SOME BOUNDS ON THE DISTRIBUTION FUNCTIONS OF THE LARGEST AND SMALLEST ROOTS OF NORMAL DETERMINANTAL EQUATIONS[1]

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While the joint density function of the roots of certain determinantal equations have been obtained, [1], [2], [3], the result is sufficiently complex that the marginal distribution functions of these statistics have not, to the author's knowledge, been tabulated. We present here a lower bound on the distribution function of the smallest root and an upper bound on the distribution function of the largest root.

These bounds may be of possible usefulness in problems of significance tests since observed values that are not "significant" according to the bounds will certainly not be "significant" with respect to the exact distribution.

Let S^1_{ij} and S^2_{ij}, $i, j = 1, \cdots, k$, be two sample covariance matrices from normal distributions having identical covariance matrices. It is well known [4] that the smallest and largest roots, say W_1 and W_k, of the equation

$$|S^1_{ij} - WS^2_{ij}| = 0$$

satisfy the inequalities

$$W_1 \leqq \frac{\sum S^1_{ij}\, x_i\, x_j}{\sum S^2_{ij}\, x_i\, x_j} \leqq W_k, \qquad \sum S^2_{ij}\, x_i\, x_j > 0$$

Let $F_i = S^1_{ii} / S^2_{ii}$. It then follows that

$$W_1 \leqq \min_i \{F_i\}; \qquad W_k \geqq \max_i \{F_i\}.$$

Since the roots are invariant under linear transformations of the underlying variables, the covariance matrix may be taken to be the identity matrix. Then the F_i are independently and identically distributed according to the well known F distribution. Denote by $F_{[1]}$ and $F_{[k]}$ the smallest and the largest of a set of k independently identically distributed F values. We then have the desired bounds.

$$P\{W_1 \leqq u\} \geqq P\{F_{[1]} \leqq u\}$$

$$P\{W_k \geqq v\} \geqq P\{F_{[k]} \geqq v\}.$$

Denote by $G(F)$ the distribution function of F (which depends, of course, on the numbers of degrees of freedom for S^1_{ij} and S^2_{ij}). The above bounds become

$$P\{W_1 \leqq u\} \geqq 1 - [1 - G(u)]^k$$

$$P\{W_k \geqq v\} \geqq 1 - [G(v)]^k.$$

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NOTE ON A MOVING SINGLE SERVER PROBLEM[1]

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1. Introduction and summary. B. McMillan and J. Riordan in [1] derived the generating function for the probability distribution of the number of items completed before absorption in a moving single server problem in two special cases. Through an analogy to the work of L. Takács [2] on busy period problems for a simple queue, McMillan and Riordan postulated a nonlinear integral equation relation for the generating function. In this note the validity of this relation is proved in general by exploiting the analogy more fully, and the generating function in the two special cases is obtained directly from the integral equation. A similar functional relation is established for the Laplace-Stieltjes transform of the distribution of time until absorption, and the transform is obtained for the two special cases.

2. Functional relations. As stated by McMillan and Riordan the moving single server problem is the following: an assembly line moving with uniform speed has items for service spaced along it. The single server available moves with the line while serving and against it with infinite velocity while transferring service to the next item in line. The line has a barrier in which the server may be said to be "absorbed" in the sense that service is disabled if the server moves into the barrier. The server with exponentially (α) distributed service time starts service on the first item when it is T time units away from the barrier. Let the spacings between items be independent random variables with the general distribution function $B(t)$.

This problem is analogous to a simple queue with a single server, Poisson arrivals ($\lambda = \alpha$), and distribution of service times $F_S(t) \equiv B(t)$. The time until absorption in the moving single server problem is equivalent to the length of a

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