A GENERALIZATION OF THE BETA-DISTRIBUTION[1]

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0. Summary. A class of distributions is defined and studied which includes as particular cases (cf. Section 13) the ordinary β-distribution, the (univariate) triangular distribution, the uniform distribution over any nondegenerate simplex, and a continuous range of other distributions over such a simplex, called *basic β-distributions* (Section 6) and immediately analogous to the ordinary β-distribution. Our class also includes (Section 13 (vi)) various (univariate and other) distributions which arise in connection with the random division of an interval.

The main results are given in Section 2 and further results for the univariate case are given in Section 8.

This paper is exclusively concerned with the mathematical theory. One application may, however, be mentioned, which will be considered in more detail elsewhere. Suppose we wish to test the hypothesis H_0 that $n - 1$ numbers $y_1, \cdots, y_{n-1}$ (all lying between 0 and 1) were drawn independently from a rectangular distribution over (0, 1). Let $u_1, \cdots, u_n$ be the lengths of the n intervals into which the y_j divide the interval (0, 1). Then H_0 is equivalent to the hypothesis that the point with vector-coordinate $\mathbf{u}$ is distributed uniformly over a certain non-degenerate simplex S, and a useful set of alternative hypotheses is the set of basic n-dimensional β-distributions. Hence (using Section 4) this theory can be used to find the power-functions of certain tests of the hypothesis H_0.

1. Definitions. Let $x^1, \cdots, x^n$ be n random variables with the joint distribution function $F = F(x^1, \cdots, x^n)$. We shall be particularly concerned with a certain integral transform ϕ_p of this distribution, defined by the equation

$$\phi_p(t; a_1, \cdots, a_n) = E\left\{\left(t - \sum_{j=1}^{n} a_j x^j\right)^{-p}\right\} \qquad (p > 0) \tag{1}$$

or, more formally, by the equation

$$\phi_p(t; a_1, \cdots, a_n) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(t - \sum_{j=1}^{n} a_j x^j\right)^{-p} dF(x^1, \cdots, x^n) \qquad (p > 0, \text{im } t \neq 0) \tag{2}$$

where $E\{\cdot\}$ denotes, as usual, an expectation. Here p is the *exponent* of the transformation and t and a_j $(j = 1, \cdots, n)$ are the $n + 1$ (homogeneous) parameters of the transformation.

If p is not an integer, it is necessary to specify which branch of the integrand is intended. In order to avoid ambiguity in this case we shall restrict the parameters a_j to real values and the parameter t to non-real values and then take the

[1] Received November 18, 1957.

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principal value of the logarithm in the formula $(t - \lambda)^{-p} = \exp(-p \log (t - \lambda))$. It is easily seen that the integral (2) converges for all such values of t and a_j.

Now suppose that there is a set of $r + rn$ real constants p_i (>0) and c_i^j $(i = 1, \cdots, r; j = 1, \cdots, n)$ such that

$$\phi_p(t; a_1, \cdots, a_n) = \prod_{i=1}^{r} \left(t - \sum_{j=1}^{n} a_j c_i^j\right)^{-p_i}, \tag{3}$$

where, taking $a_j = 0$ $(j = 1, \cdots, n)$, it is clear that

$$p = \sum_{i=1}^{r} p_i. \tag{4}$$

DEFINITION. Any joint distribution of $x^1, \cdots, x^n$ which satisfies (3) (where ϕ_p is defined in (2)) will be called an n-dimensional *β-distribution* with *indices* $p_1, \cdots, p_r$. The $n \times r$ matrix $\mathbf{C} = [c_i^j]$ will be called the *coordinate matrix* and the columns $\mathbf{c}_i$ $(i = 1, \cdots, r)$ of $\mathbf{C}$ will be called the *vertices*[1] of the β-distribution. Lastly, the sum (4) of the indices will be called the *exponent* of the β-distribution.

2. Main results.

THEOREM 1. *For any given set of $r + rn$ real constants p_i (>0) and c_i^j $(i = 1, \cdots, r; j = 1, \cdots, n)$, there is exactly one joint distribution of the variates $x^1, \cdots, x^n$ which satisfies equation (3) (subject to (2) and (4)).*

THEOREM 2. *Let $\mathfrak{D}$ be the convex hull of the r points whose vector coordinates are $\mathbf{c}_i$, suppose that $\mathfrak{D}$ is of dimension d $(1 \leqq d \leqq \min(r - 1, n))$, and let $\mathfrak{E}$ be the d-dimensional hyperplane containing $\mathfrak{D}$. Then any β-distribution with vertices $\mathbf{c}_i$ is a continuous distribution over $\mathfrak{E}$ with positive d-dimensional density at all interior points of $\mathfrak{D}$ and zero density elsewhere.*

Further if $p_i \geqq 1$ (all i) the density of the distribution is bounded, and if $p_i > 1$ (all i) the density is continuous over $\mathfrak{E}$.

The proof (Section 7) of Theorem 2 will show that if $d = 0$ the distribution is concentrated at a single point. Conversely, it is clear that any distribution which is concentrated at a single point is a β-distribution all of whose vertices have the same coordinates as the point, and whose exponent is arbitrary. On the other hand (Section 12) we shall prove

THEOREM 3. *Any β-distribution which is not concentrated at a single point admits only one possible value for its exponent,*

which, from (2) and (3), has the immediate

COROLLARY. *Any β-distribution which is not concentrated at a single point has a unique set of distinct vertices, and its corresponding indices are also uniquely determined.*

3. The univariate case. A preliminary lemma. If $n = 1$ we may drop the

[1] Thus (e.g.) two identical columns of $\mathbf{C}$ will be regarded as defining two coincident vertices of the distribution, each with its appropriate index.

superscript j. Then, writing t for t/a_1, we find that (3) may be written

$$E\{(t - x)^{-p}\} = \prod_{i=1}^{r} (t - c_i)^{-p_i} \qquad (\text{im } t \neq 0). \tag{5}$$

Applying Theorem 1 of [3], we deduce that, if $F(x)$ is the distribution function of a unidimensional β-distribution, then, for almost all values of x and y,

$$F(y) - F(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} K(t)\, dt, \tag{6}$$

where

$$K(t) = t^{-1}\left\{\prod_{i=1}^{r}\left(\frac{t}{t + y - c_i}\right)^{p_i} - \prod_{i=1}^{r}\left(\frac{t}{t + x - c_i}\right)^{p_i}\right\} \tag{7}$$

and the integral in (6), taken along the imaginary axis, converges (in particular at $t = 0$).

Now if $y > x > \max c_i$, the integrand $K(t)$ is a regular function of t in the region re $t > 0$, and so the path of integration in (6) may be deformed into the infinite semicircle $\pm i\infty$ in this region. Hence, since $|K(t)| = 0\ (|t|^{-2})$, it follows that in this case (i.e. for almost all x and y such that $y > x > \max c_i$) $F(x) = F(y) = \lim_{y\to\infty} F(y) = 1$. Using the monotonicity of $F(x)$, it follows that $F(x) = 1$ for all $x > \max c_i$, and similarly we may prove that $F(x) = 0$ for all $x < \min c_i$. This yields immediately

LEMMA 1. *Any unidimensional β-distribution is bounded.*

4. The invariance properties of the class of β-distributions. Suppose that the n-dimensional vector-variate $\mathbf{x}$ is a β-variate with vertices $\mathbf{c}_1, \cdots, \mathbf{c}_r$ and indices $p_1, \cdots, p_r$. Regard $\mathbf{x} = \{x^j\}$ as a column-vector and $\mathbf{a} = \{a_j\}$ as a row-vector. Then (3) may be written

$$E\{(t - \mathbf{ax})^{-p}\} = \prod_{i=1}^{r} (t - \mathbf{ac}_i)^{-p_i} \qquad (\text{im } t \neq 0). \tag{8}$$

Let $\mathbf{M}$ be any $m \times n$ matrix with real coefficients and $\mathbf{v}$ any real m-dimensional column-vector, and define the m-dimensional column-vectors $\boldsymbol{\xi}$ and $\boldsymbol{\gamma}_i$ by means of the equations

$$\boldsymbol{\xi} = \mathbf{Mx} + \mathbf{v}, \qquad \boldsymbol{\gamma}_i = \mathbf{Mc}_i + \mathbf{v} \qquad (i = 1, \cdots, r). \tag{9}$$

Next, let $\boldsymbol{\alpha}$ be an arbitrary real m-dimensional row-vector and τ an arbitrary non-real scalar parameter, and write $\mathbf{a} = \boldsymbol{\alpha}\mathbf{M}$, $t = \tau - \boldsymbol{\alpha}\mathbf{v}$. Then it follows that

$$\begin{aligned} \tau - \boldsymbol{\alpha\xi} &= \tau - \boldsymbol{\alpha}(\mathbf{Mx} + \mathbf{v}) = t - \mathbf{ax}, \\ \tau - \boldsymbol{\alpha\gamma}_i &= \tau - \boldsymbol{\alpha}(\mathbf{Mc}_i + \mathbf{v}) = t - \mathbf{ac}_i \quad (i = 1, \cdots, r) \end{aligned} \tag{10}$$

Comparing (10) with (8), it follows that $\boldsymbol{\xi}$, defined in (9), is an m-dimensional β-variate with vertices $\boldsymbol{\gamma}_1, \cdots, \boldsymbol{\gamma}_r$ and indices $p_1, \cdots, p_r$.

In particular, taking $\mathbf{v} = \mathbf{0}$, we deduce

LEMMA 2. *If there exists a β-distribution with indices $p_1, \cdots, p_r$ and coordinate* *matrix* $\mathbf{C}$, *then there exists a* β*-distribution with indices* $p_1, \cdots, p_r$ *and coordinate matrix* $\mathbf{MC}$, *where* $\mathbf{M}$ *is any real matrix such that the product* $\mathbf{MC}$ *is defined,* which has the immediate

COROLLARY. *If there exists a* β*-distribution with indices* $p_1, \cdots, p_r$ *and coordinate matrix* $\mathbf{I} = \mathbf{I}_{(r)}$ = *the unit matrix of order* r, *then there exists a* β*-distribution with indices* $p_1, \cdots, p_r$ *and coordinate matrix* $\mathbf{M}$, *where* $\mathbf{M}$ *is any real matrix with* r *columns.*

It has been proved that the class of β-distributions is invariant under the transformation (9), for any real $m \times n$ matrix $\mathbf{M}$ and any real m-dimensional column-vector $\mathbf{v}$. This transformation includes the following particular cases:

(i) $\mathbf{M}$ non-singular. This, the general *non-singular linear transformation,* amounts to making an arbitrary choice of cartesian coordinate axes (not necessarily rectangular). It may also be regarded as a translation, dilation and generalized "rotation" of the distribution, the axes being kept fixed.

(ii) $\mathbf{M} = \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix}$, $\mathbf{v} = \mathbf{0}$. This is the process of *embedding* the β-distribution in a space of higher dimension, giving, of course, a singular distribution in the new space.

(iii) $\mathbf{M} = [\mathbf{I}\ \mathbf{0}]$, $\mathbf{v} = \mathbf{0}$. This takes the *marginal distribution* of a given set of the original joint variates, and is a particular case of

(iv) a *parallel projection,* for which $\mathbf{M}^2 = \mathbf{M}$, $\mathbf{v} = \mathbf{0}$. (In order to include (iii) in this case, we must regard the marginal distribution as a singular distribution in the whole space, which is done by adding $n - m$ rows of zeros to the matrix $\mathbf{M} = [\mathbf{I}\ \mathbf{0}]$).

5. Boundedness and uniqueness. Case (iii) of Section 4 shows that the marginal distribution of any single component x^j of an n-dimensional β-variate $\mathbf{x}$ is itself a unidimensional β-distribution and hence (by Lemma 1, Section 3) it is a bounded distribution. But this implies that the distribution of $\mathbf{x}$ is itself bounded, and hence we have

LEMMA 3. *Any* β*-distribution is bounded.*

Write (3) in the form

$$E\left\{\left(1 - \sum_{j=1}^{n} b_j x^j\right)^{-p}\right\} = \prod_{i=1}^{r}\left(1 - \sum_{j=1}^{n} b_j c_i^j\right)^{-p_i}, \tag{11}$$

where $b_j = a_j/t$. Then, for sufficiently small values of max $|b_j|$, each side of (11) can be expanded as an absolutely convergent series in powers and products of the b_j. Equating coefficients determines all the moments[2] of the joint distribution of $x^1, \cdots, x^n$. Applying Lemma 3 and using the fact [5] that a distribution known to be bounded is uniquely determined by its moments, we have

LEMMA 4. *If there exists any* β*-distribution with a given set of indices and vertices (i.e. any distribution satisfying* (3)*), then there is only one such distribution.*

[2] e.g. the mean of a β-distribution is the centroid of masses p_i at the vertices $\mathbf{c}_i$.

6. Basic β-distributions. Consider the (singular) joint distribution of $x^1, \cdots, x^r$ which is distributed over all positive values of the variates such that $x^1 + x^2 + \cdots + x^r = 1$, with density proportional to $\prod_{j=1}^r (x^j)^{p_j - 1}$, where $p_1, \cdots, p_r$ are given positive constants. Formally we have

$$dF = C(x^1)^{p_1-1}(x^2)^{p_2-1} \cdots (x^r)^{p_r-1}\, dx^1\, dx^2 \cdots dx^{r-1} \tag{12}$$

over the range

$$x^j > 0 \; (j = 1, \cdots, r), \qquad \sum_{j=1}^r x^j = 1, \tag{13}$$

where $C = C(p_1, \cdots, p_r) = \Gamma(p)/\prod \Gamma(p_j)$ (writing, as usual, $p = \sum p_j$).

For this distribution, choosing $p = \sum p_j$ as the exponent of the fundamental transformation, it follows from (2) that

$$\begin{aligned}\phi &= \phi_p(t; a_1, \cdots, a_r) \\ &= C \iint \cdots \int \left(t - \sum_{j=1}^r a_j x^j\right)^{-\Sigma p_j} \prod_{j=1}^r (x^j)^{p_j - 1}\, dx^1\, dx^2 \cdots dx^{r-1}\end{aligned} \tag{14}$$

where (13) defines x^r in terms of $x^1, \cdots, x^{r-1}$ and the integration is extended over all positive values of the variables such that $x^1 + x^2 + \cdots + x^{r-1} < 1$.

Hence, if $\partial\phi\,(m_1, m_2, \cdots, m_r; 0)$ denotes the result of differentiating ϕ m_j times with respect to a_j $(j = 1, \cdots, r)$ and then putting $a_j = 0$ $(j = 1, \cdots, r)$, we have, on writing m for $\sum m_j$,

$$\begin{aligned}&\partial\phi(m_1, \cdots, m_r; 0) \\ &\quad = \frac{C\Gamma(p+m)}{\Gamma(p)} \iint \cdots \int t^{-p-m} \prod_{j=1}^r (x^j)^{p_j + m_j - 1}\, dx^1 \cdots dx^{r-1}\end{aligned} \tag{15}$$

taken over the same range as the integral (14). This yields at once

$$\partial\phi(m_1, m_2, \cdots, m_r; 0) = \prod_{j=1}^r \frac{\Gamma(p_j + m_j)}{\Gamma(p_j)}\, t^{-p-m} \tag{16}$$

On comparing these partial derivatives (for all values of the m_j) with the corresponding derivatives of the function on the right-hand side of (17) below, and using the principle of analytic continuation, we reach the conclusion that

$$\phi_p(t; a_1, \cdots, a_r) = \prod_{j=1}^r (t - a_j)^{-p_j} \tag{17}$$

and hence[3] we have proved

LEMMA 5. *The distribution defined in* (12) *and* (13) *is an r-dimensional β-distribution with indices $p_1, \cdots, p_r$ and coordinate matrix $\mathbf{I} = \mathbf{I}_{(r)}$ = the unit matrix of order r.*

It is convenient also to adopt the

[3] This proof, suggested by a referee, is much shorter than my original proof. This accounts for the absence of equations numbered (18) to (25).

DEFINITION. Any β-distribution whose coordinate matrix is a unit matrix will be called a *basic β-distribution*. By Lemma 4 of Section 5 it is clear that any basic β-distribution is defined by (12) and (13) for some set of indices $p_1, \cdots, p_r$.

7. Proof of Theorems 1 and 2. By Lemma 5 and the Corollary to Lemma 2 (Section 4), there exists a β-distribution with any specified set of vertices and indices. By Lemma 4 (Section 5) this distribution is unique, and so we have completed the proof of Theorem 1.

The truth of Theorem 2 for basic β-distributions follows from (12) and (13). Using the Corollary to Lemma 2 (Section 4) we see that *any* β-distribution is obtained from a basic β-distribution by a transformation of the type (9)—indeed, we may take $\mathbf{v} = \mathbf{0}$. Hence it only remains to prove that the properties of a β-distribution asserted in Theorem 2 remain invariant under any transformation of type (9) with $\mathbf{v} = \mathbf{0}$.

Now the ordinary theory of canonical matrices shows that any matrix $\mathbf{M}$ may be expressed as a product $\mathbf{PABQ}$, where the matrices $\mathbf{P}$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{Q}$ correspond to the particular cases of (9) described in (i), (ii), (iii), (i) of Section 4, and it is easily seen that the properties in question remain invariant under the transformations (i), (ii) and (iii) of Section 4. Hence it is true that the relevant properties remain invariant under the transformation (9) (with $\mathbf{v} = \mathbf{0}$), and so the proof of Theorem 2 is complete.

8. The univariate case. Analytic continuation. In Sections 9–11, I shall prove the three theorems enunciated in this paragraph, with their corollaries.

THEOREM 4. *Let $F(x)$ be the distribution function of a unidimensional β-distribution with vertices $c_1, \cdots, c_r$, indices $p_1, \cdots, p_r$, and exponent $p = \sum p_j$, and suppose that $r \geqq 2$ and $c_1 < c_2 < \cdots < c_r$. Then, for each value of k $(1 \leqq k \leqq r - 1)$ there exists a unique function $G_k(z)$ such that*

(i) *$G_k(x) = F(x)$ for $c_k < x < c_{k+1}$, and*

(ii) *$G_k(z)$ is a regular function of z throughout the complex plane, cut along the real axis from $-\infty$ to c_k and from c_{k+1} to $+\infty$.*

Further properties of the derivative $G_k'(z)$ of $G_k(z)$, and of its pth derivative $G_k^{(p)}(z)$ when p is an integer, are given in

THEOREM 5.

(i) *If $r = 2$,*

$$(26) \quad G_1'(z) = \frac{\Gamma(p)}{\Gamma(p_1)\Gamma(p_2)} \left(\frac{(z - c_1)(c_2 - z)}{c_2 - c_1} \right)^{p-1} (z - c_1)^{-p_1}(c_2 - z)^{-p_2};$$

(ii) *If p is an integer* (e.g. $p = 1$),

$$(27) \quad G_k^{(p)}(z) = (-1)^{p-1}(p - 1)!\,\pi^{-1} \sin \alpha\pi \prod_{j=1}^{k} (z - c_j)^{-p_j} \prod_{j=k+1}^{r} (c_j - z)^{-p_j},$$

where $\alpha = \sum_{j=1}^{k} p_j$.

The asymptotic properties of the function $G_k(z)$ are described in

THEOREM 6. *If the z-plane is cut along the real axis from* $-\infty$ *to* c_k *and from* c_{k+1} *to* $+\infty$, *then, as* $|z| \to \infty$,

$$G_k'(z) \sim A_k(z - c_k)^{Q-1}(c_{k+1} - z)^{P-1} \tag{28}$$

where $A_k > 0$, $P = \sum_{j=1}^{k} p_j$, $Q = \sum_{j=k+1}^{r} p_j$, *and we use that branch of the function which is positive for* $c_k < z < c_{k+1}$.

COROLLARY 1. $G_k'(z) \sim B_k z^{p-2}$ *as* $|z| \to \infty$, *where* $B_k \neq 0$.

COROLLARY 2. *Theorem 3 is true in the univariate case.*

COROLLARY 3. *If* $F_1(x)$, $F_2(x)$ *are the distribution functions of two unidimensional* β*-distributions and if, at an infinity of distinct points* x, *either* $F_1'(x) = AF_2'(x) \neq 0$ *or* $F_1(x) = AF_2(x) + B \neq 0$ *or* 1, *then the two* β*-distributions have the same exponent.*

I have been unable to prove an analogue of Corollary 3 in the multivariate case, and I therefore confine myself to offering the

CONJECTURE. *If two n-dimensional* β*-distributions have the same non-zero d-dimensional density throughout some d-dimensional region, then the two* β*-distributions have the same exponent.*

9. Proof of Theorem 4. Let $F(x)$ satisfy the conditions of Theorem 4, and select an integer k such that $1 \leqq k \leqq r - 1$.

We shall have occasion to consider two complex variables, z and t. The z-plane will be cut along the real axis from $-\infty$ to c_k and from c_{k+1} to $+\infty$, and (for any given value of z) the t-plane will have straight cuts from the origin $t = 0$ to each of the points $t = c_j - z$ $(j = 1, \cdots, r)$.

Let Γ_k be a contour in the cut t-plane which starts and finishes at $t = 0$ and encloses (positively) all the points $c_j - z$ $(1 \leqq j \leqq k)$ and none of the points $c_j - z$ $(k + 1 \leqq j \leqq r)$. We now define

$$G_k(z) = \frac{1}{2\pi i} \int_{\Gamma_k} \prod_{j=1}^{r} (t + z - c_j)^{-p_j} t^{p-1}\, dt, \tag{29}$$

using that branch of the integrand (in the cut t-plane) which $\sim t^{-1}$ as $|t| \to \infty$. Clearly $G_k(z)$ is a regular function of z in the cut z-plane.

Now the distribution $F(x)$, being a β-distribution, is bounded, and so we may apply equation (4) of [3] and indeed, if $c_k < x < c_{k+1}$, the path of integration may be deformed into Γ_k. Comparing with (29) we find $F(x) = G_k(x)$ for almost all values of x such that $c_k < x < c_{k+1}$. Using the fact that $G_k(x)$ is continuous and $F(x)$ is monotonic, this result may be extended to all values of x in the range (c_k, c_{k+1}), and so $G_k(z)$ is the required analytic continuation of $F(x)$—it is certainly unique, by the elementary theory of analytic continuation.

10. Proof of Theorem 5.

(i) *The case* $r = 2$. Starting from the basic β-distribution defined in (12) and (13), we make the substitution $x - c_1 = (c_2 - c_1)x^2$, so that $c_2 - x = (c_2 - c_1)x^1$. This yields (26), which therefore gives the density over (c_1, c_2) of the unidimensional β-distribution with vertices c_1, c_2 and indices p_1, p_2.

(ii) *The case* $p = 1$. It is convenient to write

$$\Pi = \Pi(t, z) = \prod_{j=1}^{r} (t + z - c_j)^{-p_j}.$$

Then we have, from (29),

$$(30) \qquad G_k'(z) = \frac{1}{2\pi i}\int_{\Gamma_k} \frac{\partial \Pi}{\partial z}\,dt = \frac{1}{2\pi i}\int_{\Gamma_k} \frac{\partial \Pi}{\partial t}\,dt = \frac{1}{2\pi i}[\Pi]_{\Gamma_k}$$

which is the difference between two branches at $t = 0$ of the function $\Pi(t, z)$. On evaluating this difference we reach (27).

(iii) p *an integer* > 1. Integrating by parts, we find

$$(31) \qquad G_k'(z) = \frac{1}{2\pi i}\int_{\Gamma_k} \frac{\partial \Pi}{\partial z} t^{p-1}\,dt = \frac{1-p}{2\pi i}\int_{\Gamma_k} \Pi(t, z)t^{p-2}\,dt.$$

Repeating the process p times and using (30) at the last step, we again reach (27) in this more general case.

11. Proof of Theorem 6 and its corollaries.

(i) *The case* $p > 1$. Consider the transformation $t \Rightarrow w$ defined by

$$(32) \qquad t = -z^2/(w + z).$$

The points $t = c_j - z$ are transformed into points $w = c_j + O(|z|^{-1})$ and the point $t = 0$ goes to infinity. Notice also that Γ_k leaves the first k singularities of the integrand on the left, and the remainder on the right—hence the same is true of the path of integration for w. It follows that the substitution of (32) in (31) (which still holds) yields the result, valid for all sufficiently large values of $|z|$,

$$(33) \qquad G_k'(z) = \frac{(1-p)z^{p-2}}{2\pi i}\int_{-i\infty}^{i\infty} \prod_{j=1}^{r} (c_j - w + c_j w z^{-1})^{-p_j}\,dw,$$

where there are cuts in the w-plane approximating to the segments of the real axis from $-\infty$ to c_k and from c_{k+1} to $+\infty$, and we use that branch of the integrand which has the value z^{-p} when $w = -z$.

By splitting the range of integration into three parts, defined by $|w| \gtrless |z|^{1/3}$, it may be seen that the correct asymptotic value, as $|z| \to \infty$, for the integral in (33) is given[4] by substituting $z^{-1} = 0$. Making this substitution and paying due regard to the appropriate branch of the many-valued functions involved, we find that, if $p > 1$, (33) yields (28), provided that

$$(34) \qquad A_k = \frac{p-1}{2\pi i}\int_{-i\infty}^{i\infty} \prod_{j=1}^{k} (w - c_j)^{-p_j} \prod_{j=k+1}^{r} (c_j - w)^{-p_j}\,dw$$

where the w-plane is cut along the real axis from $-\infty$ to c_k and from c_{k+1} to $+\infty$.

[4] More generally, an asymptotic series for (33) is given by expanding the integrand as a power series in z^{-1} and integrating term by term.

It is convenient to write (34) in the form

$$A_k = \frac{p-1}{2\pi i}\int e^{-h(w)}\,dw, \tag{35}$$

where the contour extends to $\pm i\infty$ in the cut w-plane, and

$$h(w) = \textstyle\sum_{j=1}^{k} p_j \log(w - c_j) + \sum_{j=k+1}^{r} p_j \log(c_j - w),$$

taking real values if $c_k < w < c_{k+1}$.

The integral (35) possesses a unique saddle-point $w = w_k$ in the cut plane, defined by $h'(w_k) = 0$, i.e.

$$\sum_{j=1}^{r} \frac{p_j}{w_k - c_j} = 0, \qquad c_k < w_k < c_{k+1}. \tag{36}$$

Taking the integral along the curve of steepest descent from this saddle-point we have, on the path of integration, $\operatorname{im}(h(w)) = 0$. Hence the integrand is positive and decreasing away from the saddle-point, and this immediately yields the required result $A_k > 0$.

By differentiating (33) and substituting from (34) we reach the further result, if $p > 1$, $p \neq 2$ and $|z| \to \infty$.

$$G_k''(z) \sim \frac{p-2}{z} A_k (z - c_k)^{Q-1}(c_{k+1} - z)^{P-1} \qquad (A_k > 0) \tag{37}$$

(ii) *The case* $p < 1$. We first establish

LEMMA 6. *If* $A_k = A_k(c_1, \cdots, c_r; p_1, \cdots, p_r)$ *is defined by*

$$A_k = \lim_{|z|\to\infty} (z - c_k)^{1-Q}(c_{k+1} - z)^{1-P} G_k'(z), \tag{38}$$

where P and Q are defined in (28), *then, if* $0 < p < 1$,

$$\frac{\partial A_k}{\partial c_s} < 0 \ (s = 1, \cdots, k-1), \qquad \frac{\partial A_k}{\partial c_s} > 0 \ (s = k+2, \cdots, r),$$

the notation implying existence, reality and finiteness.

PROOF. Consider the case $1 \leqq s \leqq k - 1$. Let $F^*(x)$ be the β-distribution with vertices and indices c_j^*, p_j^*, where $c_j^* = c_j$ (all j), $p_j^* = p_j$ $(j \neq s)$, $p_s^* = p_s + 1$, so that $P^* = P + 1$, $Q^* = Q$, $p^* = p + 1$, $1 < p^* < 2$.

Then, using (31) and (29),

$$G_k^{*\prime}(z) = \frac{1-p^*}{2\pi i}\int_{\Gamma_k} (t + z - c_s)^{-1} \prod_{j=1}^{r} (t + z - c_j)^{-p_j} t^{p-1}\,dt = \frac{-p}{p_s}\frac{\partial G_k}{\partial c_s}$$

Differentiating with respect to z, multiplying by $(z - c_k)^{1-Q}(c_{k+1} - z)^{1-P}$, letting $|z| \to \infty$ and substituting from (38) and (37) yields $(-p/p_s)(\partial A_k/\partial c_s) = -(p^* - 2)A_k^* = (1 - p)A_k^* > 0$, as required. The corresponding result when $k + 2 \leqq s \leqq r$ is proved in a precisely similar manner.

From Lemma 6 we deduce immediately

$$A_k(c_k, \cdots, c_k, c_{k+1}, \cdots, c_{k+1}; \quad p_1, \cdots, p_r) < A_k(c_1, \cdots, c_r; p_1, \cdots, p_r) < \infty, \tag{39}$$

where, on identifying coincident vertices, we have

$$A_k(c_k, \cdots, c_k, c_{k+1}, \cdots, c_{k+1}; p_1, \cdots, p_r) = A(c_k, c_{k+1}; P, Q), \tag{40}$$

dropping the subscript k from A_k in the case $r = 2$.

Now Theorem 5 (i), in conjunction with (38), shows that

$$A(c_k, c_{k+1}; P, Q) = (c_{k+1} - c_k)^{1-p}\Gamma(p)/(\Gamma(P)\Gamma(Q)) > 0,$$

which, with (39) and (40), proves that $A_k(c_1, \cdots, c_r; p_1, \cdots, p_r) > 0$ if $0 < p < 1$. Since the other cases have been dealt with in Section 11 (i) and Theorem 5 (ii), this completes the proof of Theorem 6.

(iii) *Proof of the corollaries.* Corollary 1 is immediate. To prove Corollary 2 we remark that if $G(z)$ is any nontrivial analytic continuation of $F(x)$, then, by Corollary 1, the limit $\lim_{|z|\to\infty} (\log G'(z)/\log z)$ exists and is equal to $p - 2$. Hence this limit uniquely determines the value of p.

Corollary 3. Since a β-distribution has only a finite number of vertices, an infinity of the given points x must occur in an interval lying between two consecutive vertices of each β-distribution. The corresponding analytic continuations bear a simple relation to each other, and determine the same value for p. This proves Corollary 3.

12. Proof of Theorem 3. If $F(x^1, \cdots, x^n)$ is a β-distribution which is not concentrated at a single point, then (by Section 4 (iii)) the same is true of the marginal distribution of at least one of the individual component variates $x^1, \cdots, x^n$, and furthermore the resulting unidimensional β-distribution has the same exponent as the original. Theorem 3 now follows immediately from Corollary 2 to Theorem 6 (Section 8).

13. Examples of β-distributions. In addition to the basic β-distributions defined in section 6 the following examples may be mentioned.

(i) Starting with the 2-dimensional basic β-distribution, and using the transformation (iii) of Section 4, we take the marginal distribution of x^1. Writing x, p, q for x^1, p_1, p_2, we find that this yields the familiar β-distribution

$$dF = Cx^{p-1}(1 - x)^{q-1}\,dx \qquad (0 < x < 1).$$

The case $p = q = \frac{1}{2}$ is known [7] as the "Arc Sine Law".

(ii) Taking a fixed non-degenerate n-dimensional simplex in n-dimensional space as the simplex of reference, and an arbitrary interior point as the unit-point, we may establish a system of $n + 1$ homogeneous coordinates $\xi^1, \cdots, \xi^{n+1}$. Writing $x^j = \xi^j/\sum\xi^i$, we have a generalization of 2-dimensional "areal" coordinates, subject to the identical relation $\sum x^j = 1$. Then a distribution extended over the interior of the simplex of reference with density proportional to $\prod (x^j)^{p_j-1}$ $(p_j > 0)$ is a β-distribution, since it is obtained from the $(n + 1)$-dimensional basic β-distribution by the transformations (iii) and (i) of section 4.

(iii) A particular case of (ii), given by taking $p_j = 1$ for all j and using the transformation (ii) of Section 4, is a uniform distribution over any non-degenerate simplex in space of any dimension.

(iv) An explicit expression for the unidimensional β-distribution with indices p, q, r and vertices a, b, $c(a < b < c)$ is most easily obtained as the marginal distribution of the corresponding 2-dimensional distribution over the non-degenerate triangle with vertices $(a, 0)$, (b, h), $(c, 0)$. Using a result given on p. 293 of [6] we find that, in the range $a < x < b$, the density of the distribution is given by

$$\frac{dF}{dx} = \frac{\Gamma(p + q + r)}{\Gamma(p)\Gamma(q + r)} \frac{(x - a)^{q+r-1}(c - x)^{p-1}}{(b - a)^q(c - a)^{p+r-1}} F(1 - p, q; q + r; u)$$

where $u = [(c - b)/(b - a)][(x - a)/(c - x)]$ and $F(\alpha, \beta; \gamma; t)$ is the ordinary hypergeometric function. There is; of course, a similar formula applicable in the range $b < x < c$.

The symmetry of the original definition is restored by observing that in any range the density can be expressed in terms of a solution of Riemann's P-equation as follows:

$$\frac{dF}{dx} \propto (x - a)^p(x - b)^q(x - c)^r P \begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & x \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$$

provided that $\alpha' + \beta + \gamma = p, \alpha + \beta' + \gamma = q, \alpha + \beta + \gamma' = r, \alpha + \beta + \gamma = p + q + r - 1$.

It should be noticed that, if $p = q + r = 1$, this distribution is uniform over the interval (a, b), but not over the interval (b, c). Hence this gives a nontrivial example of Corollary 3 to Theorem 6 (Section 8). We may also specifically mention the case $p = q = r = 1$, which gives

(v) the ordinary (univariate) triangular distribution.

(vi) Certain distributions arising in connection with the random division of an interval, such as that studied by Fisher [2] in 1929, are β-distributions—indeed, several investigations of this problem ([4], [1], [3]) take (3) as their starting-point (with integer values for the p_j). Another approach, adumbrated in the summary (Section 0) is to apply the transformation (iv) of Section 4 to the case (iii) of this paragraph, the transforming matrix **M** being of unit rank.

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