THE NON-EXISTENCE OF CERTAIN PBIB DESIGNS

By Manohar Narhar Vartak

University of Bombay

1. Introduction. Let N be a Partially Balanced Incomplete Block (PBIB) design, (cf. Bose and Shimamoto, [1]), with three associate classes and with parameters

$$v, b, r, k, n_i, \lambda_i, p^i_{ju}; \qquad (i, j, u = 1, 2, 3). \tag{1.1}$$

These parameters are not all independent but they are connected by the equations

$$\begin{gathered} bk = vr; \qquad \sum_{i=1}^{3} n_i = v - 1; \qquad \sum_{i=1}^{3} n_i\lambda_i = r(k-1); \\ p^i_{ju} = p^i_{uj}; \qquad n_i p^i_{ju} = n_j p^j_{iu} = n_u p^u_{ij}; \\ \sum_{u=1}^{3} p^i_{ju} = n_j - \delta_{ij} \qquad (i, j, u = 1, 2, 3); \end{gathered} \tag{1.2}$$

where $\delta_{ij} = 0$ or 1 according as $i \neq j$ or $i = j$ respectively. Additional relations among the parameters (1.1) can be derived if the association scheme of the v treatments of N is completely known. Suppose, for example, that the association scheme of the given design N is of the rectangular type; that is, let us suppose that

$$v = v_1 v_2 \qquad (v_1, v_2 \geqq 2), \tag{1.3}$$

and that the treatments $\theta_{ij} (i = 1, 2, \cdots, v_1; j = 1, 2, \cdots, v_2)$ of the design N can be arranged in the form of a $v_1 \times v_2$ rectangle

$$\begin{matrix} \theta_{11}, & \theta_{12}, & \cdots, & \theta_{1v_2} \\ \theta_{21}, & \theta_{22}, & \cdots, & \theta_{2v_2} \\ \cdots & \cdots & \cdots & \cdots \\ \theta_{v_11}, & \theta_{v_12}, & \cdots, & \theta_{v_1v_2} \end{matrix} \tag{1.4}$$

so that the first associates of any treatment θ_{ij} are the other $v_2 - 1$ treatments in the ith row; its second associates are the other $v_1 - 1$ treatments in the jth column and the remaining $(v_1 - 1)(v_2 - 1)$ treatments are its third associates. For the design N with the association scheme (1.4) it then follows that the matrices (p^i_{ju}) are given by

Received July 22, 1957; revised July 7, 1959.

$$
(1.5) \qquad
\begin{aligned}
(p^1_{ju}) &= \begin{bmatrix} v_2 - 2 & 0 & 0 \\ 0 & 0 & v_1 - 1 \\ 0 & v_1 - 1 & (v_1 - 1)(v_2 - 2) \end{bmatrix}; \\
(p^2_{ju}) &= \begin{bmatrix} 0 & 0 & v_2 - 1 \\ 0 & v_1 - 2 & 0 \\ v_2 - 1 & 0 & (v_1 - 2)(v_2 - 1) \end{bmatrix}; \\
(p^3_{ju}) &= \begin{bmatrix} 0 & 1 & v_2 - 2 \\ 1 & 0 & v_1 - 2 \\ v_2 - 2 & v_1 - 2 & (v_1 - 2)(v_2 - 2) \end{bmatrix}.
\end{aligned}
$$

The relevant additional relations among the parameters (1.1) are, in this case,

$$(1.6) \qquad p^1_{11} = n_1 - 1 = v_2 - 2; \quad p^2_{22} = n_2 - 1 = v_1 - 2; \quad n_3 = n_1 n_2 .$$

The parameters r, k, λ_1, λ_2 and λ_3 are related to v_1 and v_2 through the equation

$$(1.7) \qquad r(k - 1) = (v_2 - 1)\lambda_1 + (v_1 - 1)\lambda_2 + (v_1 - 1)(v_1 - 1)\lambda_3$$

which, in fact, is one of the equations in (1.2) rewritten in the light of (1.6).

In this paper we shall be concerned with PBIB designs with three associate classes whose parameters satisfy the conditions (1.3), (1.5), (1.6) and (1.7) in addition to (1.2). We shall call the series of these designs the series A. A design belonging to the series A will be said to be symmetric if

$$(1.8) \qquad v = b, \quad \text{and consequently,} \quad r = k.$$

It may be noted that the series A includes all PBIB designs with three associate classes which are the Kronecker product of two BIB designs (cf. Vartak [2]).

In the next section we shall show that the conditions (1.2) and (1.6) uniquely characterise the association scheme (1.4). We shall then obtain an expression for the matrix NN' for any design belonging to the series A where N is the incidence matrix of the given design and N' is the transpose of N. In Section 3 we shall calculate the characteristic roots and the determinant $|NN'|$ of the matrix NN'. We shall also calculate there the Hasse-Minkowski invariants, $c_p(NN')$, for the matrix NN' of any design belonging to the series A.

Some non-existence theorems together with illustrations are given in Section 4. These theorems are direct consequences of the results obtained in Sections 2 and 3, and consist of extensions of the results of Schützenberger [3] and Shrikhande [4] for symmetrical BIB designs, applicable to the designs of series A.

2. The uniqueness of the rectangular association scheme. We shall first prove the following theorem on the uniqueness:

THEOREM 2.1: *If the parameters of a* PBIB *design N with three associate classes satisfy the conditions* (1.2) *and* (1.6), *i.e., if the design belongs to the series* A, *then the association scheme for its treatments is uniquely determined and is of the rectangular type* (1.4).

PROOF: From (1.2) and (1.6) we have, first of all,

$$v = n_1 + n_2 + n_3 + 1 = (v_2 - 1) + (v_1 - 1) + (v_1 - 1)(v_2 - 1) + 1 = v_1v_2,$$

which is the same as (1.3). Also from (1.2) and (1.6) it follows that the matrices (p^i_{ju}) are as given in (1.5).

Let ϕ and θ be any two treatments of N which are first associates. Let $\phi_{11}, \cdots, \phi_{1n_1}$ be the n_1 first associates of ϕ and $\theta_{11}, \cdots, \theta_{1n_1}$ be the n_1 first associates of θ. Then ϕ is one of the θ_{1i}'s and θ is one of the ϕ_{1i}'s $(i = 1, 2, \cdots, n_1)$. Let us say, for the sake of definiteness, that $\phi_{11} \equiv \theta$ and $\theta_{11} \equiv \phi$. Now, since by (1.6), $p^1_{11} = n_1 - 1 = v_2 - 2$, it follows that the sets ϕ_{1i} and θ_{1i} have exactly $v_2 - 2 = n_1 - 1$ treatments in common. From this and the earlier identifications $\phi_{11} \equiv \theta$ and $\theta_{11} \equiv \phi$, it follows that the sets ϕ_{1j} and θ_{1j} $(j = 2, 3, \cdots, n_1)$ are identical, i.e., consist of the same treatments. This means that any two treatments in the set $\{\phi, \theta, \theta_{12}, \cdots, \theta_{1n_1}\}$ $(n_1 = v_2 - 1)$, of v_2 treatments, are first associates and that the remaining $v_2 - 2$ treatments are first associates of each of them. This implies that the relation of being first associates is symmetric as well as transitive for all treatments of the design N. From this it follows that the $v = v_1v_2$ treatments of the design N fall into v_1 groups of v_2 treatments each, such that the relation of being first associates is symmetric as well as transitive for the treatments of any of the v_1 groups. It is, therefore, convenient to designate these groups by

$$
\begin{array}{c}
(\theta_{11}, \ \theta_{12}, \ \cdots, \ \theta_{1v_2}) \\
(\theta_{21}, \ \theta_{22}, \ \cdots, \ \theta_{2v_2}) \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
(\theta_{v_11}, \ \theta_{v_12}, \ \cdots, \ \theta_{v_1v_2}).
\end{array}
\tag{2.1}
$$

The property satisfied by any of these groups is that the first associates of any treatment in the group are the remaining treatments in the same group.

Next, suppose that the second group in (2.1) contains two treatments θ_{2j} and θ_{2k} which are second associates of θ_{11}. This will mean that $p^2_{21} \geqq 1$, which contradicts the result $p^2_{21} = 0$ obtained earlier and referred to in (1.5). This implies that the second, and in general any of the $v_1 - 1$ groups after the first, cannot contain more than one second associate of θ_{11}. But θ_{11} has exactly $n_2 = v_1 - 1$ second associates so that the 2nd, 3rd, $\cdots$, v_1th group in (2.1) must each contain one and only one second associate of θ_{11}. The same holds for each of $\theta_{12}, \cdots, \theta_{1v_2}$. In general, therefore, the ith group contains one and only one second associate of θ_{jk} when $j \neq i$. Without any loss of generality, we can assume that $\theta_{2i}, \theta_{3i}, \cdots, \theta_{v_1i}$ are the $n_2 = v_1 - 1$ second associates of θ_{1i}.

Further, we have $p^2_{22} = n_2 - 1 = v_1 - 2$, which, by the same type of argument as before, implies that the treatments $\theta_{1i}, \theta_{2i}, \cdots, \theta_{v_1i}$ are such that the relation of being second associates is symmetric as well as transitive for them. The $v = v_1v_2$

treatments of N, therefore, can be conveniently divided into v_2 groups of v_1 treatments each, such that the relation of being second associates is symmetric as well as transitive for the treatment of any group.

The two modes of classification of the treatments of N for the relation of first and second association can be superimposed by writing the treatments in the form of a rectangular array (1.4).

The third associates of any treatment θ_{ij} are, then, by exclusion, the $n_3 = (v_2 - 1)(v_1 - 1) = n_1 n_2$ treatments θ_{kl} in the array, where $k \neq i$ and $l \neq j$.

The relation of association for the treatments of the design N can thus be described with the help of the association scheme (1.4), where the treatments occurring in the same row as θ_{ij} are its first associates, those occurring in the same column as θ_{ij} are its second associates, and the others are its third associates. In other words the association scheme is uniquely determined and is of the rectangular type.

This proves the theorem.

With the help of the association scheme (1.4), we can write down the matrix NN' of the design N belonging to the series A in a very convenient form. Let the rows of N correspond to the treatments $\theta_{11}, \theta_{12}, \cdots, \theta_{1v_2}, \theta_{21}, \cdots \theta_{2v_2}, \cdots, \theta_{v_11}, \cdots, \theta_{v_1v_2}$ respectively, in this order. Then the matrix NN' is seen to have the following structure:

$$NN' = \begin{bmatrix} A & B & \cdots & B \\ B & A & \cdots & B \\ \cdots & \cdots & \cdots & \cdots \\ B & B & \cdots & A \end{bmatrix} \tag{2.2}$$

where A is a $v_2 \times v_2$ square matrix given by

$$A = (r - \lambda_1)I_{v_2} + \lambda_1 E_{v_2} \tag{2.3}$$

and B is a $v_2 \times v_2$ square matrix given by

$$B = (\lambda_2 - \lambda_3)I_{v_2} + \lambda_3 E_{v_2}, \tag{2.4}$$

I_{v_2} being the identity matrix of order v_2 and E_{v_2} a square matrix of order v_2 with all elements equal to 1. Also the matrix NN', as written in (2.2), has v_1 rows and v_1 columns. The same result can be summarized in the form of the following theorem:

THEOREM 2.2: *The matrix NN' for a design N belonging to the series* A *is given by*

$$NN' = I_{v_1} \times (A - B) + E_{v_1} \times B \tag{2.5}$$

where '$\times$' denotes the Kronecker product of matrices and A and B are as defined in (2.3) *and* (2.4).

3. Characteristic roots, determinant and the Hasse-Minkowski invariants of NN'. Let D_{v_2} be the $v_2 \times v_2$ square matrix given by

$$
(3.1) \qquad D_{v_2} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & -1 & 0 & \cdots & 0 \\ 1 & 1 & -2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & -(v_2 - 1) \end{bmatrix}
$$

It should be observed that the matrix D_{v_2} is a modified Helmertz matrix. Moreover, the determinant $|D_{v_2}|$ of D_{v_2} is clearly

$$
(3.2) \qquad |D_{v_2}| = (-)^{v_2-1}\{v_2!\},
$$

so that D_{v_2} is non-singular. In fact D_{v_2} is a semi-orthogonal matrix in the sense that

$$
(3.3) \qquad D_{v_2}D'_{v_2} = \text{diag}\{v_2, 1.2, 2.3, \cdots, (v_2 - 1)v_2\}
$$

where $\text{diag}\{a_1, a_2, \cdots, a_m\}$ is a diagonal matrix of order m whose diagonal elements are $a_1, a_2, \cdots, a_m$ and off-diagonal elements are all zero. It is easy to verify that the matrix D_{v_2} reduces both A and B to diagonal forms. Thus

$$
(3.4) \qquad D_{v_2}AD'_{v_2} = \text{diag}\{v_2[r + (v_2 - 1)\lambda_1], 1.2\,(r - \lambda_1), 2.3\,(r - \lambda_1), \cdots, (v_2 - 1)v_2(r - \lambda_1)\}
$$

and

$$
(3.5) \qquad D_{v_2}BD'_{v_2} = \text{diag}\{v_2[\lambda_2 + (v_2 - 1)\,\lambda_3], 1.2(\lambda_2 - \lambda_3), \cdots, (v_2 - 1)v_2(\lambda_2 - \lambda_3)\}.
$$

It may be noted that, since the elements of D_{v_2} are all integral, the equations (3.4) and (3.5) can be interpreted to mean that A and B are both rationally equivalent to the diagonal forms exhibited on the right sides of (3.4) and (3.5).

Now consider the matrix

$$
(3.6) \qquad H = \begin{bmatrix} D_{v_2} & D_{v_2} & D_{v_2} & \cdots & D_{v_2} \\ D_{v_2} & -D_{v_2} & 0 & \cdots & 0 \\ D_{v_2} & D_{v_2} & -2D_{v_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ D_{v_2} & D_{v_2} & D_{v_2} & \cdots & -(v_1 - 1)D_{v_2} \end{bmatrix}
$$

where D_{v_2} is the matrix given by (3.1) and H, as written above, has v_1 rows and v_1 columns, every 0 in (3.6) being a square null matrix of order $v_2 \times v_2$. It may be noted that the matrix H is the Kronecker product $D_{v_1} \times D_{v_2}$ and hence the determinant $|H|$ of H is given by

$$
(3.7) \qquad |H| = |D_{v_1} \times D_{v_2}| = |D_{v_1}|^{v_2} \cdot |D_{v_2}|^{v_1} = (-)^{v_1+v_2}(v_1!)^{v_2}(v_2!)^{v_1}.
$$

The characteristic roots of NN' of (2.2) are the roots of the determinantal equation in θ:

$$
(3.8) \qquad |NN' - \theta I_v| = 0
$$

where $I_v(v = v_1v_2)$ is the identity matrix of order v.

From (2.5), we can write this in the form

$$|\, I_{v_1} \times \{(A - \theta I_{v_2}) - B\} + E_{v_1} \times B \,| = 0 \tag{3.9}$$

However, it is easy to verify that

$$\begin{aligned} H\{NN' - \theta I_v\} H' &= H\{I_{v_1} \times [(A - \theta I_{v_2}) - B] + E_{v_1} \times B\} H' \\ &= \operatorname{diag}\{v_1 D_{v_2}[(A - \theta I_{v_2}) + (v_1 - 1)B]D'_{v_2},\ 1.2\, D_{v_2}[(A - \theta I_{v_2}) - B] \\ &\qquad \cdot D'_{v_2},\ \cdots,\ (v_1 - 1)v_1 D_{v_2}[(A - \theta I_{v_2}) - B]D'_{v_2}\} \end{aligned} \tag{3.10}$$

and since $D_{v_2}AD'_{v_2}$, $D_{v_2}BD'_{v_2}$ and $D_{v_2}D'_{v_2}$ are themselves diagonal matrices, so are $D_{v_2}\{(A - \theta I_{v_2}) - B\}\, D_{v_2}$ and $D_{v_2}\{(A - \theta I_{v_2}) + (v_1 - 1)B\}D'_{v_2}$. Hence (3.10) reduces completely to a diagonal matrix. Writing

$$\begin{aligned} \theta_0 &= rk = r + (v_2 - 1)\lambda_1 + (v_1 - 1)\lambda_2 + (v_1 - 1)(v_2 - 1)\lambda_3, \\ \theta_1 &= r - \lambda_1 + (v_1 - 1)(\lambda_2 - \lambda_3), \\ \theta_2 &= r - \lambda_2 + (v_2 - 1)(\lambda_1 - \lambda_3), \\ \theta_3 &= r - \lambda_1 - \lambda_2 + \lambda_3, \end{aligned} \tag{3.11}$$

we find that (3.10) reduces to

$$\begin{aligned} &H\{NN' - \theta I_v\}H' \\ &\quad = \operatorname{diag}\{v_1v_2(\theta_0 - \theta), v_1\, 1.2(\theta_1 - \theta), \cdots, v_1(v_2 - 1)v_2(\theta_1 - \theta), \\ &\qquad 1.2\, v_2(\theta_2 - \theta), 1.2\cdot 1.2(\theta_3 - \theta), \cdots, 1.2(v_2 - 1)v_2(\theta_3 - \theta), \\ &\qquad \cdots \\ &\qquad (v_1 - 1)v_1v_2(\theta_2 - \theta), (v_1 - 1)v_1\, 1.2(\theta_3 - \theta), \cdots, \\ &\qquad\qquad (v_1 - 1)v_1(v_2 - 1)v_2(\theta_3 - \theta)\}. \end{aligned} \tag{3.12}$$

Hence, taking the determinants of both sides, we get

$$|\, NN' - \theta I_v \,| = (\theta_0 - \theta)(\theta_1 - \theta)^{v_2 - 1}(\theta_2 - \theta)^{v_1 - 1}(\theta_3 - \theta)^{(v_1 - 1)(v_2 - 1)}. \tag{3.13}$$

Also the determinant $|\, NN' \,|$ of the matrix NN' is the product of its characteristic roots. Hence from (3.13) and (3.11) we get the following theorem:

THEOREM 3.1:

(a) *The characteristic roots of the matrix NN' of the design N of the series* A *are θ_0, θ_1, θ_2, θ_3 given by* (3.11) *and their respective multiplicities are*

$$\alpha_0 = 1, \quad \alpha_1 = v_2 - 1 = n_1, \quad \alpha_2 = v_1 - 1 = n_2, \quad \alpha_3 = (v_1 - 1)(v_2 - 1) = n_3. \tag{3.14}$$

(b) *The determinant $|\, NN' \,|$ of the matrix NN' of the design N is given by*

$$|\, NN' \,| = \theta_0\theta_1^{v_2 - 1}\theta_2^{v_1 - 1}\theta_3^{(v_1 - 1)(v_2 - 1)}$$

$$(3.15) \quad = rk\{r - \lambda_1 + (v_1 - 1)(\lambda_2 - \lambda_3)\}^{v_2 - 1}\{r - \lambda_2 + (v_2 - 1)(\lambda_1 - \lambda_3)\}^{v_1 - 1} \cdot\{r - \lambda_1 - \lambda_2 + \lambda_3\}^{(v_1-1)(v_2-1)}.$$

To derive an expression for the Hasse-Minkowski invariant $c_p(NN')$ of the matrix NN', we note that, from (3.10),

$$HNN'H' = \text{diag}\{v_1 D_{v_2}[A + (v_1 - 1)B]D'_{v_2}, 1.2\, D_{v_2}(A - B)D'_{v_2}, 2.3\, D_{v_2}(A - B)D'_{v_2}, \cdots, (v_1 - 1)v_1 D_{v_2}(A - B)D'_{v_2}\}.$$

This can be further written as the direct sum of the matrix

$$v_1 D_{v_2}[A + (v_1 - 1)B]D'_{v_2}$$

and the Kronecker product

$$\text{diag}\{1.2, 2.3, \cdots, v_1(v_1 - 1)\} \times \{D_{v_2}(A - B)D'_{v_2}\}.$$

That is, we can write

$$(3.16) \quad HNN'H' = v_1 D_{v_2}\{A + (v_1 - 1)B\}D'_{v_2} \dotplus \text{diag}\{1.2, 2.3, \cdots, v_1(v_1 - 1)\} \times \{D_{v_2}(A - B)D'_{v_2}\},$$

where $\dotplus$ denotes the direct sum.

We now make use of the following results for the c_p invariants of the direct sum and the Kronecker product of matrices:

If P and Q are symmetric matrices with rational elements whose c_p invariants are defined and if

$$U = P \dotplus Q \text{ and } V = P \times Q$$

then

$$(3.17) \quad c_p(U) = (-1, -1)_p c_p(P) c_p(Q)(|P|, |Q|)_p,$$

and

$$(3.18) \quad c_p(V) = (-1, -1)_p^{m+n-1} \{c_p(P)\}^n \{c_p(Q)\}^m (|P|, -1)_p^{\frac{n(n-1)}{2}} (|Q|, -1)_p^{\frac{m(m-1)}{2}} (|P|, |Q|)_p^{mn-1}$$

where m and n are the orders of P and Q respectively, (cf. [5] and [6] respectively).

Further we know that if λ is a non-zero rational number and B is an $n \times n$ matrix whose Hasse-Minkowski invariants are defined, then

$$(3.19) \quad c_p(\lambda B) = c_p(B)(\lambda, -1)_p^{\frac{n(n+1)}{2}} (\lambda, |B|)_p^{n-1}$$

where $|B|$ is the determinant of B.

It should be noted that $HNN'H'$ of (3.16) is rationally equivalent to NN' and is a diagonal matrix.

We are now in a position to prove the following theorem:

THEOREM 3.2: *The Hasse-Minkowski invariant $c_p(NN')$ of the matrix NN' for the design N of the series A is given by*

$$\begin{aligned} c_p(NN') &= (-1,-1)_p(\theta_0,-v)_p(v_1\theta_0,\theta_1)_p^{v_2-1}(v_2\theta_0,\theta_2)_p^{v_1-1}(\theta_0,\theta_3)_p^{(v_1-1)(v_2-1)} \\ &\times \{(\theta_1,\theta_2)_p(\theta_2,\theta_3)_p(\theta_3,\theta_1)_p\}^{(v_1-1)(v_2-1)} \\ &\times (\theta_1,-1)_p^{\frac{v_2(v_2-1)}{2}}(\theta_2,-1)_p^{\frac{v_1(v_1-1)}{2}}(\theta_3,-1)_p^{\frac{(v_1-1)(v_2-1)(v_1+v_2-2)}{2}} \\ &\times(\theta_1,v_2)_p(\theta_2,v_1)_p(\theta_3,v_1)_p^{v_2-1}(\theta_3,v_2)_p^{v_1-1}, \end{aligned} \tag{3.20}$$

if the characteristic roots θ_0, θ_1, θ_2, θ_3 of the matrix NN', given by (3.11), are all non-zero.

PROOF: Observe, in the first place, that

$$D_{v_2}\{A+(v_1-1)B\}D'_{v_2} = \text{diag}\{v_2\theta_0, 1.2\,\theta_1, 2.3\,\theta_1, \cdots, v_2(v_2-1)\theta_1\}, \tag{3.21}$$

and that

$$D_{v_2}(A-B)D'_{v_2} = \text{diag}\{v_2\theta_2, 1.2\,\theta_3, 2.3\,\theta_3, \cdots, v_2(v_2-1)\theta_3\}. \tag{3.22}$$

Hence, when the characteristic roots θ_0, θ_1, θ_2, θ_3 are all non-zero, from (3.16) we find that all the leading principal minor determinants of the rationally equivalent diagonal form of NN' are different from zero; so that the Hasse-Minkowski invariants of this diagonal form and consequently that of the matrix NN' are defined.

A little algebra shows that

$$c_p\{\text{diag}(1.2, 2.3, \cdots, v_1(v_1-1))\} = (-1,-1)_p, \tag{3.23}$$

$$c_p\{D_{v_2}[(A+(v_1-1)B]\,D'_{v_2}\} = (-1,-1)_p(\theta_0,-v_2)_p (\theta_0,\theta_1)_p^{v_2-1}(\theta_1,v_2)_p(\theta_1,-1)_p^{\frac{v_2(v_2-1)}{2}}, \tag{3.24}$$

$$c_p\{D_{v_2}(A-B)D'_{v_2}\} = (-1,-1)_p(\theta_2,-v_2)_p (\theta_2,\theta_3)_p^{v_2-1}(\theta_3,v_2)_p(\theta_3,-1)_p^{\frac{v_2(v_2-1)}{2}}. \tag{3.25}$$

Making use of (3.23), (3.24), (3.25) and (3.17), (3.18) and (3.19), it is possible to obtain (3.20) after a little calculation.

This completes the proof of the theorem.

4. The non-existence theorems with illustrations. Let N be a design of the series A characterised by (1.3), (1.5) through (1.7). Let χ be any characteristic root of NN' for this design. Then there exists a vector $\mathbf{x}$ such that

$$\mathbf{x}'NN'\mathbf{x} = \chi \tag{4.1}$$

which shows that χ is non-negative. This gives the following theorem:

THEOREM 4.1: *For a design in the series A to exist it is necessary that*

$$\theta_1 = r - \lambda_1 + (v_1 - 1)(\lambda_2 - \lambda_3) \geqq 0,$$

$$\theta_2 = r - \lambda_2 + (v_2 - 1)(\lambda_1 - \lambda_3) \geqq 0,$$

$$\theta_3 = r - \lambda_1 - \lambda_2 + \lambda_3 \geqq 0.$$

The following examples illustrate the use of this theorem:

Example 4.1: Consider the symmetric ($v = b$ and hence $r = k$) PBIB design of the series A given by

$$v = b = 24, \quad r = k = 8, \quad n_1 = 5, \quad n_2 = 3, \quad n_3 = 15,$$

$$\lambda_1 = 4, \quad \lambda_2 = 7, \quad \lambda_3 = 1;$$

$$(p_{ju}^1) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 12 \end{bmatrix}, \quad (p_{ju}^2) = \begin{bmatrix} 0 & 0 & 5 \\ 0 & 2 & 0 \\ 5 & 0 & 10 \end{bmatrix}, \quad (p_{ju}^3) = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 0 & 2 \\ 4 & 2 & 8 \end{bmatrix}.$$

The characteristic roots of NN' for this design are

$$\theta_0 = 64, \quad \theta_1 = 22, \quad \theta_2 = 16, \ \theta_3 = -2;$$

and since $\theta_3 < 0$, Theorem 4.1 is contradicted. Hence the above PBIB is impossible.

Example 4.2: Consider the PBIB design of the series given by

$$v = 30, \quad b = 20, \quad r = 10, \quad k = 15, \quad n_1 = 4, \quad n_2 = 5, \quad n_3 = 20,$$

$$\lambda_1 = 10, \quad \lambda_2 = 8, \quad \lambda_3 = 3;$$

$$(p_{ju}^1) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 5 & 15 \end{bmatrix}, \quad (p_{ju}^2) = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 16 \end{bmatrix}, \quad (p_{ju}^3) = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 4 \\ 3 & 4 & 12 \end{bmatrix}.$$

The characteristic roots of NN' for this design are

$$\theta_0 = 150, \quad \theta_1 = 25, \quad \theta_2 = 30, \quad \theta_3 = -5;$$

and since $\theta_3 < 0$, Theorem 4.1 is contradicted. Hence the above PBIB design is impossible.

Example 4.3: Consider the PBIB design of the series A given by

$$v = 30, \quad b = 50, \quad r = 10, \quad k = 6, \quad n_1 = 4, \quad n_2 = 5, \quad n_3 = 20,$$

$$\lambda_1 = 5, \quad \lambda_2 = 6, \quad \lambda_3 = 0;$$

$$(p_{ju}^1) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 5 & 15 \end{bmatrix}, \quad (p_{ju}^2) = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 16 \end{bmatrix}, \quad (p_{ju}^3) = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 4 \\ 3 & 4 & 12 \end{bmatrix}.$$

The characteristic roots of NN' for this design are

$$\theta_0 = 60, \quad \theta_1 = 35, \quad \theta_2 = 24, \quad \theta_3 = -1;$$

and since $\theta_3 < 0$, Theorem 4.1 is contradicted. Hence the above PBIB design is impossible.

In the case of a symmetric PBIB design of the series A we have $v = b$ so that the matrix N is a square matrix of order $v = v_1 v_2$. The determinant $| NN' |$ of the matrix NN' must therefore be a perfect square when $| N | \neq 0$. This condition can be formulated in the form of the theorem:

THEOREM 4.2: *A necessary condition for the existence of a symmetric* PBIB *design of the series* A *when* $| N | \neq 0$ *is that*

(a) *if* v_1 *is even and* v_2 *is odd then* $\theta_2 = r - \lambda_2 + (v_2 - 1)(\lambda_1 - \lambda_3)$ *is a perfect square,*

(b) *if* v_2 *is even and* v_1 *is odd, then* $\theta_1 = r - \lambda_1 + (v_1 - 1)(\lambda_2 - \lambda_3)$ *is a perfect square, and*

(c) *if* v_1 *and* v_2 *are both even then* $\theta_1\theta_2\theta_3$, $(\theta_3 = r - \lambda_1 - \lambda_2 + \lambda_3)$, *is a perfect square.*

The following examples illustrate the application of this theorem:

Example 4.4: Consider the design given by

$$v = b = 66, \quad r = k = 14, \quad n_1 = 2, \quad n_2 = 21, \quad n_3 = 42,$$

$$\lambda_1 = 7, \quad \lambda_2 = 4, \quad \lambda_3 = 2$$

$$(p_{ju}^1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 21 \\ 0 & 21 & 21 \end{bmatrix}, \quad (p_{ju}^2) = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 20 & 0 \\ 2 & 0 & 40 \end{bmatrix}, \quad (p_{ju}^3) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 20 \\ 1 & 20 & 20 \end{bmatrix}.$$

Clearly this design is a symmetric design ($v = b$) from the series A. Since $v_1 = n_2 + 1 = 22$ is an even integer and $v_2 = n_1 + 1 = 3$ is an odd integer and since $\theta_2 = r - \lambda_2 + (v_2 - 1)(\lambda_1 - \lambda_3) = 20$ is not a perfect square, it follows from Theorem 4.2 that the above PBIB design is impossible. It is easy to verify that $| N | \neq 0$.

It may be observed that the parameters of the above PBIB design are obtained by taking the Kronecker product (cf. [2]) of the BIB designs

$$N_1 : v_1 = b_1 = 22, \quad r_1 = k_1 = 7, \quad \lambda_1 = 2$$

and

$$N_2 : v_2 = b_2 = 3, \quad r_2 = k_2 = 2, \quad \lambda_2 = 1,$$

of which N_1 is already known to be non-existent (cf. Shrikhande [4]).

Example 4.5: Consider the PBIB design given by

$$v = b = 48, \quad r = k = 10, \quad n_1 = 7, \quad n_2 = 5, \quad n_3 = 35$$

$$\lambda_1 = 5, \quad \lambda_2 = 4, \quad \lambda_3 = 1$$

$$(p_{ju}^1) = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 5 & 30 \end{bmatrix}, \quad (p_{ju}^2) = \begin{bmatrix} 0 & 0 & 7 \\ 0 & 4 & 0 \\ 7 & 0 & 28 \end{bmatrix}, \quad (p_{ju}^3) = \begin{bmatrix} 0 & 1 & 6 \\ 1 & 0 & 4 \\ 6 & 4 & 24 \end{bmatrix},$$

which is a symmetric ($b = v$) design from the series A. Here both v_1 and v_2 are even and the characteristic roots of NN' for this design are $\theta_0 = 100$, $\theta_1 = 20$,

$\theta_2 = 34$, $\theta_3 = 2$. This implies that $|N| \neq 0$. Moreover, $\theta_1\theta_2\theta_3 = 1360$ is not a perfect square. It follows therefore from Theorem 4.2 that the above design is impossible.

The Hasse-Minkowski invariant $c_p(NN')$ obtained in (3.19) gives us another non-existence theorem for the symmetric designs of the series A.

Let N be a symmetric design of the series A with $|N| \neq 0$. Then the matrix $NN' = B$ for this design is obviously rationally equivalent to I_v, the identity matrix of order $v = v_1v_2$. Hence $c_p(NN')$ must be $+1$ for all odd primes p. If, for any design, $c_p(NN') = -1$ for some odd prime p, then that design will be impossible.

We state this result as the following theorem:

THEOREM 4.3: *If N is a symmetrical design of the series* A *with $|N| \neq 0$, then a necessary condition for the design N to exist is that $c_p(NN') = +1$ for all odd primes p.*

The following examples illustrate the use of this theorem.

Example 4.6: Consider the PBIB design given by

$$v = b = 87, \quad r = k = 16, \quad n_1 = 28, \quad n_2 = 2, \quad n_3 = 56,$$

$$\lambda_1 = 4, \quad \lambda_2 = 8, \quad \lambda_3 = 2.$$

$$(p_{ju}^1) = \begin{bmatrix} 27 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 54 \end{bmatrix}, \quad (p_{ju}^2) = \begin{bmatrix} 0 & 0 & 28 \\ 0 & 1 & 0 \\ 28 & 0 & 28 \end{bmatrix}, \quad (p_{ju}^3) = \begin{bmatrix} 0 & 1 & 27 \\ 1 & 0 & 1 \\ 27 & 1 & 27 \end{bmatrix}.$$

This is evidently a symmetric design from the series A with $|N| \neq 0$. Further it is easy to verify that $c_p(NN')$ given by (3.19) reduces in this case to $(24, 29)_p$; further, for $p = 3$ this becomes $c_3(NN') = (2, 3)_3 = (2/3) = -1$ where (a/p) is the Legendre symbol of a with respect to the prime p. Thus Theorem 4.3 is contradicted and therefore the above design is impossible.

It may be observed that the above design has a set of parameters which could be obtained by taking the Kronecker product of the BIB designs

$$N_1{:}v_1 = b_1 = 3, \quad r_1 = k_1 = 2, \quad \lambda_1 = 1,$$

and

$$N_2{:}v_2 = b_2 = 29, \quad r_2 = k_2 = 8, \quad \lambda_2 = 2,$$

of which, N_2 is proved to be impossible (cf. Shrikhande [4]).

Example 4.7: Consider the PBIB design given by

$$v = b = 63, \quad r = k = 11, \quad n_1 = 8, \quad n_2 = 6, \quad n_3 = 48,$$

$$\lambda_1 = 4, \quad \lambda_2 = 5, \quad \lambda_3 = 1,$$

$$(p_{ju}^1) = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 0 & 6 \\ 0 & 6 & 42 \end{bmatrix}, \quad (p_{ju}^2) = \begin{bmatrix} 0 & 0 & 8 \\ 0 & 5 & 0 \\ 8 & 0 & 40 \end{bmatrix}, \quad (p_{ju}^3) = \begin{bmatrix} 0 & 1 & 7 \\ 1 & 0 & 5 \\ 7 & 5 & 35 \end{bmatrix}.$$

This is obviously a symmetric design from the series A with $|N| \neq 0$. Further it is easy to verify that the Hasse-Minkowski invariant $c_p(NN')$ given by (3.19) reduces in this case to $(30, 7)_p\,(30, -1)_p$. For $p = 3$ this becomes $c_3(NN') = (2, 3)_3 = (2/3) = -1$, where (a/p) is the Legendre symbol of a with respect to the prime p. Thus Theorem 4.3 is contradicted and therefore the above PBIB design is impossible.

5. Summary and acknowledgement. Three non-existence theorems are obtained for the PBIB designs with three associate classes and belonging to a certain series called the Series A. The first theorem makes use of the fact that the characteristic roots of the matrix NN' are always non-negative; the second is an extension of Schützenberger's result [3] and the third is an extension of Shrikhande's result [4] for symmetrical BIB designs.

I wish to express my sincere thanks to Professor M. C. Chakrabarti for his kind interest in this work. Also I am indebted to the referee for useful suggestions especially in connection with Theorem 3.2.

REFERENCES

[1] R. C. Bose and T. Shimamoto, "Classification and analysis of partially balanced incomplete block designs with two associate classes," *Jour., Amer. Stat. Assn.*, Vol. 47 (1952), pp. 151–184.

[2] M. N. Vartak, "On an application of Kronecker product of matrices to statistical designs," *Ann. Math. Stat.*, Vol. 26 (1955), pp. 420–438.

[3] M. P. Schützenberger, "A non-existence theorem for an infinite family of symmetrical block designs," *Ann. Eugenics*, Vol. 14 (1949), pp. 286–287.

[4] S. S. Shrikhande, "The impossibility of certain symmetrical balanced incomplete block designs," *Ann. Math. Stat.*, Vol. 21 (1950), pp. 106–111.

[5] R. C. Bose and W. S. Connor, "Combinatorial properties of group divisible incomplete block designs," *Ann. Math. Stat.*, Vol. 23 (1952), pp. 367–383.

[6] M. N. Vartak, "On the Hasse-Minkowski invariant of the Kronecker product of matrices," *Canadian Jour. Math.*, Vol. 10 (1958), pp. 66–72.