SOME OPTIMUM WEIGHING DESIGNS

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1. Introduction and summary. Suppose we are given N objects to be weighed in N weighings with a chemical balance having no bias. Let

$x_{ij} = 1$ if the jth object is placed in the left pan in the ith weighing,
$= -1$ if the jth object is placed in the right pan in the ith weighing,
$= 0$ if the jth object is not weighed in the ith weighing.

The Nth order matrix $X = (x_{ij})$ is known as the design matrix. Also let y_i be the result recorded in the ith weighing, ϵ_i be the error in this result and w_j be the true weight of the jth object, so that we have the N equations

$$x_{i1}w_1 + x_{i2}w_2 + \cdots + x_{iN}w_N = y_i + \epsilon_i, \qquad i = 1, \cdots, N.$$

We assume X to be a non-singular matrix. The method of Least Squares or theory of Linear Estimation gives the estimated weights $(\hat{w}_i)$ by the equation

$$\hat{w} = (X'X)^{-1}X'Y,$$

where Y is the column vector of the observations and $\hat{w}$ is the column vector of the estimated weights.

If σ^2 is the variance of each weighing, then

$$\text{Var}(\hat{w}) = (X'X)^{-1}\sigma^2 = (c_{ij})\sigma^2,$$

where (c_{ij}) is the inverse matrix of $(X'X)$.

An expository article reviewing the work done in weighing designs is given by Banerjee [2].

Kishen [4] treats the reciprocal of the increase in variance resulting from the adoption of any design other than the most efficient design, with mean variance σ^2/N, as the efficiency of the design. This efficiency can be measured by

$$1/\sum_{i=1}^{N} c_{ii}.$$

Mood [5] gives an alternative definition for the best weighing design. In his view the best weighing design should give the smallest confidence region in the $\hat{w}_i (i = 1, \cdots, N)$ space for the estimates of the weights. Hence a design will be called best if the determinant of the matrix (c_{ij}) is minimised.

In this paper we follow Kishen's definition in obtaining the best weighing designs.

Hotelling [3] proved that for the best weighing design $c_{ii} = 1/N$ and $c_{ij} = 0$ $(i \neq j)$. The weighing designs for which $c_{ii} = 1/N$ and $c_{ij} = 0$ are best in the sense of both Kishen and Mood. Later Mood proved that the above property is

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satisfied by Hadamard matrices. Plackett and Burman [6] have constructed Hadamard matrices, H_N, up to and including $N = 100$, excepting $N = 92$. It may be remarked here that a necessary condition for the existence of H_N is $N \equiv 0 \pmod 4$, with the exception of $N = 2$. It is not known whether this condition is sufficient or not.

In this paper, the best weighing designs are obtained in the cases (i) N is odd and (ii) $N \equiv 2 \pmod 4$ subject to the conditions:

i) The variances of the estimated weights are equal; and

ii) The estimated weights are equally correlated.

The 2nd condition here is the same as that of Banerjee [1].

2. Some theorems relating to the best weighing designs. With the conditions made above $(X'X)$ matrix takes the form

$$\begin{bmatrix} r & \lambda & \cdots & \lambda \\ \lambda & r & \cdots & \lambda \\ \vdots & & & \\ \lambda & \lambda & \cdots & r \end{bmatrix}. \tag{2.1}$$

Now

$$\det(X'X) = \{\det(X)\}^2,$$
$$= (r - \lambda)^{N-1}\{r + \lambda(N-1)\}.$$

Therefore,

$$\det(X) = \pm(r - \lambda)^{(N-1)/2}\{r + \lambda(N-1)\}^{\frac{1}{2}}. \tag{2.2}$$

The det (X) is real and not equal to zero. Hence we have

$$r > \lambda \tag{2.3}$$

and

$$r + \lambda(N-1) > 0. \tag{2.4}$$

Relation (2.4) holds good when λ is non negative and $\lambda = -1$. In the latter case $r = N$.

Therefore, in this paper, we consider only the values of r and λ satisfying

$$r > \lambda \geqq 0, \qquad \text{or } r = N, \qquad \lambda = -1.$$

When the matrix $(X'X)$ is of the form (2.1), the variance of the estimated weight is

$$\frac{\{r + \lambda(N-2)\}\sigma^2}{(r-\lambda)\{r + \lambda(N-1)\}}. \tag{2.5}$$

Therefore, the efficiency of the weighing design is

$$\frac{(r-\lambda)\{r + \lambda(N-1)\}}{N\{r + \lambda(N-2)\}} = f(r, \lambda), \quad \text{say}. \tag{2.6}$$

LEMMA 2.1.

i) *Let* $r = N$. *Then* λ *cannot be even (including zero) when* N *is odd and* λ *cannot be odd (including* -1*) when* N *is even.*

ii) *Let* $r = N - 1$. *Then* λ *cannot be even (including zero) when* N *is odd and* λ *cannot be odd when* N *is even.*

PROOF. Let x_i and x_j be any two column vectors of the design matrix X.

i) When $r = N$, $x_i'x_j$ will have N terms each term being either $+1$ or -1. Since $x_i'x_j = \lambda$, amongst the N terms $\{N - |\lambda|\}$ terms sum to zero. Hence N and $|\lambda|$ should either be odd or even and the statement follows.

ii) When $r = N - 1$, $x_i'x_j$ will have N terms each term being $+1$ or -1 or 0. Since $x_i'x_j = \lambda$, amongst the N terms $(N - \lambda)$ terms sum to zero. If N is odd and λ is even $(N - \lambda)$ will become odd and the $(N - \lambda)$ terms cannot sum to zero unless there is a single zero term. x_i and x_j will contribute a single zero term to $x_i'x_j$ when and only when the zeros of x_i and x_j are in the same row. This is also the case for any two columns of X. Hence, if N is odd and λ is even, we get a row of zeros in X, and in this case det $(X) = 0$, contrary to our assumption. Therefore, λ cannot be even when N is odd.

Similarly we can show that λ cannot be odd when N is even.

THEOREM 2.1. *When* N *is odd the best weighing design* X *is that for which*

$$X'X = \begin{bmatrix} N & 1 & \cdots & 1 \\ 1 & N & \cdots & 1 \\ \vdots & & & \\ 1 & 1 & \cdots & N \end{bmatrix}.$$

PROOF.

$$f(N, 1) - f(r, \lambda)$$

$$= \frac{2N-1}{2N} - \frac{(r-\lambda)\{r + \lambda(N-1)\}}{N\{r + \lambda(N-2)\}}, \qquad r, \lambda \text{ are positive, } r > \lambda, \text{ or } r = N, \lambda = -1.$$

$$(2.7) \quad = \frac{(2N-1)(N-2)\lambda + (2N-1)r - 2r^2 + 2r\lambda - 2(r-\lambda)(N-1)\lambda}{2N\{r + \lambda(N-2)\}},$$

$$= \frac{(N-1)\lambda(2N - 2r + 2\lambda - 1) + (2N - 2r - 1)(r - \lambda)}{2N\{r + \lambda(N-2)\}} > 0,$$

when $r < N$.

$$(2.8) \qquad f(N, 1) - f(r, \lambda) = \frac{\lambda(\lambda - 1)(N-2) + N(\lambda^2 - 1)}{2N\{r + \lambda(N-2)\}},$$

when $r = N$. (2.8) is again greater than zero for all values of λ excepting zero in which case it is less than zero. But Lemma 2.1 proves that λ cannot be zero since N is odd. Hence the best weighing design in this case has efficiency $f(N, 1)$. The theorem is thus proved.

THEOREM 2.2. *When* $N \equiv 2 \pmod 4$ *the best weighing design* X *is that for which*

$$X'X = \text{diag}\{(N-1), (N-1), \cdots, N \text{ terms}\}.$$

PROOF.

$$f(N-1,0) - f(r,\lambda) = \frac{(N-1)}{N} - \frac{(r-\lambda)\{r+\lambda(N-1)\}}{N\{r+\lambda(N-2)\}},$$

r, λ are positive, $r > \lambda$, or $r = N, \lambda = -1$.

$$= \frac{r(N-1) + (N-1)(N-2)\lambda - r(r-\lambda) - (N-1)(r-\lambda)\lambda}{N\{r+\lambda(N-2)\}},$$

$$= \frac{r(N-r+\lambda-1) + \lambda(N-1)(N-r+\lambda-2)}{N\{r+\lambda(N-2)\}} > 0,$$

when $r < N$.

$$f(N-1,0) - f(r,\lambda) = \frac{N(\lambda-1) + (N-1)(\lambda-2)\lambda}{N\{r+\lambda(N-2)\}}, \tag{2.9}$$

when $r = N$. (2.9) is greater than zero when $\lambda \geqq 2$ and $\lambda = -1$ and it is less than zero when $\lambda = 0$ or 1.

Lemma 2.1 proves that $\lambda = 1$ cannot exist and it is known that the Hadamard matrix cannot exist in this case and λ cannot be zero.

Therefore, the best weighing design in this case has efficiency $f(N-1, 0)$. The theorem is hence proved.

3. P_N matrices.

DEFINITION 3.1. *A P_N matrix is an Nth order matrix with elements $+1$ and -1 such that*

$$P_N'P_N = (N-1)I_N + E_{NN},$$

where I_N is the identity matrix of order N and E_{NN} is an Nth order matrix with positive unit elements everywhere.

It is obvious from theorem 2.1 above that the P_N matrix is the best weighing design whenever it exists and N must be odd.

THEOREM 3.1. *A necessary condition for the existence of P_N is that*

$$N = \frac{d^2+1}{2}$$

where d is an odd integer.

PROOF. det $\{P_N'P_N\} = \{\det P_N\}^2 = (2N-1)(N-1)^{N-1}$. Therefore det $(P_N) = (2N-1)^{\frac{1}{2}}(N-1)^{(N-1)/2}$. Also since P_N is a matrix with integral elements det (P_N) is an integer.

Hence $(2N-1)$ should be a perfect square. Let

$$2N - 1 = d^2$$

$$N = \frac{d^2+1}{2}$$

Since N is an integer, d is an odd integer and thus the theorem is proved.

THEOREM 3.2. *If a Balanced Incomplete Block Design exists with parameters*

$$v^* = b^* = N, \qquad r^* = k^* = (N \pm d)/2, \qquad \lambda^* = (N \pm 2d + 1)/4,$$

then, by changing the zeros into -1's *in the incidence matrix of the incomplete block design, we get a* P_N *matrix.*

PROOF. Let the column vectors of the incidence matrix after 0's are changed to -1 be $p_1, p_2, \cdots, p_N$.

The negative contribution to $p_i'p_j = 2(r^* - \lambda^*) = (N-1)/2$ $(i, j = 1, 2, \cdots, N; i \neq j)$.

Therefore, the positive contribution to $p_i'p_j = (N+1)/2$. Hence $p_i'p_j = 1$. Thus the theorem is established.

4. S_N matrices. Williamson [7] proved that when

$$N = p^h + 1,$$

where p is an odd prime and h is a positive integer such that $p^h \equiv 1 \pmod 4$, then a symmetric matrix S_N exists such that

$$S_N'S_N = (N-1)I_N,$$

where I_N is the Nth order identity matrix. In that case the S_N matrix can be taken as our best weighing design. The construction of the S_N matrices is based on Galois Fields and the Legendre function ζ, and it is discussed in detail in [7].

5. Numerical examples. Now we construct some designs that belong to the P_N and S_N series. Among the designs given below, P_5 is proved to be the best design by Mood. A similar type of S_6 is constructed by Banerjee [1] intuitively.

$$P_5 = \begin{bmatrix} -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 \end{bmatrix}.$$

Variance of each estimated weight $= 2\sigma^2/9$.
Covariance of each pair of estimated weights $= -\sigma^2/36$.
Efficiency $= 9/10$.

$$S_6 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 & -1 & -1 \\ 1 & -1 & 1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & 1 & 0 \end{bmatrix}.$$

Variance of each estimated weight $= \sigma^2/5$.
Covariance of each pair of estimated weights $= 0$.
Efficiency $= 5/6$.

$$S_{10} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 0 & 1 & -1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 0 & -1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 0 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & 1 & 0 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 0 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & 0 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 0 & 1 \\ 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Variance of each estimated weight $= \sigma^2/9$.
Covariance of each pair of estimated weights $= 0$.
Efficiency $= 9/10$.

$$P_{13} = \begin{bmatrix} -1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 \\ -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \end{bmatrix}.$$

Variance of each estimated weight $= 2\sigma^2/25$.
Covariance of each pair of estimated weights $= -\sigma^2/300$.
Efficiency $= 25/26$.

$$S_{14} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 0 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 0 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 0 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 0 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 & 1 & 0 & 1 & -1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & 1 & 0 & 1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 0 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 0 & 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 0 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 0 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 0 \end{bmatrix}.$$

Variance of each estimated weight $= \sigma^2/13$.
Covariance of each pair of estimated weights $= 0$.
Efficiency $= 13/14$.

$$S_{18} = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\
1 & -1 & 1 & 1 & 0 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 \\
1 & 1 & -1 & 1 & 1 & 0 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 \\
1 & -1 & 1 & -1 & 1 & 1 & 0 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & 0 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 \\
1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & 0 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & 0 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\
1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & 0 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\
1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & 0 & 1 & 1 & -1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & 0 & 1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & 0 & 1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & 0 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & 0
\end{bmatrix}$$

Variance of each estimated weight $= \sigma^2/17$.
Covariance of each pair of estimated weights $= 0$.
Efficiency $= 17/18$.

$$
P_{25} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 \\
-1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 \\
-1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
-1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 \\
-1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 \\
-1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 \\
-1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
-1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1
\end{bmatrix}
$$

Variance of each estimated weight $= 2\sigma^2/49$.
Covariance of each pair of estimated weights $= -\sigma^2/1176$.
Efficiency $= 49/50$.

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