THE WEIGHTED COMPOUNDING OF TWO INDEPENDENT SIGNIFICANCE TESTS

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1. Introduction and outline of the problem. In a recent paper on the analysis of incomplete block designs [9], the situation arose where one had two statistically independent F statistics for testing the same null hypothesis. A test was proposed for combining the two tests themselves into a single test which weighted one test relative to the other. It is the purpose of this paper to investigate numerically the power function of this proposed test as it will shed some light as to when an intra-block analysis is worthwhile.

Other situations where one has more than one independent test for testing the same null hypothesis are not uncommon. The tests may have arisen from several sets of independent data or from independent tests made on the same data. General discussions of combining independent tests can be found in Mosteller and Bush [4], Birnbaum [1], and E. S. Pearson [6]. For example a common situation in clinical experiments is that one desires to investigate the effects of two treatments (say) t_1 and t_2 on $2n + m$ people. It is known in advance that m of these people will be available for receiving only one treatment. The experiment is run by assigning t_1 to $(m + n)$ subjects and t_2 to the remaining n people. At a later time, r new people are available who receive treatment t_2. Also of the $2n$ original remaining people, the n people who first received t_1 receive t_2 and vice-versa. Thus the data consist of a cross-over design making use of $2n$ people, and also data where a person received only a single treatment. Thus it is possible to have two tests of the same null hypothesis that the treatments have no effect.[1]

The problem of combining information can be formulated as a problem in estimation. Generally for applications, this latter formulation is usually preferred as it will lead to confidence statements which are usually preferred to tests of a null hypothesis. However it seems interesting from a theoretical point of view to explore the consequences of combining the significance tests themselves.

Let there be two independent variance ratio statistics given by

$$F_j = s_{tj}^2/s_{ej}^2, \qquad j = 1, 2,$$

with degrees of freedom ν and f_j $(j = 1, 2)$ respectively used to test the same null hypothesis. The numerator and denominator mean squares will be referred to as the "treatment" and "error" mean squares and are such that $f_j s_{ej}^2/\sigma_j^2$

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[1] We are indebted to Dr. S. Geisser, National Institute of Mental Health for pointing out this example.

follows a (central) chi-square distribution and $\nu s_{tj}^2/\sigma_j^2$ follows a non-central chi-square distribution with non-central parameter δ_j, where δ_j is defined by

$$(1) \qquad \frac{E(s_{tj}^2)}{E(s_{ej}^2)} = 1 + \frac{\delta_j}{\nu}, \qquad j = 1, 2.$$

When $\delta_j = 0$, the null hypothesis is true and $\nu s_{tj}^2/\sigma_j^2$ will follow a (central) chi-square distribution.

For the purpose of combining the two tests, consider the integral transformation

$$(2) \qquad P_j = P\{F \geqq F_j \mid H_0\}, \qquad j = 1, 2,$$

which is the probability of the F-ratio exceeding the calculated F_j if the null hypothesis is true. Our proposed method for combining the two tests is to use the critical region

$$(3) \qquad \omega:\{P_1 P_2^\theta \leqq C_\alpha\}$$

where C_α is a constant depending on an α level of significance and θ is a weighting factor $(0 \leqq \theta \leqq 1)$ which weights the second test relative to the first. (We will always assume that the first test has power which is equal to or greater than the second test.) This test is closely related to the procedure suggested by Good [3] for combining independent tests. Note that when $\theta = 0$ this corresponds to only using the first test; when $\theta = 1$, then both tests are given equal weight and the procedure is equivalent to the well-known method of Fisher [2] for combining independent tests of significance. The real problem here is to determine how to choose the weighting factor θ. Our procedure for choosing θ is to let $\theta = \delta_2/\delta_1$, which in turn can be written as $\theta = (c_2/c_1)(\sigma_1^2/\sigma_2^2)$ where the c_i are known constants. It is remarkable that this choice of a weighting factor results in minimum Type II error over a wide range of the other parameters involved.

2. Distribution of the combined test.

Null distribution. It is well known that when the null hypothesis is true, the distribution of P_j will be that of a uniform random variable over the unit interval. Therefore the Type I error of the combined test is

$$(4) \qquad P(\omega \mid H_0) = P\{P_1 P_2^\theta \leqq C \mid H_0\} = \iint_\omega dP_1\, dP_2,$$

where ω denotes the critical region $\{P_1 P_2^\theta \leqq C\}$. Hence by an elementary integration we have

$$(5) \qquad P(\omega \mid H_0) = \begin{cases} C, & \text{for } \theta = 0, \\ \dfrac{C - \theta C^{1/\theta}}{1 - \theta} & \text{for } 0 < \theta < 1, \\ C(1 - \ln C), & \text{for } \theta = 1. \end{cases}$$

Therefore setting $P(\omega \mid H_0) = \alpha$ results in critical values of C_α for an α level of significance. Table I gives critical values of C_α for $\theta = 0.1(.1)1.0$ and $\alpha = .01, .05$.

TABLE I*

Critical values of C_α for $\alpha = .05, .01$

θ	$\alpha = .05$	$\alpha = .01$
0.0	.050000	.010000
0.1	.045000	.009000
0.2	.040000	.008000
0.3	.035004	.007000
0.4	.030062	.006001
0.5	.025321	.005013
0.6	.020956	.004062
0.7	.017092	.003190
0.8	.013775	.002432
0.9	.010995	.001805
1.0	.008705	.001309

$$*P\{P_1P_2^\theta \leqq C_\alpha\} = \left\{\begin{matrix} C_\alpha \\ \dfrac{C_\alpha - \theta C_\alpha^{1/\theta}}{1-\theta} \\ C_\alpha(1 - \ln C_\alpha) \end{matrix}\right\} = \alpha, \quad \text{for} \quad \begin{matrix} \theta = 0 \\ 0 < \theta < 1 \\ \theta = 1 \end{matrix}$$

Non-null distribution. If

$$x_j = \frac{\nu F_j}{f_j + \nu F_j}, \tag{6}$$

then the non-null distribution of x will have the p.d.f.

$$p(x_j \mid \delta_j) = e^{-\delta_j/2} \sum_{i=0}^{\infty} \frac{1}{B\left(\frac{\nu}{2} + i, \frac{f_j}{2}\right)} \frac{\delta_j^i}{2^i i!} x^{i+(\nu/2)-1} (1 - x_j)^{(f_j/2)-1} \quad (0 \leqq x_j \leqq 1) \tag{7}$$

and when $\delta_j = 0$, (7) reduces to the beta distribution,

$$p(x_j \mid 0) = \frac{1}{B\left(\frac{\nu}{2}, \frac{f_j}{2}\right)} x_j^{(\nu/2)-1}(1 - x_j)^{(f_j/2)-1}, \quad 0 \leqq x_j \leqq 1. \tag{8}$$

From the elementary properties of the probability integral transformation (cf. Pearson [6]) the p.d.f. of P_j when the null hypothesis is not correct is given by

$$f(P_j) = \left.\frac{p(x_j \mid \delta_j)}{p(x_j \mid 0)}\right|_{x_j = g(P_j)} \quad (0 \leqq P_j \leqq 1), \tag{9}$$

where $x_j = g(P_j)$ means the solution of x_j for a given value of P_j, where x_j and P_j are related by

$$P_j = \int_{x_j}^{1} p(x_j \mid 0)\, dx_j\,. \tag{10}$$

Hence substituting (7) and (8) in (9) results in the p.d.f. of the non-null distribution of P_j, i.e.,

$$f(P_j \mid \delta_j) = e^{-\delta_j/2} B\left(\frac{\nu}{2}, \frac{f_j}{2}\right) \sum_{i=0}^{\infty} \frac{1}{B\left(\frac{\nu}{2} + i, \frac{f_j}{2}\right)} \frac{\delta_j^i}{2^i i!} x_j^i, \tag{11}$$

where x_j is related to P_j by the incomplete beta function

$$P_j = I_{1-x_j}\left(\frac{f_j}{2}, \frac{\nu}{2}\right). \tag{12}$$

Therefore the power of the combined test for a given level of significance α is

$$P(\omega \mid H_1) = \iint_{\omega} f(P_1 \mid \delta_1) f(P_2 \mid \delta_2) \, dP_1 \, dP_2, \tag{13}$$

where the region of integration is $\omega: \{P_1 P_2^{\theta} \leqq C_\alpha\}$.

The integral given in (13) is difficult to integrate as the p.d.f. $f(P_j \mid \delta_j)$ is not an explicit function of P_j. In order to evaluate (13) numerically it is convenient to consider the integral transformation

$$\pi_j = \int_{P_j}^{1} f(P \mid \delta_j) \, dP, \qquad j = 1, 2. \tag{14}$$

Then (13) can be written

$$P(\omega \mid H_1) = \iint_{\omega^*} d\pi_1 \, d\pi_2 \tag{15}$$

where ω^* denotes the region ω in terms of π_1 and π_2. Thus to every point on the boundary $P_1 P_2^{\theta} = C_\alpha$ in the (P_1, P_2) space there will correspond a point in the (π_1, π_2) space and it will be possible to map the region ω^* entirely, even though we do not have an explicit expression in the π_1, π_2 variables for the boundary.

For this purpose it is convenient to find π_j from the non-central distribution of x_j, i.e.,

$$\pi_j = \int_{P_j}^{1} f(P \mid \delta_j) \, dP = \int_{x_j}^{1} p(x \mid \delta_j) \, dx, \qquad j = 1, 2. \tag{16}$$

Unfortunately the non-central distribution given by (16) is only tabulated for values of x_j corresponding to $P_j = .01$ and $.05$. However it is possible to use the Patnaik approximation to the non-central F (or equivalent beta) distribution [5] and find approximate values for π_j. (This approximation appears to have a maximum error of one unit in the second decimal.) For purposes of tabulation, it is more convenient to use the non-central variable $\Delta_j = \delta_j/\nu$ which is related to Tang's non-central parameter Φ, [7], by $\Phi = [\nu\Delta/(\nu + 1)]^{\frac{1}{2}}$. Then in terms of Δ_j, the Patnaik approximation can be written

$$\int_0^{x_j} p(x \mid \delta_j) \, dx \approx I_{x_j'}\left(\frac{n}{2}, \frac{f_j}{2}\right) \tag{17}$$

where

$$(18) \qquad \begin{cases} n = \dfrac{\nu(1 + \Delta_j)^2}{(1 + 2\Delta_j)} \\ x_j' = \dfrac{(1 + \Delta_j)x_j}{1 + 2\Delta_j - \Delta_j x_j}. \end{cases}$$

Table II summarizes calculations for the Type II error (P_{II}) for the parameters

$\alpha = .05$, $\theta = 0(.2)1.0$, $\Delta_1 = 1(2)5$, $\Delta_2 = 0(1)4$, $\Delta_2 < \Delta_1$,

$(\nu, f_1, f_2) = (5, 10, 5), (5, 15, 5), (5, 15, 10), (10, 10, 5), (10, 15, 5),$
$(10, 15, 10), (5, 30, 10), (5, 30, 15), (5, 30, 20), (5, 30, 25).$

$\alpha = .05$, $\theta = 0(.2)1.0$, $\Delta_1 = 1(2)3$, $\Delta_2 = 0(1)2$, $\Delta_2 < \Delta_1$,

$\nu = 10, 15$, $f_1 = 30$, $f_2 = 10(5)25$.

$\alpha = .01$, $\theta = 0(.2)1.0$, $\Delta_1 = 1(2)5$, $\Delta_2 = 0(1)4$, $\Delta_2 < \Delta_1$, $\nu = 5, 10$

$(f_1, f_2) = (10, 5), (15, 5), (15, 10)$, $f_1 = 30$, $f_2 = 10(5)25$.

$\alpha = .01$, $\theta = 0(.2)1.0$, $\Delta_1 = 1(2)5$, $\Delta_2 = 0(1)4$, $\Delta_2 < \Delta_1$, $\nu = 15$

$(f_1, f_2) = (15, 5), (15, 10)$, $f_1 = 30$, $f_2 = 10(5)25$.

$\alpha = .01$, $\theta = 0(.2)1.0$, $\Delta_1 = 7$, $\Delta_2 = 0(2)6$, $\nu = 5, 10$

$(f_1, f_2) = (10, 5), (15, 5), (15, 10)$.

Since Patnaik's approximation may be in error by one unit in the second decimal place, the accuracy of Table II is limited to at best an error of the same magnitude. Interpolation in the table on any of the degrees of freedom parameters should be made using the reciprocals, i.e., ν^{-1}, f_j^{-1}.

3. The effect of the weight factor on the Type II error. A typical Type II error curve is graphed in Fig. 1 for the parameters $\alpha = .05$, $\nu = 5$, $f_1 = 10$, $f_2 = 5$. Note that it is possible for the Type II error of the combined test to be larger than if a single test had been used alone. This corresponds to the case when the second test is given too much weight.

Note also that the minimum P_{II} is rather flat. For example for $\Delta_1 = 5$, $\Delta_2 = 1$ the minimum is between $\theta = .2$ and $\theta = .4$. This is typical of the behavior of P_{II}. Table III shows the range of θ for which the minimum P_{II} (to two decimal places) was attained. Also given in this table is the ratio $\Delta_2/\Delta_1 = \delta_2/\delta_1$ which we put forward as the weighting factor. In the entire table of P_{II} this choice of θ will result in being off by at most one unit in the second decimal from the minimum P_{II}.

TABLE II: *Values of* P_{II} *(Type II error) for* $\alpha = .05$

ν	Δ_1	Δ_2	$f_1 =$	10						15						15					
			$f_2 =$	5						5						10					
			$\theta =$	0.0	.2	.4	.6	.8	1.0	0	.2	.4	.6	.8	1.0	0	.2	.4	.6	.8	1.0
5	1	0		.793	.807	.817	.828	.842	.856	.779	.783	.793	.807	.823	.839						
	3	0		.405	.410	.429	.465	.512	.562	.322	.327	.346	.380	.424	.474						
		1			.381	.373	.384	.410	.442		.302	.297	.309	.334	.366	.322	.294	.283	.290	.310	.338
		2			.345	.306	.292	.294	.307		.272	.241	.231	.235	.248		.251	.206	.187	.185	.191
	5	0		.150	.155	.173	.206	.252	.306	.087	.090	.103	.125	.158	.200						
		1			.137	.138	.154	.181	.215		.079	.080	.092	.111	.137	.087	.076	.075	.084	.101	.124
		2			.116	.101	.102	.112	.129		.066	.058	.059	.067	.079		.058	.047	.046	.050	.058
		3			.105	.083	.078	.082	.092		.059	.047	.045	.048	.055		.049	.034	.030	.031	.035
		4			.096	.070	.061	.062	.067		.054	.039	.035	.036	.040		.042	.025	.020	.020	.021
10	1	0		.741	.744	.755	.772	.793	.814	.690	.692	.704	.724	.750	.776						
	3	0		.294	.299	.321	.360	.411	.467	.184	.188	.206	.238	.282	.333						
		1			.264	.253	.263	.286	.317		.163	.158	.167	.187	.215	.184	.153	.140	.144	.158	.179
		2			.235	.200	.190	.195	.208		.143	.121	.117	.123	.135		.122	.091	.080	.080	.085
	5	0		.071	.075	.088	.114	.151	.199	.023	.025	.030	.042	.060	.085						
		1			.061	.061	.070	.088	.112		.019	.020	.024	.032	.044	.023	.018	.017	.020	.026	.036
		2			.050	.042	.043	.051	.062		.016	.013	.014	.018	.023		.012	.009	.009	.011	.014
		3			.043	.032	.031	.034	.040		.013	.010	.010	.012	.015		.009	.006	.005	.006	.007
		4			.037	.024	.021	.022	.026		.011	.007	.007	.007	.009		.007	.003	.003	.003	.004

ν	Δ_1	Δ_2																							
		$f_1 =$	30						30						30						30					
		$f_2 =$	10						15						20						25					
		$\theta =$	0	.2	.4	.6	.8	1.0	0	.2	.4	.6	.8	1.0	0	.2	.4	.6	.8	1.0	0	.2	.4	.6	.8	1.0
5	1	0	.746	.752	.762	.777	.796	.814																		
	3	0	.235	.239	.255	.284	.322	.367																		
		1		.220	.217	.228	.250	.278	.235	.213	.206	.213	.231	.256	.235	.210	.202	.206	.223	.246	.235	.208	.197	.202	.218	.240
		2		.196	.174	.167	.172	.184		.180	.148	.135	.135	.141		.172	.135	.120	.118	.123		.167	.127	.112	.109	.113
	5	0	.040	.042	.048	.060	.079	.103																		
		1		.036	.037	.043	.054	.068	.040	.034	.034	.040	.049	.062	.040	.034	.033	.038	.047	.059	.040	.033	.033	.037	.046	.058
		2		.030	.026	.027	.031	.038		.026	.021	.021	.024	.028		.024	.019	.018	.020	.024		.023	.017	.017	.019	.022
		3		.027	.021	.020	.022	.026		.022	.015	.014	.014	.016		.019	.012	.011	.011	.013		.018	.011	.010	.010	.011
		4		.024	.017	.016	.016	.019		.018	.011	.009	.009	.010		.016	.008	.007	.007	.007		.014	.007	.006	.005	.006
10	1	0	.608	.613	.626	.650	.680	.711																		
	3	0	.082	.086	.096	.115	.142	.177																		
		1		.073	.071	.077	.090	.108	.082	.068	.062	.066	.075	.088	.082	.065	.058	.060	.068	.079	.082	.063	.055	.056	.063	.074
		2		.062	.053	.052	.056	.064		.053	.039	.035	.036	.040		.047	.032	.027	.027	[illegible]		.043	.027	.023	.022	.024
15	1	0	.518	.521	.537	.565	.601	.639																		
	3	0	.029	.030	.036	.046	.062	.084																		
		1		.024	.024	.027	.034	.043	.029	.022	.019	.021	.026	.032	.029	.020	.017	.018	.022	.027	.029	.019	.016	.016	.019	.024
		2		.020	.017	.017	.019	.024		.016	.011	.010	.011	.013		.013	.008	.007	.007	.008		.011	.006	.005	.006	.006

TABLE II (*Continued*): *Values of* P_{II} (*Type* II *error*) *for* $\alpha = .01$

ν	Δ_1	Δ_2	$f_1 =$ 10						15						15					
			$f_2 =$ 5						5						10					
			$\theta =$ 0	.2	.4	.6	.8	1.0	0	.2	.4	.6	.8	1.0	0	.2	.4	.6	.8	1.0
5	1	0	.921	.923	.930	.935	.940	.944	.904	.911	.919	.925	.932	.938						
	3	0	.714	.720	.732	.755	.788	.824	.622	.628	.641	.668	.708	.754						
		1		.705	.692	.694	.713	.742		.610	.598	.604	.628	.664	.622	.602	.583	.582	.600	.632
		2		.680	.636	.612	.611	.626		.582	.542	.523	.526	.546		.560	.500	.462	.452	.461
	5	0	.455	.459	.477	.514	.570	.638	.310	.314	.331	.366	.421	.491						
		1		.434	.426	.440	.477	.529		.293	.289	.305	.340	.393	.310	.287	.277	.288	.319	.366
		2		.402	.364	.353	.367	.400		.268	.242	.236	.252	.283		.250	.211	.197	.204	.226
		3		.383	.331	.306	.309	.330		.254	.216	.201	.207	.228		.229	.176	.151	.148	.159
		4		.368	.303	.268	.262	.276		.242	.195	.173	.172	.186		.211	.147	.117	.109	.113
	7	0	.248	.253	.271	.310	.372	.452	.122	.125	.138	.164	.209	.272						
		2		.206	.183	.182	.201	.237		.098	.087	.088	.102	.128	.122	.089	.073	.071	.077	.098
		4		.181	.140	.124	.128	.146		.084	.064	.058	.062	.074		.069	.044	.036	.037	.043
		6		.160	.107	.084	.080	.086		.073	.047	.037	.037	.042		.053	.026	.018	.017	.018
10	1	0	.896	.903	.912	.919	.926	.934	.868	.877	.886	.895	.907	.919						
	3	0	.629	.637	.651	.680	.722	.771	.474	.482	.498	.531	.583	.645						
		1		.608	.588	.587	.609	.645		.451	.436	.441	.468	.512	.474	.437	.408	.402	.420	.455
		2		.579	.529	.503	.504	.524		.423	.382	.365	.372	.398		.392	.325	.288	.280	.291
	5	0	.322	.327	.346	.387	.450	.529	.148	.152	.166	.196	.245	.314						
		1		.295	.284	.296	.332	.385		.133	.129	.140	.167	.209	.148	.125	.116	.122	.142	.177
		2		.267	.234	.226	.241	.274		.118	.102	.100	.113	.138		.102	.078	.071	.076	.091
		3		.250	.204	.185	.190	.212		.108	.086	.079	.085	.101		.088	.057	.047	.047	.054
		4		.234	.176	.148	.146	.158		.099	.072	.061	.062	.072		.075	.041	.029	.028	.030
	7	0	.133	.137	.153	.187	.243	.321	.032	.034	.040	.053	.078	.119						
		2		.101	.087	.087	.102	.130		.023	.020	.021	.027	.039						
		4		.083	.057	.049	.053	.064		.018	.012	.011	.013	.018						
		6		.070	.040	.030	.030	.035		.014	.008	.006	.007	.010						

15	1	0							.842	.854	.864	.875	.890	.906						
	3	0							.372	.376	.393	.430	.487	.557						
		1								.343	.328	.334	.363	.409	.372	.326	.297	.291	.309	.344
		2								.317	.280	.266	.277	.304		.284	.222	.192	.188	.199
	5	0							.082	.085	.096	.119	.159	.219						
		1								.071	.069	.077	.097	.129	.082	.065	.059	.062	.076	.101
		2								.061	.052	.052	.062	.080		.050	.036	.033	.037	.046
		3								.054	.041	.038	.042	.053		.040	.023	.019	.020	.024
		4								.049	.033	.028	.030	.037		.033	.016	.012	.012	.014

ν	Δ_1	Δ_2	$f_1 =$	30						30						30						30					
			$f_2 =$	10						15						20						25					
			$\theta =$	0	.2	.4	.6	.8	1.0	0	.2	.4	.6	.8	1.0	0	.2	.4	.6	.8	1.0	0	.2	.4	.6	.8	1.0
5	1	0		.883	.892	.901	.909	.918	.927																		
	3	0		.500	.502	.515	.543	.587	.640																		
		1			.483	.474	.483	.510	.550	.500	.476	.461	.463	.485	.521	.500	.472	.453	.453	.473	.507	.500	.469	.449	.447	.465	.498
		2			.457	.424	.409	.416	.440		.438	.387	.358	.354	.367		.427	.367	.332	.322	.331		.420	.353	.315	.303	.310
	5	0		.166	.169	.181	.206	.246	.303																		
		1			.156	.155	.166	.193	.233	.166	.152	.148	.156	.180	.216	.166	.150	.144	.152	.173	.208	.166	.149	.142	.149	.170	.203
		2			.140	.126	.125	.137	.160		.130	.109	.103	.109	.126		.124	.100	.092	.097	.111		.121	.095	.086	.090	.102
		3			.132	.111	.104	.110	.125		.118	.089	.077	.077	.086		.110	.077	.065	.064	.070		.104	.071	.058	.056	.061
		4			.125	.099	.088	.090	.100		.107	.073	.058	.056	.060		.097	.060	.046	.042	.045		.091	.053	.039	.035	.037
10	1	0		.819	.825	.836	.849	.867	.886																		
	3	0		.270	.273	.287	.316	.364	.426																		
		1			.250	.242	.250	.276	.316	.270	.240	.224	.224	.243	.275	.270	.234	.213	.210	.225	.253	.270	.230	.206	.200	.213	.240
		2			.230	.205	.198	.208	.231		.210	.170	.151	.150	.163		.196	.148	.126	.121	.129		.187	.134	.110	.104	.109
	5	0		.031	.032	.037	.047	.064	.093																		
		2			.024	.020	.021	.025	.034																		
		4			.019	.013	.012	.013	.016																		

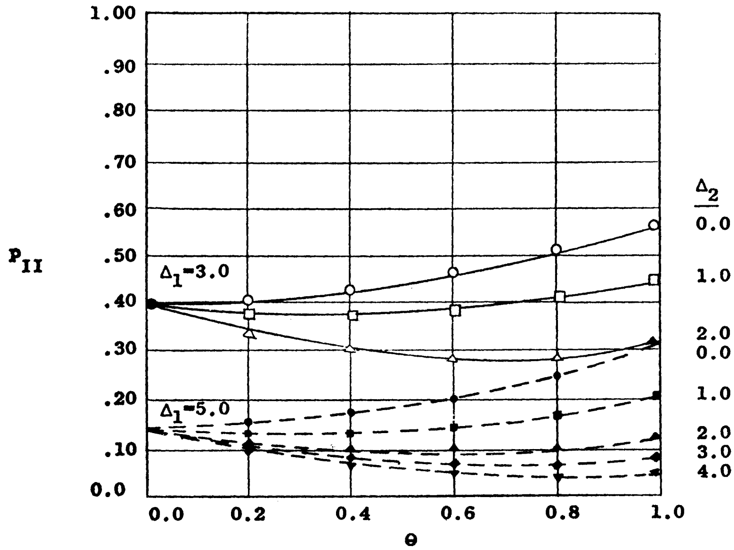

FIG. 1. Type II error (P_{II}) for $\nu = 5, f_1 = 10, f_2 = 5, \alpha = .05$.

TABLE III: *Range of θ for which minimum P_{II} is attained for $\nu = 5, 10$ and $\alpha = .01, .05$*

α	$\nu - f_1 - f_2$	$\Delta_1 =$ 3		5				7		
	$\Delta_2 =$	1	2	1	2	3	4	2	4	6
	$\frac{\Delta_2}{\Delta_1} =$	.33	.67	.20	.40	.60	.80	.29	.57	.86
.01	5–10– 5	.4–.6	.6–.8	.2–.4	.6	.6–.8	.8	.4–.6	.6	.6–.8
	15– 5	.4–.6	.6	.2–.4	.4–.6	.6	.6–.8	.4–.6	.6–.8	.6–.8
	10	.4–.6	.8	.4	.6–.8	.6–.8	.8–1.0	.4–.6	.6–.8	.6–.8
	10–10– 5	.4–.6	.6–.8	.4	.4–.6	.6	.6–.8	.4–.6	.6–.8	.6–.8
	15– 5	.4–.6	.6	.2–.4	.4–.6	.6	.6–.8	.4–.6	.4–.6	.6–.8
	10	.6	.8	.4–.6	.6	.6–.8	.6–.8	.4–.6	.4–.6	.4–.6
	5–30–10	.4	.6–.8	.2–.4	.4	.6	.4–.8			
	15	.4–.6	.8	.2–.4	.6	.6–.8	.6–1.0			
	20	.4–.6	.8	.4	.6	.6–.8	.8–1.0			
	25	.4–.6	.8	.4	.6–.8	.6–1.0	.6–1.0			
	10–30–10	.4	.4–.6		.2–.8		.4–.8			
	15	.4–.6	.6–.8							
	20	.4–.6	.8							
	25	.6	.8							
.05	5–10– 5	.4	.6–.8	.2–.4	.4–.6	.4–.6	.6–.8			
	15– 5	.2–.4	.6	.2–.4	.4–.6	.6	.4–1.0			
	10	.4	.6–1.0	.2–.6	.4–.8	.4–.8	.6–1.0			
	10–10– 5	.4	.6–.8	.2–.4	.4–.6	.4–.8	.4–1.0			
	15– 5	.2–.4	.4–.8	.2–.4	.4–.6	.2–.8	.2–1.0			
	10	.4–.6	.6–.8	.2–.6	.2–1.0	.6	.4–1.0			
	5–30–10	.2–.4	.4–.8	.2–.4	.2–.8	.4–.8	.2–1.0			
	15	.2–.6	.6–1.0	.2–.4	.4–.8	.6–.8	.4–1.0			
	20	.2–.6	.6–1.0	.2–.4	.2–1.0	.4–1.0	.4–1.0			
	25	.4–.6	.6–1.0	.2–.4	.2–1.0	.4–1.0	.2–1.0			
	10–30–10	.2–.4	.4–.6							
	15	.4	.4–1.0							
	20	.4–.6	.4–1.0							
	25	.2–.8	.6–1.0							

In general the non-centrality parameter can be written as

$$\delta_j = \frac{c_j}{\sigma_j^2}\mu^2, \qquad j = 1, 2, \tag{19}$$

where c_j is a known constant which depends on how the observations were taken, σ_j^2 is an underlying population variance, and μ^2 is a non-negative constant which depends on the particular hypothesis involved and is only equal to zero if the null hypothesis is true. Hence,

$$\theta = \frac{\delta_2}{\delta_1} = \frac{c_2}{c_1}\frac{\sigma_1^2}{\sigma_2^2}, \tag{20}$$

which is a function only of the known constants c_j and the ratio of the population variances. Of course in many practical situations the ratio of the variances σ_1^2/σ_2^2 may not be known. In this case we believe that the estimate for σ_1^2/σ_2^2 can be used in the weighting factor. This recommendation is based on the fact that the weighting factor need not be known accurately in order to achieve a minimum P_{II}. However it should be pointed out that this latter procedure will result in a change in the significance level and power of the test.

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