SOME TESTS OF PERMUTATION SYMMETRY[1]

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Summary. The two-sample sign test is viewed as a test for the permutation symmetry of a bivariate distribution, and extensions to k-variate distributions are sought. Friedman's rank test, [1], although originally intended as a substitute for the F-test in a two-way classification, is such an extension. Study of the family of two-sample sign tests obtained by comparing the k coordinates pairwise has yielded a statistic with an asymptotic Chi-square distribution from which a further test of symmetry can be constructed. The statistic is based on more degrees of freedom than Friedman's and is sensitive to a greater variety of alternatives. This extension is analogous to that obtained by Terpstra [2] from the Wilcoxon test.[2] In this case, however, the limiting distribution turns out to be non-singular. The argument leading to the test is not restricted to the case of complete symmetry but may be carried through with any specified degree of asymmetry. The coordinates may also be compared m at a time, $2 \leqq m \leqq k$. The argument can be extended and, with a slight modification, includes the derivation of Friedman's test. Thus a hierarchy of tests of permutation symmetry are available: Friedman's test corresponds to the case, $m = 1$; when $m = k$, the corresponding test turns out to be Pearson's Chi-square.

1. Introduction. Given n pairs of observations, ... sometimes called two "matched" samples, ... the sign test statistic for comparing the populations from which the two matched samples were drawn is the number of cases in which the first observation of a pair is greater than the second; a simple count. The statistic and its distribution are easily computed, the test requires minimal assumptions about the underlying probability distributions, and when these distributions are normal, the efficiency of the test relative to the t-test is high [3]. It is natural, therefore, to explore extensions of the test to three or more matched samples.

In the case of three samples, J. W. Tukey has suggested the following approximate test. To make the test at level α', conduct ordinary two-tailed sign tests comparing each of the three pairs of samples, at level $\alpha = \alpha'/3$. If one or more of these three tests yields a significant result, the combined test is significant. This is a convenient approximation, but it does not appear to be worth while to extend this method to the case of more than three samples.

In the general case of k matched samples of n—that is to say, n observations on k-variate distributions—one extension has been given by Friedman [1]. The

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[2] The author is indebted to the editors for drawing his attention to the work of Terpstra.

k coordinates of each observation are ranked in order of magnitude, and the mean ranks calculated. The sum of squares of deviations of these k mean ranks from their general mean is proportional to a statistic X_r^2 which, under the appropriate null hypothesis, has asymptotically a Chi-square distribution with $k - 1$ degrees of freedom. Another extension is suggested in the preceding paragraph. Consider the family of two-sample sign tests obtained by comparing the k coordinates pairwise; the corresponding statistics form a set of C_2^k simple counts, the number of times the ith coordinate exceeds the jth in the sample. This paper is primarily concerned with this set of simple counts.

Underlying much discussion of the sign test is an intuitive picture of two independent and similar, if not identical, populations which are to be compared for differences in location. Friedman's rank test is the natural generalization. However, regarding the two paired samples as a single sample from a bivariate distribution, the sign test becomes a test of the permutation symmetry of the distribution; the null hypothesis states that both orderings of the coordinates are equally probable. In the case of a k-variate distribution, the most general non-parametric test of permutation symmetry is based on the statistic (of dimension, $k!$) giving the number of times each of the k possible orderings occurs in the sample. While such a test is of use against any non-symmetric alternative, it would appear that, unless the sample size is of the order $k!$, only the most extreme departures from symmetry would be detected. If only certain kinds of asymmetry are of interest then a more specific test is required; Friedman's rank is an example of such a specific test.

Suppose one wishes to determine whether a card-shuffling device is acceptable. Ideally, no matter what the order of the cards before shuffling, all orderings should be equally probable after shuffling. It would be acceptable, however, if it were practically impossible for a card-player, knowing the initial order of the cards, and the final position of some of them, to draw inferences about the position of the remainder. A bridge player holding the Queen of Spades should not be able to infer from the previous hand that the King of Spades is more likely to be on his right than on his left. If the initial position of a card is i, let X_i denote its final position after shuffling. Then we are concerned with the symmetry of the multivariate distribution of the $\{X_i\}$. The most general test may very well require an impossibly large experiment—52! is a very large number. On the other hand, Friedman's rank test is too specific. A single cut, provided all possible places for the cut are equally likely, (including no cut at all), is sufficient to ensure that the expectation of X_i is the same for all i; and, as is shown below, this implies that the expectation of X_r^2 is the same as it would be under the null hypothesis of complete symmetry. Clearly, something intermediate is required.

2. Notation. Let

$$X^{(a)} = (X_1^{(a)}, X_2^{(a)}, \cdots, X_k^{(a)}), \qquad a = 1, 2, \cdots, n,$$

represent n k-variate real-valued random variables. Assume, as null hypothesis, that

(2.1) $$\Pr\{X_{i_1}^{(a)} < X_{i_2}^{(a)} < \cdots < X_{i_k}^{(a)}\} = 1/k!$$

for all permutations $(i_1, i_2, \cdots, i_k)$ of the subscripts $(1, 2, \cdots, k)$, and all a.

Let

(2.2) $$Y_{ij}^{(a)} = 1 \text{ if } X_i^{(a)} > X_j^{(a)} \qquad i \neq j$$
$$= 0 \text{ otherwise,}$$

and define

(2.3) $$\beta_{ij} = \frac{2}{\sqrt{n}} \sum_{a=1}^{n} (Y_{ij}^{(a)} - \tfrac{1}{2}).$$

Since $\beta_{ij} + \beta_{ji} = 0$, we need consider only one of each pair, $\{\beta_{ij}, \beta_{ji}\}$. It will appear (Lemma 1), that the choice does not affect our conclusions and we consider, therefore, only the set $\{\beta_{ij}\}$ for which $i < j$.

The exact probability at any point $\{\beta_{ij}\}$ is given by a sum of multinomial terms.

THEOREM I:

(2.4) $$\Pr(\beta_{ij}) = \frac{1}{(k!)^n} \sum \frac{n!}{\prod_{h=1}^{k!} \alpha_h}$$

where the sum is taken over all sets of non-negative integers $\{\alpha_h\}$, $(h = 1, 2, \cdots, k!)$, *satisfying*

(2.5) $$\sum_{h=1}^{k!} \alpha_h = n$$

and a set of C_2^k *linear equations of the form*

(2.6) $$2 \sum_{h=1}^{k!} c_{ijh}\alpha_h = \sqrt{n}\beta_{ij} - n, \qquad c_{ijh} = 1 \text{ or } 0 \text{ as required.}$$

The moments of the β_{ij} can be computed directly. $E\{\beta_{ij}\} = 0$. For any admissible choice of $\{\beta_{ij}\}$, the covariance matrix is non-singular and can be inverted.

THEOREM II: *Let* $(\sigma_{ii',jj'})$ *be the covariance matrix of the* C_2^k*-dimensional random variate* $\{\beta_{ij}\}$. *Let* D *be its determinant and let* $(\sigma^{ii',jj'})$ *be its inverse. Then for all permissible choices of the coordinates* $\{\beta_{ij}\}$,

(2.7) $$D = (k+1)^{k-1}/3^{\frac{k(k-1)}{2}};$$

if	$i = j$,	$i' = j'$,	$\sigma_{ii',jj'} = 1$	$\sigma^{ii',jj'} = \dfrac{3(k-1)}{k+1}$,
if	$i = j'$,	$i' = j$,	$= -1$	not defined
if	$i = j$,	$i' \neq j'$,		
or	$i \neq j$,	$i' = j'$,	$= \frac{1}{3}$	$= \dfrac{-3}{k+1}$

if	$i = j'$,	$i' \neq j$,		
or	$i \neq j'$,	$i' = j$,	$= -\frac{1}{3}$	$= \dfrac{3}{k+1}$
otherwise,			$= 0$	$= 0.$

Outline of proof: In the determinant, D, rows can be replaced by linear combinations of rows to introduce blocks of zeros. Precisely, we replace $\sigma_{ii',jp}$ by

$$\tau_{ii',jp} = \sigma_{ii',jp} - \frac{1}{p} \sum_{\substack{\nu \\ (\nu < p)}} \sigma_{ii',j\nu}, \qquad \text{all } (ii'). \tag{2.8}$$

D can then be written as a product of principal minors,

$$D = \prod_{p=2}^{k} d_p, \tag{2.9}$$

where d_p is of order $(p - 1)$ with elements $(2(p + 1))/3p$ on the diagonal, and $(p + 1)/3p$, elsewhere.

$$d_p = p\left[\frac{p+1}{3p}\right]^{p-1}. \tag{2.10}$$

Hence, $D = (k + 1)^{k-1}/3^{(k(k-1))/2}$.

To verify the remainder of the theorem, we multiply the covariance matrix by its stated inverse, and evaluate the sums, $\sum_{(ii')}\sigma_{hh',ii'}\sigma^{ii',jj'}$. A term of such a sum will differ from zero only if the pair of indices (ii') has an element in common with each of the pairs, (hh') and (jj').

For a diagonal element of the product, $(hh') = (jj')$. There is one term in the sum with $(ii') = (hh')$ of value, $(3(k - 1))/(k + 1)$; and $2(k - 2)$ terms in which (ii') has one element in common with (hh'), each of value $-1/(k + 1)$; all other terms are zero, and the sum is unity.

For non-diagonal elements of the product, where (hh') and (jj') have one or zero elements in common, we have, for example:

(i) $\displaystyle\sum_{(ii')} \sigma_{12,ii'}\sigma^{ii',13} = \sigma_{12,12}\sigma^{12,13} + \sigma_{12,13}\sigma^{13,13} + \sigma_{12,23}\sigma^{23,13} + \sum_{i=4}^{k} \sigma_{12,1i'}\sigma^{1i',13} = 0$

(ii) $\displaystyle\sum_{(ii')} \sigma_{12,ii'}\sigma^{ii',34} = \sigma_{12,13}\sigma^{13,34} + \sigma_{12,14}\sigma^{14,34} + \sigma_{12,23}\sigma^{23,34} + \sigma_{12,24}\sigma^{24,34} = 0.$

3. The statistic χ_t^2. We now define the statistic on which our first test of permutation symmetry is based. Let

$$\chi_t^2 = \sum_{i<i'} \sum_{j<j'} \sigma^{ii',jj'} \beta_{ii'}\beta_{jj'} \tag{3.1}$$

χ_t^2 is thus the quadratic form associated with the inverse of the covariance matrix of β, for a particular choice of the coordinates of β. This particular choice is only

TABLE I

Distribution of χ_t^2 for $k = 3$; $n = 2, 3, 4, 5, 6$

x	Pr $(\chi_t^2 \leq x)$	x	Pr $(\chi_t^2 \leq x)$	x	Pr $(\chi_t^2 \leq x)$
$n = 2$		$n = 4$		$n = 6$	
0	.167	0	.0694	0	.0399
1.5	.833	1.5	.5139	1.0	.3485
3.0	1.000	3.0	.6435	2.0	.4507
		4.5	.8657	3.0	.6977
$n = 3$		6.0	.9193	4.0	.7748
1.0	.417	7.5	.9954	5.0	.8828
3.0	.639	12.0	1.0000	6.0	.8983
5.0	.972			7.0	.9600
9.0	1.000	$n = 5$		8.0	.9685
		0.6	.2392	9.0	.9891
		1.8	.4244	10.0	.9968
		3.0	.7639	13.0	.9999
		5.4	.8912	18.0	1.0000
		6.6	.9529		
		7.8	.9838		
		10.2	.9992		
		15.0	1.0000		

a notational convenience; the same statistic is obtained with any other admissible choice, (Lemma 1).

The exact distribution of χ_t^2, for n finite, can be computed from the distribution of β as given in Theorem I. This is an arduous process, except when k and n are both small. A few values are given in Table I, for $k = 3$.

The asymptotic distributions of β and χ_t^2 are given by

THEOREM III: *The vector random variate, $\beta = \{\beta_{ij}\}$, has asymptotically a non-singular multivariate normal distribution with density function $Ce^{-\frac{1}{2}\chi_t^2}$. The statistic χ_t^2 is asymptotically distributed as Chi-square with C_2^k degrees of freedom.*

PROOF: β is the standardized sum of n identically and independently distributed vector random variates. Since all second moments are finite, the simplest conditions for the central limit theorem in its multivariate form, [4], are satisfied. Therefore, as n increases, the distribution of β tends to the multivariate normal. The covariance matrix is non-singular and independent of n; hence, the density function exists.

It is well-known that the exponent of the density function is distributed as Chi-square, [5]. Hence, χ_t^2 is asymptotically Chi-square with C_2^k degrees of freedom.

The null hypothesis should be rejected whenever χ_t^2 is large.

An indication of the accuracy of the asymptotic approximation is given in Table II for $k = 3$ and small n.

TABLE II

Comparison of X_t^2 with approximating Chi-square, X_3^2, with 3 d.f. ($k = 3$)

x	$\Pr(X_t^2 \geq x)$	$\Pr(X_3^2 \geq x)$	x	$\Pr(X_t^2 \geq x)$	$\Pr(X_3^2 \geq x)$
	$n = 3$			$n = 5$	
9.0	.028	.029	10.2	.0162	.0170
5.0	.361	.172*	7.8	.0471	.0504
			6.6	.1088	.0859*
	$n = 4$			$n = 6$	
12.0	.0046	.0074	9.0	.0315	.0295
7.5	.0807	.0576*	8.0	.0400	.0461
6.0	.1343	.1117	7.0	.1017	.0720*

* Only in the cases marked by an asterisk is the approximation improved by a continuity correction.

4. Friedman's rank test. There exists a hierarchy of tests of permutation symmetry, one of which is the X_t^2-test of the previous section. Another such test, lying at one end of the chain, is Friedman's rank test [1].

In the rank test, the k coordinates of an observation are ranked in increasing order of magnitude. If $r_{i\nu}$ denotes the rank of coordinate X_i, in the νth observation, then $r_{i\nu} - 1 =$ no. of coordinates less than X_i. Let

$$\bar{\rho}_i = \frac{1}{n} \sum_{\nu=1}^{n} \left(r_{i\nu} - \frac{k+1}{2} \right). \tag{4.1}$$

Friedman proposed the statistic

$$\chi_r^2 = \frac{12n}{k(k+1)} \sum_{i=1}^{k} \bar{\rho}_i^2 \tag{4.2}$$

for testing the null hypothesis. Large values of χ_r^2 lead to rejection of the hypothesis.

Friedman tabulated the exact distribution of χ_r^2 for small n and k, and gave a proof that χ_r^2 is asymptotically distributed as Chi-square with $(k - 1)$ degrees of freedom.

We note that

$$\bar{\rho}_i = \frac{1}{2\sqrt{n}} \sum_{\substack{j \\ (j \neq i)}} \beta_{ij} . \tag{4.3}$$

Hence,

$$\chi_r^2 = \frac{3}{k(k+1)} \sum_{i=1}^{k} \left[\sum_{\substack{j \\ (j \neq i)}} \beta_{ij} \right]^2 \tag{4.4}$$

and

$$\chi_t^2 = 3 \sum_{\substack{ij \\ (i<j)}} \beta_{ij}^2 - k\chi_r^2 \tag{4.5}$$

We also define

$$\chi_\Delta^2 = \chi_t^2 - \chi_r^2 . \tag{4.6}$$

From Fisher's Lemma [6], we can deduce that χ_Δ^2 is asymptotically Chi-square with $C_2^k - (k - 1) = C_2^{k-1}$ degrees of freedom, and independent of χ_r^2.

Friedman also showed that, for finite n,

$$E\chi_r^2 = k - 1 \qquad \text{Var}\, \chi_r^2 = \frac{n-1}{n} 2(k - 1). \tag{4.8}$$

Similarly, we obtain, by direct calculation,

$$\begin{aligned} E\chi_t^2 &= C_2^k & \text{Var}\, \chi_t^2 &= \frac{n-1}{n} k(k-1) \\ E\chi_\Delta^2 &= C_2^{k-1} & \text{Var}\, \chi_\Delta^2 &= \frac{n-1}{n} (k-1)(k-2). \end{aligned} \tag{4.9}$$

For each of these statistics, the mean is independent of n, and the variance is an increasing function of n. This strongly suggests that the use of the asymptotic distribution, (without a continuity correction), for defining the critical region will lead to errors in the so-called "safe" direction; i.e., the true size of the critical region will be smaller than the significance level. The computations carried out by Friedman, and by the writer, support this.

χ_r^2 and χ_t^2 provides tests of the same null hypothesis. In most practical applications, the alternatives of interest—e.g., one or more coordinates tending to be consistently higher than the remainder—can be distinguished by either test. In the writer's opinion, χ_r^2 is usually the preferred test, and χ_t^2 should be reserved for special situations. An experimenter may, on occasion, wish to make two tests based on χ_r^2 and χ_Δ^2. These two tests are asymptotically independent and are sensitive to two distinct classes of alternatives. The classification of alternatives is discussed in Section 9.

5. First extension: arbitrary null hypothesis. Although the argument has been presented with one particular hypothesis as null hypothesis—viz., all orderings of the $\{X_i\}$ equally likely—it could just as easily be carried through with an asymmetric null hypothesis:

$$\Pr\{X_{i_1} > X_{i_2} > \cdots > X_{i_k}\} = p_{i_1 i_2 \cdots i_k} > 0 \tag{5.1}$$

where $p_{i_1 i_2 \cdots i_k}$ is a set of $k!$ positive numbers which sum to unity.

As a notational convenience, we introduce symbols representing sums of the constants in the null hypothesis.

$$p_{i_1 i_2 \cdots i_m} = \Pr\{X_{i_1} > X_{i_2} > \cdots > X_{i_m}\}, \qquad 2 \leqq m < k. \tag{5.2}$$

Random variables $Y_{ij}^{(a)}$ are defined as before, (2.2), and the standardized variates, β_{ij}, are defined similarly by

$$\beta_{ij} = \frac{1}{\sqrt{np_{ij}(1 - p_{ij})}} \sum_{a=1}^{n} (Y_{ij}^{(a)} - p_{ij}). \tag{5.3}$$

The vector variate, β, is defined by C_2^k coordinates, β_{ij}, which satisfy no linear relation.

Let $\sigma_{ii',jj'} = \text{Covar}\{\beta_{ii'}, \beta_{jj'}\}$. As before, $\text{Var}(\beta_{ii'}) = 1$, and when i, i', j, j' are all distinct, $\sigma_{ii',jj'} = 0$. However, the covariance of two coordinates with one subscript in common is more complex; e.g.

$$\begin{aligned} \sigma_{12,13} = {} & \sqrt{\frac{(1 - p_{12})(1 - p_{13})}{p_{12}p_{13}}} \cdot (p_{123} + p_{132}) - \sqrt{\frac{(1 - p_{12})p_{13}}{p_{12}(1 - p_{13})}} \cdot p_{312} \\ & - \sqrt{\frac{p_{12}(1 - p_{13})}{(1 - p_{12})p_{13}}} \cdot p_{213} + \sqrt{\frac{p_{12}p_{13}}{(1 - p_{12})(1 - p_{13})}} \cdot (p_{231} + p_{321}). \end{aligned} \tag{5.4}$$

However, all variances and covariances are finite, and the central limit theorem still applies; therefore, β has asymptotically a multivariate normal distribution.

It will be shown (Theorem IV) that the rank of the matrix $(\sigma_{ii',jj'})$ of order C_2^k, is also C_2^k; hence, its inverse $(\sigma^{ii',jj'})$ exists. We can therefore define

$$Q_2 = \sum_{\substack{ii' \\ (i<i')}} \sum_{\substack{jj \\ (j<j')}} \sigma^{ii',jj'} \beta_{ii'} \beta_{jj'}. \tag{5.5}$$

Q_2 is the exponent of the density function of the asymptotic distribution and we deduce, as before, that Q_2 is asymptotically Chi-square with C_2^k degrees of freedom.

To test the null hypothesis, reject when Q_2 is large. The critical region can be determined easily, and approximately, with the asymptotic distribution.

6. Second extension: the Q_m-statistics. Instead of comparing the random variables, $\{X_i\}$, two at a time, we could compare them m at a time ($2 \leqq m \leqq k$).

We define new random variables of order m, by

$$\begin{aligned} Z_{i_1 i_2 \cdots i_m}^{(a)} &= 1 \text{ if } X_{i_1}^{(a)} > X_{i_2}^{(a)} > \cdots > X_{i_m}^{(a)} \\ &= 0 \text{ otherwise} \end{aligned} \tag{6.1}$$

where $(i_1, i_2, \cdots, i_m)$ is a subset of the first k positive integers.

There are $k(k - 1) \cdots (k - m + 1) = \theta_m$, say, such random variables of order m.

If n observations are made, then $\sum_{a=1}^{n} Z_{i_1 i_2 \cdots i_m}^{(a)} =$ the number of times in which the ordering $X_{i_1} > X_{i_2} > \cdots > X_{i_m}$ occurs.

We define the standardized random variate of order m by

$$\gamma_{i_1 i_2 \cdots i_m} = \frac{1}{\sqrt{np_{i_1 i_2 \cdots i_m}(1 - p_{i_1 \cdots i_m})}} \sum_{a=1}^{n} (Z_{i_1 i_2 \cdots i_m}^{(a)} - p_{i_1 \cdots i_m}). \tag{6.2}$$

For any fixed set of integers $(i_1 i_2 \cdots i_m)$, the sum, over all permutations of the set, of the $Z^{(a)}_{i_1 i_2 \cdots i_m}$, is unity.

The standardized random variates of order m satisfy $\tau_m = C^k_m$ independent linear relations, $(m \geqq 2)$.

When $m = 2$, $\gamma_{i_1 i_2} = \beta_{i_1 i_2}$ as defined in Section 5.

When $m = 1$, we extend the definition by defining

$\gamma_i =$ standardized mean rank

$\tau_1 = 1$ (not C^k_1, since the γ_i satisfy only one linear relation).

For every fixed m, a vector random variate $\gamma'_m = \{\gamma_{i_1 i_2 \cdots i_m}\}$ of dimension θ_m, is defined.

By the central limit theorem, the distribution of γ'_m, approaches the multivariate normal distribution. However, because the coordinates satisfy τ_m linear relations the distribution is singular in the full θ_m-space.

We reduce the space to dimension $\theta_m - \tau_m$ by omitting one of the coordinates appearing in each of the τ_m linear relations. (No coordinate appears in more than one such relation, so that exactly τ_m coordinates are omitted.) We denote the resulting vector random variate of dimension $(\theta_m - \tau_m)$ by γ_m. It will be shown in Theorem IV that the covariance matrix of γ_m is non-singular and therefore, (at least in theory), can be inverted. Hence, we can define the statistic $Q_m =$ the quadratic form in $\gamma_{i_1 i_2 \cdots i_m}$ associated with the inverse of the reduced covariance matrix. Q_m is asymptotically Chi-square with $(\theta_m - \tau_m)$ degrees of freedom.

LEMMA 1: *For fixed m, and a given simple null hypothesis, Q_m is a uniquely defined function of the original sample.*

PROOF: Non-uniqueness could only occur when reducing the θ_m-space to dimension $(\theta_m - \tau_m)$ by different choices of the coordinates to be retained.

Let Q_m be defined in terms of one set $\gamma_{i_1 \cdots i_m}$ of coordinates, and let $\bar{Q}_m$ be defined in terms of a second set, $\bar{\gamma}_{i_1 \cdots i_m}$, of these coordinates. Using the known linear relations, each coordinate of the second set can be expressed as a linear combination of coordinates in the first set; thus, $\bar{Q}_m$ is also a quadratic form in the first set of coordinates. We must show, then, that corresponding coefficients in the two forms are equal. It is clearly sufficient to consider only the asymptotic distributions for, since the coefficients are independent of n, equality of coefficients in the limit implies equality for all n.

Consider, therefore, two quadratic forms, Q, $\bar{Q}$, in the random variates U_1, $U_2, \cdots, U_s$ where $U' = \{U_1, U_2, \cdots, U_s\}$ has an s-variate normal distribution and $Q = U'AU$, $\bar{Q} = U'\bar{A}U$, have Chi-square distributions with s degrees of freedom. This implies that the forms Q, $\bar{Q}$ are of full rank and their associated matrices A, $\bar{A}$ are non-singular.

Therefore, there exist linear transformations

$$V = CU, \qquad \bar{V} = \bar{C}U \tag{6.3}$$

transforming Q, $\bar{Q}$ into sums of squares of s independent standard normal variates, where V, $\bar{V}$ denote s-dimensional column vectors and C, $\bar{C}$ non-singular $s \times s$ matrices

$$(6.4) \qquad Q = V'V, \qquad \bar{Q} = \bar{V}'\bar{V}.$$

Clearly, $V = C\bar{C}^{-1}\bar{V} = P\bar{V}$, say. Since the coordinates of both V and $\bar{V}$ are independent standard normal variates, P is an orthogonal matrix. Thus $V'V = \bar{V}'\bar{V}$ and $Q = V'V = \bar{V}'\bar{V} = \bar{Q}$.

Essential for the validity of the proof, and for the truth of the lemma, is the fact that both quadratic forms are of full rank. Otherwise, C and $\bar{C}$ are singular matrices without inverses, and V is not, in general, a linear transform of $\bar{V}$.

7. Rank of Q_m: Expectation of Q_m. We have already used the fact that the rank of the quadratic form Q_m is $\theta_m - \tau_m$. We now give a proof of this statement.

THEOREM IV: *Let γ_m be the vector random variate of dimension $(\theta_m - \tau_m)$ defined in Section 6 and let A_m denote its covariance matrix of rank, r_m. Then $r_m = \theta_m - \tau_m$.*

LEMMA 2: *$r_m < \theta_m - \tau_m$ if, and only if, a linear relation holds among the coordinates of γ_m with probability one.* Proof omitted.

LEMMA 3: *Each random variate $\gamma_{i_1 i_2 \cdots i_m}$ of order m can be expressed as a linear combination of random variates of order $(m + 1)$.*

PROOF: $Z_{i_1 i_2 \cdots i_m} = 1$ whenever the ordering $X_{i_1} > X_{i_2} > \cdots > X_{i_m}$ occurs, i.e. whenever one of the orderings $(X_j > X_{i_1} > \cdots > X_{i_m})$, $(X_{i_1} > X_j > X_{i_2} > \cdots > X_{i_m})$, $\cdots$, $(X_{i_1} > \cdots > X_{i_m} > X_j)$ occurs, for fixed j not a member of the set $(i_1, i_2, \cdots, i_m)$. $Z_{i_1 i_2 \cdots i_m} = Z_{j i_1 \cdots i_m} + Z_{i_1 j i_2 \cdots i_m} + \cdots + Z_{i_1 i_2 \cdots i_m j}$. The γ-variates are linear combinations of the z-variates of the same order. The lemma follows.

LEMMA 4: $r_k = k! - 1$.

PROOF: When $m = k$ there is only one choice for the set of integers $i_1 i_2 \cdots i_m$. Let h index the permutations of this set. By direct calculations we obtain

$$(7.1) \qquad \text{Var}(\gamma_h) = 1, \qquad \text{Covar}(\gamma_h, \gamma_{h'}) = -\sqrt{\frac{p_h p_{h'}}{(1 - p_h)(1 - p_{h'})}}.$$

We omit the coordinate corresponding to $h = k!$ to obtain A_k.

The determinant of A_k can be evaluated directly.

$$(7.2) \qquad | A_k | = \prod_{h=1}^{k!-1} \frac{p_1}{(1 - p_h)} \neq 0$$

since by assumption, $p_1 \neq 0$

$$(7.3) \qquad \therefore r_k = k! - 1.$$

PROOF OF THEOREM IV: Suppose $r_m < \theta_m - \tau_m$ for some m.

By Lemma 2 there exists a linear relation among the random variates $\gamma_{i_1 i_2 \cdots i_m}$ of order m represented in A_m. But, by Lemma 3 each of these can be expressed as a linear combination of random variates of order $(m + 1)$ represented in A_{m+1}. Thus there exists a linear relation among the variates of order $(m + 1)$ and $r_{m+1} < \theta_{m+1} - \tau_{m+1}$. By induction, $r_k < k! - 1$. But this contradicts Lemma 4, hence $r_m = \theta_m - \tau_m$.

The expected value of the statistic, Q_m, for finite n, is given by

THEOREM V: $E\{Q_m\} = \theta_m - \tau_m$.

PROOF: We can write

$$Q_m = \sum_{(i_1 \cdots i_m)} \sum_{(j_1 \cdots j_m)} \sigma^{i_1 \cdots i_m, j_1 \cdots j_m} \gamma_{i_1 \cdots i_m} \gamma_{j_1 \cdots j_m}, \tag{7.4}$$

where

$$\sigma^{i_1 \cdots i_m, j_1 \cdots j_m} = \frac{\text{Cofactor of } E\{\gamma_{i_1 \cdots i_m} \cdot \gamma_{j_1 \cdots j_m}\} \text{ in } A_m}{| A_m |}$$

and each summation is over the $(\theta_m - \tau_m)$ sets of m integers $(i_1, i_2, \cdots, i_m)$ which appear as subscripts of the $\gamma_{i_1 \cdots i_m}$.

$$\begin{aligned} E\{Q_m\} &= \sum_{(i_1 \cdots i_m)} \frac{1}{| A_m |} \sum_{(j_1 \cdots j_m)} [\text{Cofactor of } E\{\gamma_{i_1 \cdots i_m} \cdot \gamma_{j_1 \cdots j_m}\}] \\ &\qquad \cdot E\{\gamma_{i_1 \cdots i_m} \cdot \gamma_{j_1 \cdots j_m}\} \\ &= \sum_{(i_1 \cdots i_m)} \cdot \frac{| A_m |}{| A_m |} = \theta_m - \tau_m. \end{aligned} \tag{7.5}$$

8. The Case $m = k$. Let h index the permutations of $(1, 2, \cdots, k)$. Let n_h be the number of times that ordering, h, occurs in a sample of n, and let p_h be the probability of that ordering.

Then

$$\gamma_h = \frac{(n_h - n \cdot p_h)}{\sqrt{n p_h (1 - p_h)}} \tag{8.1}$$

and

$$\sum_{h=1}^{k!} (1 - p_h) \gamma_h^2 = \sum_h \frac{(n_h - n \cdot p_h)^2}{n \cdot p_h}$$

which is immediately recognizable as Pearson's Chi-square statistic.

THEOREM VI: $Q_k = \sum_{h=1}^{k!} (1 - p_h) \gamma_h^2$.

PROOF: It has been shown that Q_k is asymptotically Chi-square with $(k! - 1)$ degrees of freedom, and it is well-known that Pearson's statistic has the same limiting distribution. Regarding one of the γ_h as a linear combination of the remainder, both statistics are quadratic forms of rank $(k! - 1)$ in the same $(k! - 1)$ variates with the same asymptotic Chi-square distribution. This is exactly the situation considered in Lemma 1, (Section 6), and by an identical argument, the two statistics are equal.

$$\therefore Q_k = \sum_h (1 - p_h) \gamma_h^2.$$

9. Consistency and the classification of alternatives. To make a symmetry test of order m, we compute Q_m and reject the null hypothesis if Q_m is large. But, in a particular case, what order should the test be? The answer to this question requires a consideration of the alternative hypothesis. Intuitively, the tests of low order such as Friedman's X_r^2, provide a relatively high sensitivity to a small class of alternatives, whereas the high order tests give a low sensitivity—thus requiring a large sample—against a large class of alternatives. This concept of a classification of alternatives is made more precise by the following definition and Theorem VII.

Definition: An alternative is *distinct* from the null hypothesis *at level m* if, under the alternative

$$E\{Z_{i_1 i_2 \cdots i_m}\} \neq p_{i_1 i_2 \cdots i_m}$$

for at least one choice and permutation of the digits $(i_1, i_2, \cdots, i_m)$.

Remark: Since $Z_{i_1 i_2 \cdots i_m}$ can be written as a sum of such counter variables of higher order, it is immediate that, if an alternative is distinct at level m, it is distinct at any level $m' > m$.

THEOREM VII: *The symmetry test of order m is consistent against any alternative which is distinct at level m, and only against such alternatives.*

PROOF: The null hypothesis is rejected whenever $Q_m > c$, i.e., whenever the point $\gamma_m = \{\gamma_{i_1 \cdots i_m}\}$ lies outside the region, S, defined by

$$S = [\gamma_m \mid Q_m \leqq c]. \tag{9.1}$$

c is chosen so that the measure of S, computed with the asymptotic distribution under the null hypothesis is $1 - \alpha$. S is then a fixed, finite region. Under any hypothesis the measure of S tends to a specific value, P, as n approaches infinity: we wish to show (1) that this value is zero under alternatives distinct at level m, and (2) that this value is different from zero under alternatives not distinct at level m.

The measure of S can be written in the form $P + \eta$, where P is the measure of S under the approximating normal distribution with the same mean and variance-covariance matrix as the given distribution, and η is a correction term which approaches zero as n approaches infinity. The variances and covariances of γ_m are independent of n; and the mean of γ_m, and of the approximating normal variate, has coordinates

$$E\{\gamma_{i_1 \cdots i_m}\} = \sqrt{\frac{n}{p_{i_1 \cdots i_m}(1 - p_{i_1 \cdots i_m})}} [E\{Z_{i_1 \cdots i_m}\} - p_{i_1 \cdots i_m}] \tag{9.2}$$

Under any alternative not distinct at level m, these coordinates are all zero: the approximating normal distribution, and therefore P, is independent of n. Clearly, $P \neq 0$, thus establishing the second part of the theorem.

Under an alternative distinct at level m, however, the distance, d, from the mean of the distribution to the origin given by

$$d^2 = n \sum_{(i_1 \cdots i_m)} \frac{[E\{Z_{i_1 \cdots i_m}\} - p_{i_1 \cdots i_m}]^2}{p_{i_1 \cdots i_m}(1 - p_{i_1 \cdots i_m})} \neq 0. \tag{9.3}$$

Thus the distance from the mean to the origin approaches infinity as n tends to infinity.

It is well-known that the ordinate of a normal distribution tends to zero as the distance from the mean increases; hence, the measure, P, of the fixed, bounded set S, under the sequence of approximating normal distributions, tends to zero. Since both P and η tend to zero under an alternative distinct at level m, as n approaches infinity, the consistency of the test against such alternatives is established.

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