NOTES

SUMS OF SMALL POWERS OF INDEPENDENT RANDOM VARIABLES

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1. Introduction and summary. Let (x_{nk}), $k = 1, 2, \cdots k_n$; $n = 1, 2, \cdots$ be a double sequence of infinitesimal random variables which are rowwise independent (i.e. $\lim_{n\to\infty} \max_{1\leq k\leq k_n} P(|x_{nk}| > \epsilon) = 0$ for every $\epsilon > 0$, and for each n, $x_{n1}, \cdots, x_{nk_n}$ are independent). Let $S_n = x_{n1} + \cdots + x_{nk_n} - A_n$ where the A_n are constants and let $F_n(x)$ be the distribution function of S_n.

In a previous paper [3] the system of infinitesimal, rowwise independent random variables $(|x_{nk}|^r)$ was studied for $r \geqq 1$. Specifically, let

$$S_n^r = |x_{n1}|^r + \cdots + |x_{nk_n}|^r - B_n(r),$$

where the $B_n(r)$ are suitably chosen constants. Let $F_n^r(x)$ be the distribution function of S_n^r. Necessary and sufficient conditions for $F_n^r(x)$ to converge $(n \to \infty)$ to a distribution function $F^r(x)$ and for $F^r(x)$ to converge $(r \to \infty)$ to a distribution function $H(x)$ were given, together with the form that $H(x)$ must take.

In Section 2 of this paper we consider the system $(|x_{nk}|^r)$ for $0 < r < 1$. Results similar to the above are found, replacing $(r \to \infty)$ by $(r \to 0^+)$. However different assumptions must be made at certain points. Various remarks are made in this paper to show where the results here differ from [3]. In particular it is shown that, if $F^r(x)$ converges $(r \to 0^+)$ to a distribution function $H(x)$, then $H(x)$ will be the distribution function of the sum of two independent random variables, one Poisson and the other Gaussian. Furthermore, while the Gaussian summand may or may not be degenerate, the Poisson summand will be nondegenerate in all but one special case.

2. Small powers of random variables.[1] In the remainder of the paper we use the notation of [3].

Theorem 1. *Let* $\lim_{n\to\infty} F_n^r(x) = F^r(x)$ *for* $0 < r < 1$ *and* $\lim_{r\to 0^+} F^r(x) = H(x)$. *Then* $H(x)$ *is the distribution function of the sum of two independent random variables, one Gaussian and the other Poisson.*

We require the following lemma.

Lemma 1. *If we add to the hypothesis of Theorem* 1 *the condition that* $\lim_{n\to\infty} F_n(x) = F(x)$, *the conclusion of Theorem* 1 *holds.*

The proof of this lemma follows the same lines as Lemma 1 of [3] except that

$$N^*(x) = \begin{cases} N(+\infty) - M(-\infty) = 0, & x < 1 \\ N(0^+) - M(0^-), & 0 < x < 1, \end{cases}$$

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[1] The proofs in this section are similar to those given in [3], and hence they are condensed or omitted.

which implies that $N(0^+)$ and $M(0^-)$ are finite. Thus $N^*(x)$ is either identically 0 or takes one jump at $x = 1$. In fact $N^*(x)$ is identically 0 if and only if $N(0^+) = M(0^-) = 0$; i.e. if and only if $F(x)$ is Gaussian.

Proof of Theorem 1. Take $0 < s < 1$ and let $y_{nk} = | x_{nk} |^s$. Then

$$| x_{nk} |^r = | y_{nk} |^{r/s},$$

and, for $r/s < 1$, under the conditions of Theorem 1, the conditions of Lemma 1 are satisfied with the system (x_{nk}) replaced by (y_{nk}).

Remark. As can be seen from the above, if $F_n(x) \to F(x)$ then, under the conditions of Theorem 1, $H(x)$ is Gaussian if and only if $F(x)$ is Gaussian. That is, the (nondegenerate) Poisson summand will be present except when $F(x)$ is Gaussian.

LEMMA 2. *If* $\lim_{n\to\infty} F_n(x) = F(x)$ *and if* $M(x)$ *and* $N(x)$ *are bounded*[2], *then, for suitably chosen constants* $B_n(r)$, $F_n^r(x)$ *converges to a distribution function* $F^r(x)$ *if and only if*

$$\lim_{\epsilon\to 0} \lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \sum_{k=1}^{k_n} \left\{ \int_0^\epsilon x^{2r}\, d[F_{nk}(x) - F_{nk}(-x-)] - \left(\int_0^\epsilon x^r\, d[F_{nk}(x) - F_{nk}(-x-)]\right)^2 \right\} = \sigma_r^2 < \infty \tag{2.1}$$

The proof of this lemma is similar to Lemma 2 of [3] and will be omitted.

THEOREM 2. *If* $\lim_{n\to\infty} F_n(x) = F(x)$ *then a necessary and sufficient condition for* $\lim_{n\to\infty} F_n^r(x) = F^r(x)$ *and for* $\lim_{r\to 0^+} F^r(x) = H(x)$ *for suitably chosen constants* $B_n(r)$ *is that*

(2.2) $M(x)$ *and* $N(x)$ *are bounded,* (2.1) *holds, and*

$$\lim_{r\to 0^+} \sigma_r^2 = (\sigma^*)^2, \text{ a finite constant.}$$

Furthermore, $H(x)$ *is Gaussian if and only if* $F(x)$ *is Gaussian;* $H(x - m)$ *is nondegenerate Poisson if and only if* $F(x)$ *is not Gaussian and* $\sigma^* = 0$ *where* m *is a constant; otherwise* $H(x)$ *is the sum of two independent random variables, one Gaussian and the other Poisson.*

PROOF. Necessity. It follows from the proof of Lemma 1 that $M(x)$ and $N(x)$ are bounded and by Lemma 2 that (2.1) holds. We also see (Theorem 2 page 88 of [1]) that if σ^* is the non-negative constant associated with the infinitely divisible distribution $H(x)$ that

$$\lim_{\epsilon\to 0} \lim_{r\to 0^+} \begin{matrix}\sup\\ \inf\end{matrix} \left\{ \int_{-\epsilon}^0 u^2\, dM^r(u) + \sigma_r^2 + \int_0^\epsilon u^2\, dN^r(u) \right\} = (\sigma^*)^2. \tag{2.3}$$

[2] This could be replaced by a weaker condition; however, this condition appears in the proof of Lemma 1 and will appear as a necessary and sufficient condition in Theorem 2.

Now since $M^r(u) \equiv 0$ and $N^r(u) = N(u^{1/r}) - M(-u^{1/r})$ we see that

$$\int_{-\epsilon}^{0} u^2\, dM^r(u) + \int_0^{\epsilon} u^2\, dN^r(u) \leqq \epsilon^2 \int_0^{\epsilon} d[N(u^{1/r}) - M(-u^{1/r})],$$

and, since $M(u)$ and $N(u)$ are bounded, we see that (2.2) holds.

Sufficiency. By Lemma 2 we have $\lim_{n\to\infty} F^r_n(x) = F^r(x)$. Also, analogously to [3], $\lim_{r\to 0^+} M^r(x) \equiv 0 \equiv M^*(x)$ and

$$\lim_{r\to 0^+} N^r(x) = N^*(x) = \begin{cases} 0, & x > 1 \\ N(0^+) - M(0^-), & 0 < x < 1. \end{cases}$$

Furthermore $\int_{-\epsilon}^{0} x^2\, dM^*(x) + \int_0^{\epsilon} x^2\, dN^*(x) < \infty$ and since

$$\lim_{\epsilon\to 0} \lim_{r\to 0^+} \begin{matrix}\sup \\ \inf\end{matrix} \left\{ \int_{-\epsilon}^{0} x^2\, dM(x) + \int_0^{\epsilon} x^2\, dN^r(u) \right\} = 0$$

(as in the necessity proof) we see that (2.3) holds. Now if we replace $r \to \infty$ by $r \to 0^+$, the remainder of the proof is the same as that of Theorem 2 of [3].

Remark. In [3] the conditions imposed on $M(x)$ and $N(x)$, (and hence on $F(x)$) required $F(x)$ to have moments of all orders (c.f. [2] and [1] page 83). In the present paper our conditions on $M(x)$ and $N(x)$ are different and in particular do not require $F(x)$ to have any moments.

THEOREM 3. If $\lim_{n\to\infty} F_n(x) = F(x)$, $M(x)$ and $N(x)$ are bounded, and if for some $\epsilon > 0$

$$\sum_{k=1}^{k_n} \int_{|x|<\epsilon} |x|^s\, dF_{nk}(x) \tag{2.4}$$

is bounded in n for any fixed $s > 0$, then, for suitably chosen constants $B_n(r)$, $F^r_n(x)$ converges $(n \to \infty)$ to a distribution function $F^r(x)$ and $F^r(x)$ converges $(r \to 0^+)$ to the Poisson distribution.

PROOF: We first show that (2.4) implies (2.1) with $\sigma_r = 0$. This follows since for any $\epsilon > 0$, r fixed and $s < 2r$ we have

$$\sum_{k=1}^{k_n} \left(\int_0^{\epsilon} x^{2r}\, d[F_{nk}(x) - F_{nk}(-x-)] \right) \leqq \epsilon^{2r-s} \sum_{k=1}^{k_n} \int_{|x|\leqq\epsilon} |x|^s\, dF_{nk}(x).$$

Now using Theorem 2, since $\sigma^* = 0$, we see that (by proper choice of $B_n(r)$) $H(x)$ is a Poisson distribution (possibly degenerate).

REFERENCES

[1] B. V. GNEDENKO AND A. N. KOLMOGOROV, *Limit Distributions for Sums of Independent Random Variables*, translation by K. L. Chung, Addison-Wesley, 1954.

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[3] J. M. SHAPIRO, "Sums of powers of independent random variables," *Ann. Math. Stat.*, Vol. 29 (1958), pp. 515–522.