In this note an alternative proof is given which entails little computation and is self-contained.

Replace the interval (0, 1) by the reals modulo 1, considered as a circle of circumference 1. Let c be an arbitrary point on the circle. Moving from c in the direction corresponding to increasing values (0, 1), one meets successively the points $U_{k+1}, U_{k+2}, \cdots, U_n, U_1, \cdots, U_k$ where k, so defined, is a r.v. depending on c. Rename these points $U_1^c, U_2^c, \cdots, U_n^c$ respectively. Define $i = i(j)$ by $U_j^c = U_i$. Let u_j^c denote the (arc) distance of U_j^c from c taken in the increasing direction. Therefore,

$$i = k + j; \qquad u_j^c = U_{k+j} - c \qquad \text{for } j = 1, \cdots, n - k$$

$$i = k + j - n; \qquad u_j^c = U_{k+j-n} + 1 - c \qquad \text{for } j = n - k + 1, \cdots, n$$

With the indicated relation between i and j observe that

$$j/n - u_j^c = (i - k)/n - U_i + c = i/n - U_i + c - k/n.$$

For a fixed c and a given sample, c and k are constants and hence $j/n - u_j^c$ attains its maximum at the same point $U^* = U_{i^*}$ as does $i/n - U_i$.

Given a sample $U_1, \cdots, U_n$, the point U^* on the circle of reals mod. 1 is therefore independent of the choice of the initial point c taken instead of 0 on this circle. Since the distribution of X mod. 1 is uniform, that is, is invariant under translations, the distribution of U^* mod. 1 is also invariant under translations. Thus U^* has a uniform distribution on (0, 1). q.e.d.

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QUASI-RANGES OF SAMPLES FROM AN EXPONENTIAL POPULATION

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In a study of the use of ranges and quasi-ranges in estimating the standard deviation of a population, Harter [4] has compared the results for samples from a normal population with those for samples from certain other populations, including the exponential. In this note are given the distributions of quasi-ranges from the exponential population and also formulas for the cumulants of these quasi-ranges.

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Let $x_1, x_2, \cdots, x_n$ be a sample, and suppose that $x_1 \leqq x_2 \leqq \cdots \leqq x_n$. The quasi-range of order r of this sample is the statistic

$$w_r = x_{n-r} - x_{r+1}, \tag{1}$$

w_0 being the range itself.

The population to be considered is

$$f(x) = e^{-x}, \qquad 0 \leqq x, \tag{2}$$

which has mean and variance each equal to 1. The cumulative distribution function for this population is

$$F(x) = \int_0^x e^{-x}\,dx = 1 - e^{-x},$$

and the probability that r members of the sample are below x_{r+1}, r above x_{n-r}, and the remaining values between x_{r+1} and x_{n-r}, is proportional to

$$(1 - e^{-x_{r+1}})^r (e^{-x_{r+1}} - e^{-x_{n-r}})^{n-2r-2} e^{-rx_{n-r}}\,dx_{r+1}\,dx_{n-r}.$$

Replacing x_{n-r} by $x_{r+1} + w_r$, integrating with respect to x_{r+1} between the limits 0 and ∞, and supplying the proper multiplicative constant, we find the distribution of the rth quasi-range, w_r, to be

$$\frac{\Gamma(n-r)}{\Gamma(n-2r-1)\Gamma(r+1)} (1 - e^{-w_r})^{n-2r-2} e^{-(r+1)w_r}\,dw_r. \tag{3}$$

Obviously we must have $n \geqq 2r + 2$.

Upon multiplying (3) by e^{tx} and integrating between the limits 0 and ∞, we have ([2], p. 144) for the moment generating function of the rth quasi-range,

$$M(t) = \frac{\Gamma(n-r)\Gamma(r+1-t)}{\Gamma(r+1)\Gamma(n-r-t)} = \prod_{j=r+1}^{n-r-1} \left(1 - \frac{t}{j}\right)^{-1}. \tag{4}$$

To find the cumulants of the distribution (3) we note that

$$K(t) = \ln M(t) = -\sum_{j=r+1}^{n-r-1} \ln\left(1 - \frac{t}{j}\right) = \sum_{j=r+1}^{n-r-1} \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{t}{j}\right)^k. \tag{5}$$

It follows at once that the cumulant of order p is

$$\kappa_p = (p-1)! \sum_{j=r+1}^{n-r-1} \frac{1}{j^p}. \tag{6}$$

In particular, we have for the mean and variance respectively,

$$\kappa_1 = \sum_{j=r+1}^{n-r-1} \frac{1}{j}, \qquad \kappa_2 = \sum_{j=r+1}^{n-r-1} \frac{1}{j^2}. \tag{7}$$

Thus the mean of the quasi-range w_r, being equal to the sum of a harmonic series, diverges with sample size, although very slowly, while the variance approaches a finite value. For the case $r = 0$, that is for the range itself, the variance approaches the value $\pi^2/6 = 1.6449$, and somewhat rapidly. For example, the variance of the range of samples of size 10 is 1.4977.

For $r = 0$, the values of κ_3 and κ_4 approach 2.4041 and $\pi^4/15 = 6.4939$ respectively as n becomes infinite. (Values can be obtained from tables of the Riemann zeta function, e.g. [3].) The ratios $\kappa_3/\kappa_2^{3/2}$ and κ_4/κ_2^2 approach 1.1395 and 12/5 respectively. For a normal distribution these ratios are, of course, both zero.

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V. N. Murty has kindly pointed out to me that the result of my note, "A Note on Balanced Incomplete Block Designs," (*Ann. Math. Stat.*, Vol. 28 (1957), p. 1054), was given previously by K. Kishen and C. R. Rao in "An Examination of Various Inequality Relations Among Parameters of the Balanced Incomplete Block Design" (*Journal of the Indian Society of Agricultural Statistics*, Vol. IV, No. 2 (1952), pp. 137–144).

CORRECTION TO "RANDOM ORTHOGONAL TRANSFORMATIONS AND THEIR USE IN SOME CLASSICAL DISTRIBUTION PROBLEMS IN MULTIVARIATE ANALYSIS"

BY ROBERT A. WIJSMAN

In footnote 3 of the paper cited in the title (*Ann. Math. Stat.* Vol. 28 (1957), pp. 415–423), for χ^2 read χ.