ON THE MUTUAL INDEPENDENCE OF CERTAIN STATISTICS

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1. Summary and introduction. The results of this note yield the mutual independence of certain matrices, characteristic roots, Hotelling's T^2 or Mahalanobis' D^2 statistics, and C. R. Rao's R statistic. The result concerning the mutual independence of certain of Hotelling's T^2 statistics has been proved by K. S. Rao [1]. The results mentioned here occur (by implication and as a by-product) in [5], [6] and [7] in the course of investigations on some specific problems in statistical inference but are not explicitly stated. These results can be utilised in statistical inference and especially in simultaneous tests and simultaneous confidence interval estimation, and also in other problems.

2. Certain known results.

(2.1) Let $X{:}p \times n (n \geqq p)$ be a matrix of p rows and n columns. (A column vector $x{:}p \times 1$ is denoted as $\underset{\sim}{x}{:}p \times 1$). Let X have a distribution $f(X)$. Then XX' is symmetric positive definite.

(2.2) If $S{:}p \times p$ is symmetric positive definite, then $S = \tilde{T}\tilde{T}'$, where $\tilde{T}$ is a triangular matrix with zero's above the principal diagonal, $t_{ii} > 0$, and $t_{ji} = | A_{ji} | / \sqrt{|A_{ii}| * | A_{i-1,i-1} |}$, where

$$A_{ji} = \begin{pmatrix} s_{11} & \cdots & s_{1i} \\ \multicolumn{3}{c}{\cdots\cdots\cdots\cdots\cdots} \\ s_{i-1,1} & \cdots & s_{i-1,i} \\ s_{j1} & \cdots & s_{j1} \end{pmatrix}, \qquad j \geqq i.$$

(See [2].)

(2.3) The roots of XX' are the roots of $X'X$ except for some zero roots. (See [5], [7].)

(2.4) If X_j and Y_j are transformed to X_{j+1} and Y_{j+1} respectively, $j = 1, \cdots k$, by the following matrix transformations,

$$X_j = F_j(X_{j+1}, Y_j) \text{ and } Y_j = G_j(X_{j+1}, Y_{j+1}) \quad (j = 1, 2, \cdots k),$$

then the Jacobian of the transformation X_1 to X_{k+1} and Y_1 to Y_{k+1} is

$$J(X_1, Y_1; X_{k+1}, Y_{k+1}) = \textstyle\prod_{j=1}^{k} J(X_j; X_{j+1}) J(Y_j; Y_{j+1}).$$

(See [2], [5], [7].)

(2.5) The Jacobian of the transformation $X = AYB$ $(X{:}p \times q,\ Y{:}p \times q,\ A{:}p \times p,\ B{:}q \times q)$ is $J(X; Y) = | A |^q | B |^p$. (See [3].)

(2.6) The Jacobian of the transformation $S = GRG'$ $(S, R{:}p \times p$ symmetric matrices, $G{:}p \times p)$ is $J(S; R) = | G |^{p+1}$. (See [3].)

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(2.7) The Jacobian of the transformation $S = R^{-1}(S, R:p \times p$ symmetric matrices) is $J(S; R) = | R |^{-(p+1)}$. (See [3].)

$$\int_{\substack{S \leqq XX' \leqq S+(ds_{ij}) \\ X:p\times n \, (n \geqq p)}} \cdots \int dX = \pi^{\{2pn-p(p-1)\}/4} \prod_{i=1}^{p} \left\{ \Gamma\left(\frac{n-i+1}{2}\right) \right\}^{-1} \cdot | S |^{(n-p-1)/2} \, dS, \tag{2.8}$$

where dQ is the product of the differentials of the variables. (See [4], [5], [7].)

3. Theorem I. *If $S:p \times p$ and $X:p \times q$ are independently distributed as Wishart $(n, p; \Sigma; S)$* and $MN(0, \Sigma)$ *respectively (where $MN(0, \Sigma)$ is called multivariate normal and has a density which is a multiple of* exp $[-(\text{tr } \Sigma^{-1}XX')/2])$, *then $S_1 = S + XX'$ and $Z = \tilde{T}^{-1}X$ (where $S = \tilde{T}\tilde{T}'$) are independent with distributions given by Wishart $(n + q, p; \Sigma; S_1)$ and the density $c(I - ZZ')^{(n-p-1)/2}$ respectively. Here*

$$c = \prod_{i=1}^{p} \Gamma\left(\frac{n+q-i+1}{2}\right) \left\{ \Gamma\left(\frac{n-i+1}{2}\right) \right\}^{-1} \pi^{-pq/2}.$$

Proof. Transform by the relations $S = S_1 - XX'$ and $X = \tilde{T}Z$, where $S_1 = \tilde{T}\tilde{T}'$; then by (2.4) and (2.5), the Jacobian of the transformation is $| \tilde{T} |^q = | S_1 |^{q/2}$; also $| S | = | S_1 | \, | I - ZZ' |$.

Since S and X are independently distributed, we can easily see that the joint distribution of S_1 and Z is

$$f(S_1, Z) = f_1(S_1) f_2(Z), \tag{1}$$

where

$$\begin{aligned} f_1(S_1) &= \text{Wishart } (n + q, p; \Sigma; S_1), \\ f_2(Z) &= c \, | I - ZZ' |^{(n-p-1)/2}, \end{aligned} \tag{2}$$

c being the same as defined in the Theorem.

Corollary 1. *If $S:p \times p$, $X_i:p \times q_i$, $i = 1, 2, \cdots m$, are independently distributed as Wishart $(n, p; \Sigma; S)$ and $MN_i(0, \Sigma)$ respectively, then*

$S_m = S + \sum_{j=1}^{m} X_j X_j'$ and $Z_i = \tilde{T}_i^{-1} X_i$ (where $S_i = \tilde{T}_i \tilde{T}_i' = S + \sum_{j=1}^{i} X_j X_j'$)

are mutually independent and distributed with the respective densities Wishart $(n + e_m, p; \Sigma; S_m)$ and

$$\pi^{-pq_i/2} \prod_{j=1}^{p}$$

$$\cdot \Gamma\left(\frac{n+e_i-j+1}{2}\right) \left\{ \Gamma\left(\frac{n+e_{i-1}-j+1}{2}\right) \right\}^{-1} | I - Z_i Z_i' |^{(n+e_{i-1}-p-1)/2},$$

$$e_i = \sum_{j=1}^{i} q_j, \qquad i = 1, 2, \cdots m. \tag{3}$$

The proof can be obtained in a similar manner as above.

Corollary 2. *In Corollary 1, suppose $p \geqq q_i$. Then the distribution of*

$V_i = Z_i' Z_i = X_i'(S + \sum_{j=1}^{i} X_j X_j')^{-1} X_i$ *is given by*

$$c_1 \mid V_i \mid^{(p-q_i-1)/2} \mid I - V_i \mid^{(n+e_{i-1}-p-1)/2}, \tag{4}$$

$$c_1 = \prod_{j=1}^{q_i}$$

$$\cdot \Gamma\left(\frac{n+e_i-j+1}{2}\right)\left\{\Gamma\left(\frac{p-j+1}{2}\right)\Gamma\left(\frac{n+e_i-j-p+1}{2}\right)\right\}^{-1} \pi^{-q_i(q_i-1)/4}.$$

Proof. The distribution of V_i can at once be obtained by the use of the integral (2.8) in formula (3).

Note that the distribution of $W_i = V_i(I - V_i)^{-1} = X_i'(S + \sum_{j=1}^{i-1} X_j X_j')^{-1} X_i$ is obtained by using the Jacobian (2.6) for $(I - V_i) = (I + W_i)^{-1}$ and we find it to be

$$c_1 \mid W_i \mid^{(p-q_i-1)/2} \mid I + W_i \mid^{-(n+e_i)/2}. \tag{5}$$

Similarly if $p \leqq q_i$, the distribution of $Z_i Z_i'$ and $Z_i Z_i'(I - Z_i Z_i')^{-1}$ can be obtained from (3).

Corollary 3. *If in Corollary* 1, $q_i \equiv 1$ *for all* i, *then*

$$T_i^2 = \underset{\sim}{x}_i' \, (S + \sum_{j=1}^{i-1} \underset{\sim}{x}_j \underset{\sim}{x}_j')^{-1} \underset{\sim}{x}_i, \qquad i = 1, 2 \cdots m$$

and

$$S_m = S + \sum_{j=1}^{m} \underset{\sim}{x}_j \underset{\sim}{x}_j',$$

are mutually independent with distributions given by

$$\text{const.}\ (T_i^2)^{(p-2)/2}(1 + T_i^2)^{-(n+i)/2}, \qquad i = 1, 2, \cdots m$$

and Wishart $(m + n, p; \Sigma; S_m)$.

Corollary 4. *If* $S{:}p \times p$, $X_i{:}p \times q_i$, $i = 1, 2, \cdots m$ *are independently distributed as Wishart* $(n, p; \Sigma; S)$ *and* $MN_i(0, \Sigma)$, *then the characteristic roots of* $A_i(S + \sum_{j=1}^{i-1} A_j)^{-1}$ *are distributed independently of those of* $S_m = S + \sum_{j=1}^{m} A_j$ (*where* $A_j = X_j X_j'$).

Proof. By Corollary 1, we have Z_i and S_m, $i = 1, 2, \cdots m$, independently distributed and hence roots of $Z_i Z_i'$ and S_m are independently distributed; i.e. by the application of (2.3) we have the corollary.

Theorem 2. *Let* $S{:}p \times p$, $\underset{\sim}{x}{:}p \times 1$ *be independently distributed as Wishart* $(n, p; \Sigma; S)$ *and multivariate normal* $(\underset{\sim}{\mu}, \Sigma)$, $\underset{\sim}{\mu} \neq 0$, *respectively, and let*

$$\underset{\sim}{x}{:}\begin{pmatrix}\underset{\sim}{x}_1\\ \underset{\sim}{x}_2\end{pmatrix}\begin{matrix}q\\ p-q\end{matrix}, \quad S{:}\begin{array}{c}\begin{pmatrix}S_{11} & S_{12}\\ S_{21} & S_{22}\end{pmatrix}\begin{matrix}q\\ p-q\end{matrix}\\ \quad q \qquad p-q\end{array}, \quad \Sigma = \begin{array}{c}\begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}\begin{matrix}q\\ p-q\end{matrix}\\ \quad q \qquad p-q\end{array},$$

$$\underset{\sim}{\mu} = \begin{pmatrix}\underset{\sim}{\mu}_1\\ \underset{\sim}{\mu}_2\end{pmatrix}\begin{matrix}q\\ p-q\end{matrix}, \qquad \Delta_q^2 = \underset{\sim}{\mu}_1' \Sigma_{11}^{-1} \underset{\sim}{\mu}_1 \text{ and } \Delta_p^2 = \underset{\sim}{\mu}' \Sigma^{-1} \underset{\sim}{\mu}.$$

If $\Delta_p^2 = \Delta_q^2$, *then* $R = (1 + \underline{x}_1' S_{11}^{-1} \underline{x}_1)/(1 + \underline{x}' S^{-1} \underline{x})$ *and* $S_1 = S + \underline{x}\underline{x}'$ *are independently distributed and their densities are constant* $R^{(n-p-1)/2}(1 - R)^{(p-q-2)/2}$ *and non-central Wishart* $(n + 1, p, 1; \Sigma, \underline{\mu}; S_1)$ *with* 1 (1 *is trivially the rank of* $\underline{\mu}$) *non-central parameter* (for a discussion of the R statistic, see [8]).

Proof. Let $2\Sigma = \tilde{B}\tilde{B}'$. Make the transformations $V_0 = \tilde{B}^{-1} S \tilde{B}'^{-1}$, $\underline{y} = \tilde{B}^{-1}\underline{x}$, $V_1 = V_0 + \underline{y}\underline{y}'$ and $\tilde{T}^{-1}\underline{y} \equiv \underline{w}$ where $V_1 = \tilde{T}\tilde{T}'$. By (2.5) and (2.6) the Jacobian of the transformation is $| \tilde{B} |^{p+2} | \tilde{T} | = | \Sigma |^{(p+2)/2} | V_1 |^{1/2}$. Also

$$| S | = | V_1 |.| I - \underline{w}\underline{w}' | = | \Sigma |.| V_1 |.(1 - \underline{w}'\underline{w}).$$

Let

$$\underline{\delta} = \tilde{B}^{-1}\underline{\mu} = \begin{pmatrix} \underline{\delta}_1 \\ \underline{\delta}_2 \end{pmatrix}\begin{matrix} q \\ p-q \end{matrix}, \quad \tilde{B} = \begin{pmatrix} \tilde{B}_1 & \cdot \\ \tilde{B}_2 & \tilde{B}_3 \end{pmatrix}\begin{matrix} q \\ p-q \end{matrix}, \quad \underline{w} = \begin{pmatrix} \underline{w}_1 \\ \underline{w}_2 \end{pmatrix}\begin{matrix} q \\ p-q \end{matrix}, \quad \tilde{T} = \begin{pmatrix} \tilde{T}_1 & \cdot \\ T_2 & \tilde{T}_3 \end{pmatrix}\begin{matrix} q \\ p-q \end{matrix}.$$
$$\qquad\qquad\qquad\qquad\quad q \quad p-q \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad q \quad p-q$$

Then we get $2\Sigma_{11} = \tilde{B}_1\tilde{B}_1'$, $\underline{\delta}_1'\underline{\delta}_1 = \underline{\mu}_1' \Sigma_{11}^{-1} \underline{\mu}_1/2 = \Delta_q^2/2$ and $\underline{\delta}'\underline{\delta} = \Delta_p^2/2$. Hence if $\Delta_p^2 = \Delta_q^2$, we have

$$\underline{\delta}_2'\underline{\delta}_2 = 0 \text{ i.e. } \underline{\delta}_2 = \underline{0}. \tag{7}$$

Hence we can write down the joint distribution of V_1 and $\underline{w}$ as

$$f(V_1, \underline{w}) = \frac{(1 - \underline{w}'\underline{w})^{(n-p-1)/2} | V_1 |^{(n-p)/2} \exp(-\operatorname{tr} V_1 + 2\underline{\delta}_1' \tilde{T}_1 \underline{w}_1 - \Delta_q^2/2)}{\pi^{p(p+1)/4} \prod_{i=1}^{p} \Gamma\left(\frac{n - i + 1}{2}\right)}. \tag{8}$$

Apply the transformation $\underline{z}_2 \equiv \underline{w}_2/\sqrt{(1 - \underline{w}_1'\underline{w}_1)}$. The Jacobian is $(1 - \underline{w}_1'\underline{w}_1)^{(p-q)/2}$. Hence equation (8) is

$$f(V_1, \underline{w}_1, \underline{z}_2) \equiv f_1(V_1, \underline{w}_1) f_2(\underline{z}_2) \tag{9}$$

where

$$f_1(V_1, \underline{w}_1) = c_2(1 - \underline{w}_1'\underline{w}_1)^{(n-q-1)/2} | V_1 |^{(n-p)/2} \exp(\text{-tr } V_1 + 2\underline{\delta}_1'\tilde{T}_1\underline{w}_1 - \Delta_q^2/2), \tag{10}$$

$$c_2 \text{ being } \pi^{-(2q+p^2-p)/4}\, \Gamma\left(\frac{n - p + 1}{2}\right) \left\{\Gamma\left(\frac{n - q + 1}{2}\right)\right\}^{-1} \prod_{i=1}^{p} \left\{\Gamma\left(\frac{n - i + 1}{2}\right)\right\}^{-1}$$

and

$$f_2(\underline{z}_2) = \pi^{-(p-q)/2}\, \Gamma\left(\frac{n - q + 1}{2}\right) \left\{\Gamma\left(\frac{n - p + 1}{2}\right)\right\}^{-1} (1 - \underline{z}_2'\underline{z}_2)^{(n-p-1)/2}. \tag{11}$$

Hence $\underline{z}_2$ and V_1 are independently distributed; i.e. V_1 and $\underline{z}_2'\underline{z}_2$ are independently distributed. Note that $\underline{z}_2'\underline{z}_2 = 1 - R = (\underline{w}'\underline{w} - \underline{w}_1'\underline{w}_1)/(1 - \underline{w}_1'\underline{w}_1)$ for $1 - \underline{w}'\underline{w} = 1 - \underline{y}'V_1^{-1}\underline{y} = 1/(1 + \underline{y}'V_0^{-1}\underline{y})$. Hence the distribution of R can be easily obtained by the use of the integral (2.8) in (11). By integrating over $\underline{w}_1$ in (10), it can be easily shown that the distribution of V_1 (and so of S_1) is non-central Wishart with $(n + 1)$ d.f. and one (1 is the rank of $\underline{\mu}$) non-central parameter.

Corollary 5. *Let* $S: p \times p$, $\underline{x}_i : p \times 1$, $i = 1, 2, \cdots m$, *be independently* *distributed as Wishart* $(n, p; \Sigma; S)$ *and multivariate normals* $(\underline{\mu}_i, \Sigma)$ *and let*

$$\underline{x}_i = \begin{array}{c}\begin{pmatrix} \underline{x}_{1\cdot i} \\ \underline{x}_{2\cdot i} \end{pmatrix}\end{array}\begin{array}{l} q \\ p-q \end{array}, \qquad S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}\begin{array}{l} q \\ p-q \end{array}, \qquad \underline{\mu}_i = \begin{pmatrix} \underline{\mu}_{1\cdot i} \\ \underline{\mu}_{2\cdot i} \end{pmatrix}\begin{array}{l} q \\ p-q \end{array},$$

$$\Sigma = \begin{array}{cc} \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} & \begin{array}{l} q \\ p-q \end{array} \\ \begin{array}{cc} q & p-q \end{array} & \end{array}, \qquad \Delta^2_{q\cdot i} = \underline{\mu}'_{1\cdot i}\, \Sigma^{-1}_{11} \underline{\mu}_{1\cdot i} \text{ and } \Delta^2_{p\cdot i} = \underline{\mu}'_i\, \Sigma^{-1} \underline{\mu}_i \,.$$

If

$$\Delta^2_{p\cdot i} = \Delta^2_{q\cdot i}, \quad \text{then} \quad R_i = \frac{1 + \underline{x}'_{1\cdot i}\left(S_{11} + \sum_{j=1}^{i-1} \underline{x}_{1\cdot j}\, \underline{x}'_{1\cdot j}\right)^{-1} \underline{x}_{1\cdot i}}{1 + \underline{x}'_1\left(S + \sum_{j=1}^{i-1} \underline{x}_j\, \underline{x}'_j\right)^{-1} \underline{x}_i},$$

$i = 1, 2, \cdots m$ *and* $S_m = S + \sum_{j=1}^{m} \underline{x}_j\, \underline{x}'_j$ *are mutually independent and their distributions are given by const.* $R_i^{(n-p+i-2)/2}(1 - R_i)^{(p-q-2)/2}$ $i = 1, 2, \cdots m$ *and non-central Wishart* $(m + n, p, t; \Sigma, \mu; S_m)$ *with* t*(*t *is the rank of* $\mu = (\underline{\mu}_1, \cdots \underline{\mu}_m)$*) non-central parameters.*

The proof can be obtained in a similar manner as above.

Note: During the time of revision, the author has obtained the most general form of Theorem II which yields Theorem I as a corollary.

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[1] I am thankful to the referee for suggesting these useful references.