TRUNCATION AND TESTS OF HYPOTHESES[1]

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1. Summary. This paper examines the loss of power when using tests based on the assumption that the variable being sampled has a "complete" normal distribution when in fact the distribution is a "truncated" one. The cases considered here are for small sample sizes and "symmetric" truncation, while the hypothesis considered is the one-sided testing for the mean of a normal distribution. Some tables are computed and it appears that an appreciable loss occurs only in the size of the test. The loss in power is found to decrease very rapidly with the distance of the alternative value of the mean from the one tested and also with the distance of the truncation from the mean.

2. Introduction. In sampling from a normal distribution the assumption that the random variable X is defined over $(-\infty, \infty)$ is an unrealistic one, and "a sample of n from a normal distribution" is in reality a sample of n from a "truncated" normal distribution. This problem has been dealt with from various points of view in several recent papers (see references). However, one aspect that seems to have been neglected is that of the tests of hypotheses. We shall attempt to examine the results of applying some usual tests of hypotheses to the case when the available sample is known to have come from a truncated population.

We call a normal distribution 'symmetrically truncated' at the 'terminus point' a if its density is given by

$$f(x) = \frac{c}{\sqrt{2\pi}\sigma} \exp\left[-\tfrac{1}{2}(x-\mu)^2/\sigma^2\right], \quad \text{for } |x-\mu| < a\sigma, \tag{2.1}$$

$$= 0 \qquad \text{otherwise,}$$

where c is given by

$$\frac{1}{c} = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-t^2/2}\, dt. \tag{2.2}$$

We shall confine our attention to the problems of symmetric truncation only, with a and σ^2 known.

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3. Distribution of sample means. Suppose a sample $X_1, \cdots, X_n$ of size n is available from a distribution of the form (2.1). The sampling distribution of $\bar{X} = 1/n\sum_{i=1}^{n} X_i$ for arbitrary n is very complicated and no general formula giving the distribution of $\bar{X}$ explicitly is available. However, by using convolutions of distributions, it is quite easy to derive the distribution of $\bar{X}$ for small values of n. The results for $n = 1, 2, 3$ and 4 are given below where without loss of generality $\mu = 0$, $\sigma = 1$. The density function of $\bar{X}$ is denoted by $f_n(x)$.

Case $n = 1$. From (2.1) the density is given by

$$f_1(x) = \begin{cases} \dfrac{c}{\sqrt{2\pi}} \exp(-x^2/2), & \text{for } |x| < a, \\ 0 \quad \text{otherwise}, \end{cases} \tag{3.1}$$

where c is given by (2.2).

Case $n = 2$. Using convolution on (3.1) we obtain

$$f_2(x) = \begin{cases} \dfrac{\sqrt{2}c^2}{\pi} e^{-x^2} \displaystyle\int_0^{\sqrt{2}(a-|x|)} e^{-t^2/2}\, dt, & \text{for } |x| < a, \\ 0 \quad \text{otherwise}. \end{cases} \tag{3.2}$$

Case $n = 3$. Convoluting (3.1) and (3.2) it can be verified that

$$f_3(x) = \begin{cases} \dfrac{\sqrt{6}c^3}{2\pi^{3/2}} e^{-(3/2)x^2} \displaystyle\int_{(-\sqrt{6/4})(a-x)}^{(\sqrt{6/4})(a+x)} \int_0^{\sqrt{2}(a-|(u/\sqrt{6})+x|)} e^{-\frac{1}{2}(u^2+t^2)}\, dt\, du, \\ \qquad\qquad \text{for } 0 \leqq |x| \leqq \dfrac{a}{3}, \\ \dfrac{\sqrt{6}c^3}{2\pi^{3/2}} e^{-(3/2)x^2} \displaystyle\int_{(-\sqrt{6/4})(a-x)}^{\sqrt{6}(a-x)} \int_0^{\sqrt{2}(a-|(u/\sqrt{6})+x|)} e^{-\frac{1}{2}(u^2+t^2)}\, dt\, du, \\ \qquad\qquad \text{for } \dfrac{a}{3} < |x| < a, \\ 0 \quad \text{otherwise}. \end{cases} \tag{3.3}$$

Case $n = 4$. Applying the convolution law to the density (3.2) it is found that

$$f_4(x) = \begin{cases} \dfrac{4}{\pi^2} c^4 e^{-2x^2} \displaystyle\int_0^{2(a-|x|)} \int_0^{\sqrt{2}(a-|(u/2)+x|)} \int_0^{\sqrt{2}(a-|(u/2)-x|)} \\ \qquad\qquad \cdot\, e^{-\frac{1}{2}(u^2+v^2+w^2)}\, dw\, dv\, du, & \text{for } |x| < a, \\ 0 \quad \text{otherwise}. \end{cases} \tag{3.4}$$

For sufficiently large n, Birnbaum and Andrews [1] have pointed out that $n\bar{X}$ has a limiting normal distribution. Thus for large n one may obtain an approximate cumulative distribution of $\bar{X}$ from (4.2) in [1]. However, in this paper we shall confine our attention to only those cases where $n \leqq 4$.

4. Tests of hypotheses under truncation. In this section we consider the effect of truncation on size and power of tests of hypotheses concerning the means of parent populations.

TABLE I

Values of $P_u(\mu)$, $P(\mu, a)$, $L(\mu, a)$ and loss in power expressed as percentage of $P_u(\mu)$ for $\alpha = 0.05$

n	a	1.5						2.0					
	μ	0	0.5	1.0	1.5	2.0	2.5	0	1.0	1.5	2.0	2.5	3.0
1	P_u	.0500	.1261	.2595	.4424	.6387	.8037	.0500	.2595	.4424	.6387	.8037	.9123
	P	.0000	.0684	.2224	.4335	.6601	.8506	.0285	.2480	.4396	6453	.8182	.9319
	L	.0500	.0577	.0371	.0089	−.0214	−.0469	.0215	.0115	.0028	−.0066	−.0145	−.0196
	% Loss	100.0	45.8	14.3	2.0	−3.4	−5.8	43.0	4.4	0.6	−1.0	−1.8	−2.1
2	P_u	.0500	.1742	.4087	.6831	.8817	.9707	.0500	.4087	.6831	.8817	.9707	.9953
	P	.0094	.1108	.3839	.7289	.9422	.9984	.0297	.4005	.6989	.9068	.9864	.9997
	L	.0406	.0634	.0248	−.0458	−.0605	−.0277	.0203	.0082	−.0158	−.0251	−.0157	−.0044
	% Loss	81.2	36.4	6.1	−6.7	−6.9	−2.9	40.6	2.0	−2.3	−2.8	−1.6	−0.4
3	P_u	.0500	.2180	.5347	.8297	.9656	.9964	.0500	.5347	.8297	.9656	.9964	.9998
	P	.0112	.1533	.5449	.8961	.9951	1.0	.0301	.5384	.8566	.9820	.9996	1.0
	L	.0388	.0647	−.0102	−.0664	−.0295	−.0036	.0199	−.0037	−.0269	−.0164	−.0032	−.0002
	% Loss	77.6	29.7	−1.9	−8.0	−3.1	−0.4	39.8	−0.7	−3.2	−1.7	−0.3	—
4	P_u	.0500	.2595	.6387	.9123	.9907	.9996	.0500	.6387	.9123	.9907	.9996	1.0−
	P	.0118	.1977	.6792	.9664	.9998	1.0	.0303	.6539	.9374	.9972	1.0−	1.0
	L	.0382	.0618	−.0405	−.0541	−.0091	−.0004	.0197	−.0152	−.0251	−.0065	−.0004	—
	% Loss	76.4	23.8	−6.3	−5.9	−0.9	—	39.4	−2.4	−2.8	−0.7	—	—

n	a	2.5						3.0					
	μ	0	0.75	1.50	2.25	3.00	3.75	0	0.5	1.0	2.0	3.0	4.0
1	P_u	.0500	.1854	.4424	.7274	.9123	.9824	.0500	.1261	.2595	.6387	.9123	.9907
	P	.0443	.1814	.4416	.7303	.9175	.9884	.0488	.1251	.2588	.6391	.9134	.9921
	L	.0057	.0040	.0008	−.0029	−.0052	−.0060	.0012	.0010	.0007	−.0004	−.0011	−.0014
	% Loss	11.4	2.2	0.2	−0.4	−0.6	−0.6	2.4	0.8	0.3	−0.1	−0.1	−0.1
2	P_u	.0500	.2795	.6831	.9379	.9953	.9999	.0500	.1742	.4087	.8817	.9953	1.0−
	P	.0426	.2742	.6875	.9456	.9978	1.0	.0480	.1725	.4082	.8826	.9962	1.0−
	L	.0074	.0053	−.0044	−.0077	−.0025	−.0001	.0020	.0017	.0005	−.0009	−.0009	—
	% Loss	14.8	1.9	−0.6	−0.8	−0.3	—	4.0	1.0	0.1	−0.1	—	—
3	P_u	.0500	.3647	.8297	.9878	.9998	1.0−	.0500	.2180	.5347	.9656	.9998	1.0−
	P	.0425	.3605	.8348	.9917	1.0−	1.0−	.0479	.2161	.5350	.9675	.9999	1.0
	L	.0075	.0042	−.0051	−.0039	−.0002	—	.0021	.0019	−.0003	−.0019	−.0001	—
	% Loss	15.0	1.2	−0.6	−0.4	—	—	4.2	0.9	−0.1	−0.2	—	—
4	P_u	.0500	.4424	.9123	.9978	1.0−	1.0−	.0500	.2595	.6387	.9907	1.0−	1.0−
	P	.0425	.4403	.9213	.9989	1.0−	1.0	.0478	.2575	.6399	.9918	1.0−	1.0
	L	.0075	.0021	−.0090	−.0011	—	—	.0022	.0020	−.0012	−.0011	—	—
	% Loss	15.0	0.5	−1.0	−0.1	—	—	4.4	0.8	−0.2	−0.1	—	—

Consider a sample of size n from a normal distribution $N(\mu, 1)$. Then ā Uniformly Most Powerful (UMP) test of the one-sided hypothesis testing problem

$$H:\mu = \mu_0, \qquad \text{Alt}:\mu > \mu_0, \tag{4.1}$$

is given by (we assume without loss of generality that $\mu_0 = 0$),

$$\text{Reject } H \text{ if } \bar{X} > Z_\alpha/\sqrt{n}; \qquad \text{accept } H \text{ otherwise,} \tag{4.2}$$

where Z_α is the point exceeded with probability α using the distribution of the standard normal variable. Now, if sampling from $N_a(\mu, 1)$ where $N_a(\mu, 1)$ is the density (2.1) with $\sigma = 1$, and test procedure (4.2) is used, the predetermined size α of this 'usual' test is really not obtained. The actual size is given by $\alpha' = \Pr(Z_t > Z_\alpha/\sqrt{n})$, where Z_t is the random variable with density function $f_n(x)$ of the last section.

Now, the 'usual' power function of the test (4.2) is given by

$$\begin{aligned} P_u(\mu) &= \Pr\{\bar{X} > Z_\alpha/\sqrt{n} \mid \bar{X} \sim N(\mu, 1/n)\} \\ &= \Pr\{Z > Z_\alpha - \mu\sqrt{n} \mid Z \sim N(0, 1)\}, \end{aligned} \tag{4.3}$$

if sampling is from a "complete" normal distribution. However, if the sampling is from a truncated distribution, $N_a(\mu, 1)$, the actual power function of the 'usual' size α test is given by

$$\begin{aligned} P(\mu, a) &= \Pr\{\bar{X} > Z_\alpha/\sqrt{n} \mid \bar{X} \sim f_n(x, \mu)\} \\ &= \Pr\{Z_t > Z_\alpha/\sqrt{n} - \mu \mid Z_t \sim f_n(x)\}, \end{aligned} \tag{4.4}$$

where $f_n(x, \mu)$ is the density of $\bar{X}$ when sampling from $N_a(\mu, 1)$, and Z_t is the random variable with density $f_n(x) = f_n(x, 0)$.

We denote the difference of (4.4) and (4.3) by

$$L(\mu, a) = P_u(\mu) - P(\mu, a). \tag{4.5}$$

For $\mu = 0$, L equals $\alpha - \alpha'$, while for all other values of μ, L is the "loss of power" if the usual procedure is followed, while sampling is actually from a truncated distribution. Values of $P_u(\mu)$, $P(\mu, a)$, and the loss in power expressed as percentage of $P_u(\mu)$ for different values of μ and four terminus points 'a' are given in Table I for $\alpha = 0.05$ and $n = 1, 2, 3$ and 4.

It can be easily verified that (4.5) reduces to

$$\begin{aligned} L(\mu, a) &= \Pr\left\{\bar{X} - \mu > \frac{Z_\alpha}{\sqrt{n}} - \mu \,\middle|\, \bar{X} \sim N\left(0, \frac{1}{n}\right)\right\} \\ &\quad - \Pr\left\{\bar{X} - \mu > \frac{Z_\alpha}{\sqrt{n}} - \mu \,\middle|\, \bar{X} \sim f_n(x, 0)\right\}, \end{aligned} \tag{4.6}$$

and by graphical considerations one may see that $L(\mu, a)$ and $Z_\alpha/\sqrt{n} - \mu$ have the same sign. Thus, as soon as μ exceeds $Z_\alpha/\sqrt{n}$, there will be a change of sign from positive to negative in the loss of power, $L(\mu, a)$. This is borne out by the actual computations in Table I.

TABLE II

Upper 100 α% points of $f_n(x)$

α	a	n = 1	2	3	4
0.10	1.0	.749	.510	.408	.351
	1.5	1.022	.693	.559	.482
	2.0	1.184	.813	.659	.569
	2.5	1.254	.875	.711	.615
	3.0	1.275	.898	.732	.634
0.05	1.0	.868	.636	.516	.445
	1.5	1.226	.871	.708	.613
	2.0	1.472	1.028	.838	.725
	2.5	1.593	1.114	.909	.786
	3.0	1.633	1.150	.938	.812
0.025	1.0	.932	.731	.603	.524
	1.5	1.350	1.011	.831	.722
	2.0	1.679	1.204	.987	.857
	2.5	1.868	1.315	1.076	.933
	3.0	1.939	1.365	1.115	.966
0.01	1.0	.972	.821	.695	.609
	1.5	1.436	1.155	.965	.843
	2.0	1.848	1.396	1.154	1.006
	2.5	2.142	1.545	1.266	1.100
	3.0	2.279	1.611	1.318	1.143
0.005	1.0	.986	.870	.751	.664
	1.5	1.467	1.238	1.049	.922
	2.0	1.919	1.515	1.262	1.104
	2.5	2.285	1.687	1.392	1.212
	3.0	2.493	1.775	1.455	1.263

Now, suppose the sampling is from $N_a(\mu, 1)$. By applying the Neyman-Pearson Fundamental Lemma, a UMP test of (4.1) of size α is

$$\text{(4.7)} \qquad \begin{cases} \text{Reject } H \text{ if } \bar{X} > K_\alpha(a, n), \\ \text{Accept } H \text{ otherwise,} \end{cases}$$

where $K_\alpha(a, n)$ is the point exceeded with probability α using the distribution whose density is $f_n(x)$. Table II gives the significance points for the test (4.7) for different n, α and a. That is, if sampling from a truncated normal distribution, (4.7) gives the 'correct' test for problem (4.1), and Table II gives the correct significance points for this problem.

The power of this 'correct' test (4.7) is given by

$$\text{(4.8)} \qquad P_c(\mu) = \Pr(Z_t > K_\alpha(a, n) - \mu),$$

where Z_t is the random variable with density $f_n(x)$. The gain in power, $G(\mu, a) = P_c(\mu) - P(\mu, a)$, is the gain that would result if one uses the correct test rather than the usual test. The values of $P_c(\mu)$, $G(\mu, a)$ and the gain in power expressed

TABLE III

Values of $P_c(\mu)$; $G(\mu, a)$ and gain in power expressed as percentage of $P(\mu, a)$ for $\alpha = 0.05$

n	a	1.5					2.0				
	μ	.5	1.0	1.5	2.0	2.5	1.0	1.5	2.0	2.5	3.0
1	P_c	.1929	.3968	.6246	.8238	.9601	.3098	.5117	.7108	.8646	.9576
	G	.1245	.1744	.1911	.1637	.1095	.0618	.0721	.0655	.0464	.0257
	% Gain	182.0	78.4	44.1	24.8	12.9	24.9	16.4	10.2	5.7	2.8
2	P_c	.2508	.5923	.8761	.9880	1.0	.4828	.7684	.9393	.9935	.9999
	G	.1400	.2084	.1472	.0458	.0016	.0823	.0695	.0325	.0071	.0002
	% Gain	126.4	54.3	20.2	4.9	0.2	20.5	9.9	3.6	0.7	—
3	P_c	.3199	.7450	.9684	.9997	1.0	.6220	.9010	.9905	.9999	1.0
	G	.1666	.2001	.0723	.0046	—	.0836	.0444	.0085	.0003	—
	% Gain	108.7	36.7	8.1	0.5	—	15.5	5.2	0.9	—	—
4	P_c	.3836	.8468	.9931	1.0−	1.0	.7305	.9611	.9988	1.0−	1.0
	G	.1859	.1676	.0267	.0002	—	.0766	.0237	.0016	—	—
	% Gain	94.0	24.7	2.8	—	—	11.7	2.5	0.2	—	—

n	a	2.5					3.0				
	μ	.75	1.50	2.25	3.00	3.75	.5	1.0	2.0	3.0	4.0
1	P_c	.1958	.4625	.7475	.9256	.9906	.1276	.2627	.6436	.9152	.9924
	G	.0144	.0209	.0172	.0081	.0022	.0025	.0039	.0045	.0018	.0003
	% Gain	7.9	4.7	2.4	0.9	0.2	2.0	1.5	0.7	0.2	—
2	P_c	.2986	.7124	.9534	.9983	1.0	.1773	.4156	.8873	.9965	1.0
	G	.0244	.0249	.0078	.0005	—	.0048	.0074	.0047	.0003	—
	% Gain	8.9	3.6	0.8	—	—	2.8	1.8	0.5	—	—
3	P_c	.3881	.8558	.9933	.9999	1.0	.2221	.5431	.9689	.9999	1.0
	G	.0276	.0210	.0016	—	—	.0060	.0081	.0014	—	—
	% Gain	7.7	2.5	0.2	—	—	2.8	1.5	0.1	—	—
4	P_c	.4703	.9320	.9992	1.0−	1.0	.2644	.6477	.9922	1.0−	1.0
	G	.0300	.0107	.0003	—	—	.0069	.0078	.0004	—	—
	% Gain	6.8	1.2	—	—	—	2.7	1.2	—	—	—

as percentage of $P(\mu, a)$ for different μ, a and n and for $\alpha = 0.05$ are given in Table III.

5. Conclusions. An examination of the tables indicates that serious losses occur in the size of the test rather than its power. For example, if the truncation occurs at 1.5 times the standard deviation on either side of the mean, and a usual 5% significance test is used, one is really using approximately 1% significance test rather than 5%. If the truncation occurs at twice the standard deviation on either side of the mean, the usual 5% significance test gives only approximately 3% significance level. Thus one consequence of applying the usual test is to err on the conservative side in making it much more difficult to reject the hypothesis. As expected, however, when the truncation is at about three times the standard deviation on either side of the mean, there is hardly any difference between the usual and the correct test. Even when the truncation occurs at less than twice the standard deviation away from the mean, there is not much change in the value of the power function beyond one standard deviation away from the value of the mean specified by the null hypothesis. Hence it would appear that unless there is severe truncation and unless the alternative value of the mean is quite near the value specified by the null hypothesis, the usual test would be satisfactory. The results given here are only for a usual 5% significance level test. It is proposed to give extensive tables of the distribution of the mean of samples from truncated distributions and to examine the tests at other than 5% significance levels in another paper.

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