\frac{1}{2}}\right] = \Psi(\lambda) \tag{40}$$

or, since $\log n - \sum_{r=1}^{n} 1/r \to c$,

$$\lim_{N\to\infty} P\left[\max_{k/N\epsilon I_{1/N}}\left|\ln\frac{1}{Z_k^*} - \sum_{l=k}^{n}\frac{1}{l}\right| < \lambda N^{-\frac{1}{2}}\right] = \Psi(\lambda). \tag{41}$$

The random variables $\ln(1/Z_k^*)$ are not independent since they fulfill the inequalities

$$\ln\left(\frac{1}{Z_N^*}\right) < \ln\left(\frac{1}{Z_{N-1}^*}\right) < \cdots < \ln\left(\frac{1}{Z_1^*}\right).$$

However, they do form an additive Markov chain (cf [24]), i.e., their differences

$$\ln\left(\frac{1}{Z_{k-1}^*}\right) - \ln\left(\frac{1}{Z_k^*}\right)$$

are mutually independent. The variables

$$U_l = (N+1-l)\left(\ln\frac{1}{Z_{N+1-l}^*} - \ln\frac{1}{Z_{N+2-l}^*}\right), \qquad l = 1, \cdots, N,$$

have the distribution

$$P[U_l \leqq x] = 1 - e^{-x}, \qquad 0 \leqq x < \infty.$$

On the other hand we obtain

$$\ln\left(\frac{1}{Z_k^*}\right) = \sum_{l=1}^{N+1-k} \frac{U_l}{N+1-l}$$

and

$$\ln\left(\frac{1}{Z_k^*}\right) - \sum_{l=k}^{N}\frac{1}{l} = \sum_{l=1}^{N+1-k}\frac{U_l - 1}{N+1-l}.$$

Some moments of the variables

$$V_l = N^{\frac{1}{2}}\frac{U_l - 1}{N+1-l}$$

are

$$E(V_l) = 0, \qquad E(V_l^2) = \frac{N}{(N+1-l)^2}, \; E(|V_l|^3) = \left(\frac{12}{e} - 2\right)\frac{N^{\frac{3}{2}}}{(N+1-l)^3}.$$

Let the set of integers j, for which $(N+1-j)/N \,\varepsilon\, I_{1/N}$, be

$$\{j_0 = 0, 1, \cdots, j_1; j_2, j_2+1, \cdots, j_3; \cdots; j_{2n}, j_{2n}+1, \cdots, j_{2n}+1\}.$$

The j_i are defined such that $j_i/N \to 1 - f_{2n+1-i}$ as $N \to \infty$, for $i = 0, 1, \cdots, 2n+1$.

According to well-known rules for conditional distributions, we have

$$\begin{aligned} P\left[\max_{k/N\varepsilon I_{1/N}}\left|\ln\left(\frac{1}{Z_k^*}\right) - \sum_{l=k}^{N}\frac{1}{l}\right| < \lambda N^{-\frac{1}{2}}\right] &= P\left[\max_{k/N\varepsilon I_{1/N}}\left|\sum_{l=1}^{N+1-k} V_l\right| < \lambda\right] \\ &= \int\cdots\int_{\substack{|x_i|<\lambda \\ i=0,\cdots,2n}} \prod_{\nu=0}^{n} d_{x_{2\nu}} P\left[\max_{l=j_{2\nu}+1,\cdots,j_{2\nu+1}}\left|\sum_{i=1}^{l} V_i\right| < \lambda,\right. \\ &\qquad \left.\sum_{i=1}^{j_{2\nu+1}} V_i \leqq x_{2\nu}\,\middle|\, \sum_{i\leqq j_{2\nu}} V_i = x_{2\nu-1}\right] \\ &\qquad \cdot \prod_{\nu=1}^{n} d_{x_{2\nu-1}} P\left[\sum_{i=1}^{j_{2\nu}} V_i \leqq x_{2\nu-1}\,\middle|\, \sum_{i=1}^{j_{2\nu-1}} V_i = x_{2\nu-2}\right], \end{aligned} \tag{42}$$

where $x_{-1} = 0$. The limits of the probabilities which occur in this integral can be calculated by a limit theorem for partial sums of random variables.

LEMMA. *(See [13], [14].) Let $Y_{M1}, \cdots, Y_{Mm_M}$ be m_M independent random variables with*

$$E(Y_{Mk}) = 0, \qquad \sum_1^{m_M} E(Y_{Mk}^2) = 2t_M.$$

Assume that for all k

$$\frac{E(|Y_{Mk}|^3)}{E(Y_{Mk}^2)} < \mu(M) \tag{43}$$

where $\mu(M) \to 0$ as $M \to \infty$, and let a, b, ξ, and η be any numbers such that $a < 0$, $b > 0$, and $a \leqq \xi < \eta \leqq b$. Then

$$\lim_{M\to\infty} P\left[a < \sum_1^k Y_{Mi} < b, k = 1, \cdots, m_M; \xi < \sum_1^{m_M} Y_{Mi} < \eta\right] = u(0, 0), \tag{44}$$

where $u(s, t)$ is the solution of the differential equation

$$\frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial s^2} \tag{45}$$

for which the boundary conditions

$$\begin{aligned} u(s, T) &= 0, & a < s < \xi, \quad & \eta < s < b, \\ u(s, T) &= 1, & \xi < s < \eta, & \\ u(a, t) &= 0, & 0 < t < T, & \\ u(b, t) &= 0, & 0 < t < T, & \end{aligned} \tag{46}$$

are fulfilled.

We can apply this lemma to the variables $Y_{j_{2\nu}+1}, Y_{j_{2\nu}+2}, \cdots, Y_{j_{2\nu+1}}$, for $\nu = 0, 1, \cdots, n$, because these variables satisfy (43), with

$$\mu(N) = \frac{3}{f_0 N^{\frac{1}{2}}}.$$

The sum of the second moments

$$2t_M^{(2\nu+1)} = \sum_{j_{2\nu}+1}^{j_{2\nu+1}} \frac{N}{(N+1-k)^2} = N \sum_{N+1-j_{2\nu+1}}^{N} \frac{1}{k^2} - N \sum_{N+1-j_{2\nu}}^{N} \frac{1}{k^2}$$

tends towards

$$2T^{(2\nu+1)} = \frac{1 - f_{2n-2\nu}}{f_{2n-2\nu}} - \frac{1 - f_{2n-2\nu+1}}{f_{2n-2\nu+1}} = \frac{f_{2n-2\nu+1} - f_{2n-2\nu}}{f_{2n-2\nu+1}\, f_{2n-2\nu}}, \tag{47}$$

and the boundaries for the partial sums are now

$$a = -\lambda - x_{2\nu-1}, \qquad b = \lambda - x_{2\nu-1}, \qquad \xi = -\lambda - x_{2\nu-1}, \qquad \eta = x_{2\nu} - x_{2\nu-1}, \tag{48}$$

where $x_{-1} = 0$. The solution of (45), which satisfies the boundary conditions (46), (48), is

$$u(s, t) = \frac{1}{2\sqrt{\pi(T^{(2\nu+1)} - t)}} \int_{-\lambda - x_{2\nu-1}}^{x_{2\nu} - x_{2\nu-1}} \sum_{j=-\infty}^{+\infty} (-1)^j \cdot \exp\left[- \frac{(s + x_{2\nu-1} - (-1)^j(x + x_{2\nu-1}) - 2\lambda j)^2}{4(T^{(2\nu+1)} - t)} \right] dx. \tag{49}$$

Hence we have

$$\begin{aligned} &\lim_{N\to\infty} P\left[\max_{l = j_{2\nu}+1, \cdots, j_{2\nu+1}} \left| \sum_{i=1}^{l} V_i \right| < \lambda, \sum_{i=1}^{j_{2\nu+1}} V_i \leqq x_{2\nu} \,\Big|\, \sum_{i \leqq j_{2\nu}} V_i = x_{2\nu-1} \right] \\ &= -\frac{1}{2\sqrt{\pi T^{(2\nu+1)}}} \int_{-\lambda}^{x_{2\nu}} \sum_{j=-\infty}^{+\infty}{}' (-1)^j \exp\left[- \frac{(x - (-1)^j x_{2\nu-1} - 2\lambda j)^2}{4T^{(2\nu+1)}} \right] dx. \end{aligned} \tag{50}$$

On the other hand, we apply the Central Limit theorem to the variables $V_{j_{2\nu-1}+1}$, $V_{j_{2\nu-1}+2}, \cdots, V_{j_{2\nu}}$, obtaining

$$\begin{aligned} &\lim_{N\to\infty} P\left[\sum_{i=1}^{j_{2\nu}} V_i \leqq x_{2\nu-1} \,\Big|\, \sum_{i=1}^{j_{2\nu-1}} V_i = x_{2\nu-2} \right] \\ &= \frac{1}{2\sqrt{\pi T^{(2\nu)}}} \int_{-\infty}^{x_{2\nu-1} - x_{2\nu-2}} e^{-x^2/4T^{(2\nu)}} dx, \end{aligned} \tag{51}$$

where the $T^{(2\nu)}$ are defined in the same was as the $T^{(2\nu+1)}$ in (47).

In view of (42), (50) and (51) it follows that

$$\begin{aligned} \lim_{N\to\infty} P\left[\max_{k/N \epsilon I_{1/N}} \left| \sum_{l=1}^{N+1-k} V_l \right| < \lambda \right] &= \frac{1}{2^{2n+1} \pi^{n+\frac{1}{2}} \prod_{j=1}^{2n+1} (T^{(j)})^{\frac{1}{2}}} \\ &\cdot \sum_{p_0, \cdots, p_n = -\infty}^{+\infty} \int \cdots \int_{|x_i| < \lambda} \exp\left[-\sum_{\nu=0}^{n} \frac{(x_{2\nu} - (-1)^{p_\nu} x_{2\nu-1} - 2\lambda p_\nu)^2}{4T^{(2\nu+1)}} \right. \\ &\left. - \sum_{\nu=0}^{n-1} \frac{(x_{2\nu+1} - x_{2\nu})}{4T^{(2\nu+2)}} \right] dx_0 \cdots dx_{2n}, \end{aligned} \tag{52}$$

where $x_{-1} = 0$. This expression is $\Psi(\lambda)$. This proves (37) and consequently Theorem 3.

Theorem 4 can be proved in the same way. In the lemma of Kolmogorov we replace a and ξ by $-\infty$. The solution of the boundary problem is now

$$u(s, t) = \frac{1}{2\sqrt{\pi(T^{(2\nu+1)} - t)}} \int_{-\infty}^{x_{2\nu} - x_{2\nu-1}} \sum_{j=0}^{1} (-1)^j \cdot \exp\left[-\frac{(s + x_{2\nu-1} - (-1)^j(x + x_{2\nu-1}) - 2\lambda j)^2}{4(T^{(2\nu+1)} - t)} \right] dx, \tag{53}$$

and we obtain

$$\lim_{N\to\infty} P\left[\max_{l=j_{2\nu}+1,\cdots,j_{2\nu+1}} \left(\sum_{i=1}^{l} V_i\right) < \lambda, \left(\sum_{i=1}^{j_{2\nu+1}} V_i\right) \leqq x_{2\nu} \middle| \left(\sum_{i\leqq j_{2\nu}} V_i\right) = x_{2\nu-1}\right] \tag{54}$$
$$= \frac{1}{2\sqrt{\pi T^{(2\nu+1)}}} \int_{\infty}^{x_{2\nu}} \sum_{j=0}^{1} (-1)^j \exp\left[-\frac{(x-(-1)^j x_{2\nu-1} - 2\lambda j)^2}{4T^{(2\nu+1)}}\right] dx.$$

From that Theorem 4 follows.

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