THE LIMITING DISTRIBUTION OF THE SERIAL CORRELATION COEFFICIENT IN THE EXPLOSIVE CASE II

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Introduction and summary.[1] A standard linear regression model is

$$x_t = \alpha y_t + u_t \qquad (t = 1, 2, 3, \cdots T) \tag{1}$$

where α is an unknown parameter, the y's are known parameters and the u's are NID $(0, \sigma^2)$.

The maximum likelihood estimators for α and σ^2 are

$$\hat{\alpha} = \frac{\sum x_t y_t}{\sum y_t^2}, \qquad \hat{\sigma}^2 = \frac{\sum (x_t - \hat{\alpha} y_t)^2}{T}. \tag{2}$$

The statistic

$$\frac{(\hat{\alpha} - \alpha)}{\hat{\sigma}} (\textstyle\sum y_t^2)^{\frac{1}{2}} \left(\frac{T-1}{T}\right)^{\frac{1}{2}} \tag{3}$$

then has a t distribution with $T - 1$ d.f. and its limiting distribution is $N(0, 1)$.

One approach to time-series analysis is to set $y_t = x_{t-1}$, $y_1 = x_0 =$ a constant. The model (1) is then transformed into the stochastic difference equation.

$$x_t = \alpha x_{t-1} + u_t. \qquad (t = 1, 2, \cdots, T) \tag{4}$$

The maximum likelihood estimators for α and σ^2 in (4) are

$$\hat{\alpha} = \frac{\sum x_t x_{t-1}}{\sum x_{t-1}^2}, \qquad \hat{\sigma}^2 = \frac{\sum (x_t - \hat{\alpha} x_{t-1})^2}{T} \tag{5}$$

which are exactly the values one would obtain by substituting $y_t = x_{t-1}$ in (2).

In this paper it is shown that the limiting distribution of

$$W = \frac{(\hat{\alpha} - \alpha)}{\hat{\sigma}} (\textstyle\sum x_{t-1}^2)^{\frac{1}{2}}, \tag{6}$$

which is the analogue of (3), has a limiting $N(0, 1)$ distribution, except perhaps when $|\alpha| = 1$. This result is well-known for $|\alpha| < 1$ and was proved by Mann and Wald [1] under much more general conditions. The feature of the proof presented here is that it also holds in the explosive case ($|\alpha| > 1$).

The limiting distribution. We define the quadratic forms

$$R = \frac{1}{g\sigma^2} (\textstyle\sum x_t x_{t-1} - \alpha \sum x_{t-1}^2), \qquad S = \frac{1}{g^2\sigma^2} \sum x_{t-1}^2$$

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[1] The author wishes to acknowledge several comments of the referee which simplified the presentation of these results.

where

$$g = g(T, \alpha) = \left(\frac{T}{1 - \alpha^2}\right)^{\frac{1}{2}} \quad \text{for} \quad |\alpha| < 1,$$

$$= \frac{|\alpha|^T}{\alpha^2 - 1} \quad \text{for} \quad |\alpha| > 1.$$

It has been shown [2] that the limit of the joint characteristic function of R and S is

$$\phi(u, v) = \exp\left(iv - \frac{u^2}{2}\right) \qquad \text{for } |\alpha| < 1,$$

$$= \exp\left\{\frac{x_0^2(\alpha^2 - 1)(2iv - u^2)}{2(1 + u^2 - 2iv)}\right\} \Big/ (1 + u^2 - 2iv)^{\frac{1}{2}} \qquad \text{for } |\alpha| > 1.$$

Let r and s be random variables with joint characteristic function $\phi(u, v)$. Then the limiting distribution of

$$(\hat{\alpha} - \alpha)\left(\frac{\sum x_{t-1}^2}{\sigma}\right)^{\frac{1}{2}} = \frac{R}{\sqrt{S}}$$

is the same as the distribution of $r/\sqrt{\ }$. To obtain the distribution of $r/\sqrt{s}$ we must invert $\phi(u, v)$.

For $|\alpha| < 1$ we see from the form of $\phi(u, v)$ that r is $N(0, 1)$ and Prob $(s = 1) = 1$. Therefore $r/\sqrt{\ }$ is also $N(0, 1)$.

For $|\alpha| > 1$, the joint distribution of r and s is not obvious. However, if we set

$$p = \tfrac{1}{2}x_0^2(\alpha^2 - 1),$$

we may expand $\phi(u, v)$ as

$$\phi(u, v) = e^{-p} \sum_{j=0}^{\infty} (p^j/\Gamma(j + 1)(1 + u^2 - 2iv)^{j+\frac{1}{2}}).$$

Inverting $\phi(u, v)$ first with respect to v we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivs}\phi(u, v)\, dv = e^{-p} \sum_{j=0}^{\infty} \frac{s^{j-\frac{1}{2}} \exp(-s[1 + u^2]/2)p^j}{2^{j+\frac{1}{2}}\Gamma(j + \frac{1}{2})\Gamma(j + 1)}.$$

Inverting next with respect to u we have

$$f(r, s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iur}\left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivs}\phi(u, v)\, dv\right) du,$$

$$= \frac{e^{-p}}{2\sqrt{\pi s}} \exp\left(\frac{-r^2}{2s} - \frac{s}{2}\right) \sum_{j=0}^{\infty} \frac{(ps/2)^j}{\Gamma(j + \frac{1}{2})\Gamma(j + 1)}.$$

To obtain the distribution of $r/\sqrt{s}$ we make the change of variable $w = r/\sqrt{s}$ in $f(r, s)$ and then integrate out s. We have

$$f(w, s) = \exp\left(-p - \frac{w^2}{2} - \frac{s}{2}\right)\sum_{j=0}^{\infty}\frac{(ps/2)^j}{2\sqrt{\pi s}\,\Gamma(j+\frac{1}{2})\Gamma(j+1)},$$

$$f(w) = \int_0^{\infty} f(w, s)\, ds = \frac{e^{-w^2/2}}{\sqrt{2\pi}}\, e^{-p}\sum_{j=0}^{\infty}\frac{p^j}{\Gamma(j+1)},$$

$$= \frac{e^{-w^2/2}}{\sqrt{2\pi}}.$$

Thus $r/\sqrt{s}$ is again $N(0, 1)$.

To obtain the limiting distribution of W we note that

$$\frac{\sum (x_t - \alpha x_{t-1})^2}{T} = \frac{\sum u_t^2}{T} \xrightarrow{p} \sigma^2,$$

by the law of large numbers, and therefore

$$\hat{\sigma}^2 = \frac{\sum(x_t - \hat{\alpha}x_{t-1})^2}{T} = \frac{\sum(x_t - \alpha x_{t-1})^2}{T} - \frac{(\hat{\alpha}-\alpha)^2}{T}\sum x_{t-1}^2 \xrightarrow{p} \sigma^2.$$

Hence, the limiting distribution of

$$\frac{(\hat{\alpha}-\alpha)(\sum x_{t-1}^2)^{\frac{1}{2}}}{\hat{\sigma}} = W$$

is the same as that of

$$\frac{(\hat{\alpha}-\alpha)(\sum x_{t-1}^2)^{\frac{1}{2}}}{\sigma} = \frac{R}{\sqrt{S}},$$

and hence W is $N(0, 1)$.

Applications. For large samples W approximately N (0, 1) and hence may be used to construct confidence intervals for α. For example, a symmetric 95% confidence interval for α would be

$$\hat{\alpha} - 1.96\frac{\hat{\sigma}}{(\sum x_{t-1}^2)^{\frac{1}{2}}} \leqq \alpha \leqq \hat{\alpha} + 1.96\frac{\hat{\sigma}}{(\sum x_{t-1}^2)^{\frac{1}{2}}}$$

The likelihood ratio criterion for testing the hypothesis $H : \alpha = \alpha_0$ against the alternative hypothesis $\bar{H} : \alpha \neq \alpha_0$ is

$$\lambda = \left[1 - \frac{(\hat{\alpha}-\alpha_0)^2\sum x_{t-1}^2}{\sum(x_t - \alpha_0\, x_{t-1})^2}\right]^{T/2}$$

and asymptotically

$$\lambda \cong \left(1 - \frac{W^2}{T}\right)^{T/2} \cong e^{-W^2/2}$$

$$-2\log\lambda \cong W^2.$$

Hence, the limiting distribution of $-2 \log \lambda$ is a chi-squared distribution with 1 d.f.

For testing this hypothesis a large sample critical region which might be used is

$$|W| > z_{1-p/2}$$

where p is the level of significance and $z_{1-p/2}$ is the 100 $(1 - p/2)$ percentile point of the normal distribution.

It should be noted that the above results probably do not hold for $|\alpha| = 1$

REFERENCES

[1] H. B. MANN AND A. WALD, "On the statistical treatment of linear stochastic difference equations," *Econometrica*, Vol. 11 (1943), pp. 173–200.

[2] J. S. WHITE, "The limiting distribution of the serial correlation coefficient in the explosive case I," *Ann. Math. Stat.*, Vol. 29 (1958), pp. 1188–1197.