THE ALGEBRA OF A LINEAR HYPOTHESIS[1]

By Henry B. Mann

The Ohio State University

Introduction. Let $y = (y_1, \cdots, y_N)$ be a random vector. We consider the following sequence of hypotheses:

$$\begin{aligned} &A \text{ (assumption)}: && E(y_\alpha) = \textstyle\sum_{j=1}^{s} p_{\alpha j}\beta_j, && \alpha = 1, \cdots, N. \\ &H_1: && \beta_1 = \cdots = \beta_{s_1} = 0, \\ &\vdots \\ &H_r: && \beta_{s_1+s_2+\cdots+s_{r-1}+1} = \cdots = \beta_{s_1+\cdots+s_r} = 0, \end{aligned}$$

where $s_1 + s_2 + \cdots + s_r = s$.

For $\alpha = 1, \cdots, N$, $l = 1, \cdots, r$ we put

$$\begin{aligned} &p_{\alpha j}^{(l)} = p_{\alpha j} \qquad j = s_1 + \cdots + s_{l-1} + 1, \cdots, s_1 + \cdots + s_l. \\ (1) \qquad &p_{\alpha j}^{(l)} = 0 \qquad \text{otherwise} \\ &p_l = (p_{\alpha j}^{(l)}) \end{aligned}$$

We consider the algebra $\mathfrak{A}$ generated over a real field by the matrices $p_l p_l'$ where A' denotes the transpose of A. It will be seen that this algebra is closely related to the analysis of variance of our linear hypotheses. In particular all tests of sequences of hypotheses correspond to a decomposition of $\mathfrak{A}$ into left ideals. Thus the study of the decomposition of $\mathfrak{A}$ sheds considerable light on the analysis of variance appropriate to the linear hypothesis. The algebra $\mathfrak{A}$ was first considered by A. T. James [1] for the important case that the matrices $p_l p_l'$ are relationship matrices. James also pointed out that $\mathfrak{A}$ is semisimple and hence a direct sum of complete matrix algebras.

In this paper we shall first consider the general problem and show that the tests appropriate to the sequence of hypotheses $H_1^* = H_1$, $H_2^* = H_1 \& H_2, \cdots,$ $H_r^* = H_1 \& H_2 \cdots \& H_r$ lead to a decomposition of $\mathfrak{A}$ into (not necessarily simple) left ideals. We shall then consider the case where $\mathfrak{A}$ is generated by two generators p_1p_1', p_2p_2' and where moreover $(p_ip_i')^2 = \mu(p_ip_i')$. (Throughout this paper Greek letters will denote scalars.) In this case we shall obtain the complete decomposition of $\mathfrak{A}$ into principal components. This case includes in particular all those incomplete block designs in which each block contains the same number of experimental units while each treatment is replicated the same number of times. We shall then be able to establish a relation between the decomposition of $\mathfrak{A}$ into principal components and the power function of our tests. Finally we

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shall illustrate our methods by decomposing the algebra of an s-dimensional cubic lattice into its principal components.

In the following the term matrix will always mean a matrix with real elements.

1. General Theorems.

THEOREM 1. *Let p be any matrix. There exists a matrix A such that $p'pA = p'$, where p' denotes the transpose of p. The matrix pA is moreover idempotent and symmetric.*

PROOF: Let p have N rows and s columns. Consider the indeterminate N dimensional column vector y. Put $E(y) = pb$ where b is an s dimensional column vector. Then for any choice of y the expression $Q = \sum_\alpha (y_\alpha - E(y_\alpha))^2$ must have a minimum with respect to b. Differentiating with respect to each b, we obtain the equation

$$p'y = p'pb \tag{2}$$

which must have a solution since Q has a minimum. Since (2) is a system of linear equations the b's must be linear functions of the y's. Hence $b = Ay$ and therefore

$$p' = p'pA. \tag{3}$$

Multiplying (3) from the left by A' we get $A'p' = A'p'pA$.

Hence $A'p'$ and therefore also pA is symmetric. Furthermore,

$$(pA)^2 = A'p'pA = A'p' = pA.$$

COROLLARY: *If pp' is an idempotent matrix then $p'pp' = p'$, $pp'p = p$.*

THEOREM 2. *If $a_0 + a_1x + \cdots + a_sx^s$ is the minimal polynomial of a symmetric matrix then either $a_0 \neq 0$ or $a_1 \neq 0$.*

Theorem 2 is an immediate consequence of the fact that a symmetric matrix may be transformed into a diagonal matrix by an orthogonal transformation.

THEOREM 3. *The matrix pA of Theorem 1 is uniquely determined.*

If $a_0 + a_1x + \cdots + a_sx^s$ is the minimal polynomial for $p'p$, then

$$\begin{aligned} pA &= -(a_1pp' + \cdots + a_s(pp')^s) \quad \text{if} \quad a_0 = 1. \\ pA &= -(a_2pp' + \cdots + a_spp')^{s-1}) \quad \text{if} \quad a_0 = 0, a_1 = 1. \end{aligned} \tag{4}$$

Let $a_0 = 1$ then $I = -(a_1p'p + \cdots + a_s(p'p)^s)$ where I is the unit matrix. Multiplying this equation from the right by A and from the left by p we obtain the first equation of (4).

Let $a_0 = 0$, $a_1 = 1$, then $p'p = -(a_?(p'p)^2 + \cdots + a_s(p'p)^s)$ and we obtain the second equation of (4) by multiplying left by A' and right by A.

COROLLARY: *The matrix pA of Theorem 1 lies in the algebra generated by pp'.*

We now consider the sequence of hypotheses H_1, H_1 & H_2, $\cdots$, H_1 & H_2 & $\cdots$ & H_r. Put

$$(5)\qquad \begin{aligned} P_1 &= p_1 + p_2 + \cdots + p_r, \\ P_2 &= p_2 + \cdots + p_r, \\ &\vdots \\ P_r &= p_r. \end{aligned}$$

We solve

$$(6)\qquad P_i' = P_i' P_i A_i.$$

The vectors $Y^{(i)} = P_i A_i y$ are the regression values corresponding to the hypotheses H_1 & H_2 & $\cdots$ H_i, $(i = 1 \cdots r)$. The decomposition

$$(7)\qquad \sum y_i^2 = y'(I - P_1A_1)y + y'(P_1A_1 - P_2A_2)y + \cdots + y'(P_{r-1}A_{r-1} - P_rA_r)y + y'P_rA_ry$$

where I denotes the identity is the proper decomposition of the sum of squares for testing the hypotheses H_1 & $\cdots$ & H_i [2, p. 33].

If the parameters β_j of our linear hypothesis are subject to restrictions and if there exists a solution $b = A_i y$ satisfying the restrictions then since P_iA_i is unique by Theorem 3 the decomposition (7) will still be the appropriate decomposition for the analysis of variance, although the degrees of freedom will have to be adjusted. Thus all our results will remain applicable to this case. If the least square equations are solved by the method of Lagrange operators the existence of solutions of the least square equations which satisfy the restrictions means that the Lagrange operators may be ignored. A very important case of this type is treated in Theorem 4.4 of [2].

Corresponding to (7) we have, as we shall show, a decomposition of $\mathfrak{A} \cup I$ into left ideals.

We have

$$I = (I - P_1A_1) + (P_1A_1 - P_2A_2) + \cdots + (P_{r-1}A_{r-1} - P_rA_r) + P_rA_r.$$

We have $p_iP_j' = p_ip_i'$ for $i \geqq j$ hence from (6) we get

$$(8)\qquad p_ip_i' = p_ip_i'P_jA_j = P_jA_jp_ip_i' \quad \text{for} \quad i \geqq j.$$

Now by Theorem 3 P_jA_j is a polynomial in $P_jP_j' = \sum_{i=j}^{r} p_ip_i'$. Hence for $i \geqq j$ we have $P_iA_i\, P_jA_j = P_jA_j\, P_iA_i = P_iA_i$ and thus the idempotents $e_i = P_iA_i - P_{i+1}A_{i+1}(i = 0, \cdots, r, P_0A_0 = I, P_{r+1}A_{r+1} = 0)$ are a set of orthogonal idempotents. Hence [3, p. 147, Problem 4] the decomposition $\mathfrak{A} = \mathfrak{A}_1e_1 + \cdots + \mathfrak{A}_re_r$ is a representation of $\mathfrak{A}$ as a direct sum of left ideals. These left ideals are however not always simple left ideals.

THEOREM 4. *The algebra $\mathfrak{A}$ generated by p_ip_i' $i = 1 \cdots r$ has the unit element P_1A_1. The matrix P_1P_1' has an inverse in $\mathfrak{A}$.*

Equation (8) shows that P_1A_1 is the unit element of $\mathfrak{A}$. Equation (4) may be written

$$P_1A_1 = P_1P_1'(-a_1P_1A_1 - a_2P_1P_1' + \cdots - a_s(P_1P_1')^{s-1})$$

if P_1P_1' is nonsingular and

$$P_1A_1 = P_1P_1'(-a_2P_1A_1 + \cdots - a_s(P_1P_1')^{s-2})$$

if P_1P_1' is singular.

Let now $\mathfrak{A}$ be generated by one matrix pp'. We assume first that pp' is a diagonal matrix. Let $\lambda_1, \cdots, \lambda_n$ be its distinct characteristic roots. Then

$$E_1 = \frac{(pp' - \lambda_2 I)\cdots(pp' - \lambda_n I)}{(\lambda_1 - \lambda_2)\cdots(\lambda_1 - \lambda_n)} = \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ 0 & \ddots & & & \\ \cdot & & \ddots & & \\ \cdot & & 1 & & \\ \cdot & & & 0 & \\ \cdot & & & & \ddots \\ 0 & & & & 0 \end{pmatrix}, \tag{9}$$

$$E_i = \frac{(pp' - \lambda_1 I)\cdots(pp' - \lambda_{i-1} I)(pp' - \lambda_{i+1} I)\cdots(pp' - \lambda_n I)}{(\lambda_1 - \lambda_i)\cdots(\lambda_{i-1} - \lambda_i)(\lambda_{i+1} - \lambda_i)\cdots(\lambda_n - \lambda_i)}$$

are orthogonal idempotents and the decomposition

$$\mathfrak{A} = \mathfrak{A}E_1 + \cdots + \mathfrak{A}E_n = \{\mu_1 E_1\} + \cdots + \{\mu_k E_k\} \tag{10}$$

is a decomposition of $\mathfrak{A}$ into principal components.

If pp' is not diagonal let T be an orthogonal matrix such that $Tpp'T'$ is a diagonal matrix. The isomorphism $pp' \to T\ pp'\ T'$ is a faithful isomorphism. Hence the decomposition (10) with the E_i given by (9) is still a decomposition of $\mathfrak{A}$ into principal components.

In considering the general problem we may therefore assume that the matrices p_ip_i' are idempotent. We shall also assume $r > 1$. Let $\mathfrak{A}$ be generated by the idempotent matrices $p_1p_1', p_2p_2', \cdots, p_rp_r'$.

THEOREM 5. *If for $G \neq 0$, $G \varepsilon \mathfrak{A}$, $G_1 \varepsilon \mathfrak{A}$ we have $p_ip_i'G = \mu G$, $p_ip_i'G_1 = \mu_1 G_1$ for $i = 1 \cdots, r$ then $\mu = 1$, $G_1 = \alpha G$.*

We have $p_ip_i'G = (p_ip_i')^2G = \mu G = \mu^2 G$. With P_1A_1 defined by (6) we have by Theorem (4) $P_1A_1G = G$ hence $\mu \neq 0$ and $\mu^2 = \mu$ implies $\mu = 1$.

If B is any element of $\mathfrak{A}$ we may write

$$B = \sum \alpha_i B_i$$

where the B_i are monomials in $p_1p_1', \cdots, p_rp_r'$ and

$$BG = \alpha G; \qquad \alpha = \sum \alpha_i . \tag{11}$$

Since $\mathfrak{A}$ is generated by symmetric matrices, $A \varepsilon \mathfrak{A}$ implies $A' \varepsilon \mathfrak{A}$ and so

$$G'G = \lambda G. \tag{12}$$

For any matrix $M \neq 0$ we have $M'M \neq 0$ hence in (12) $\lambda \neq 0$ and (12) shows that G is a symmetric matrix. Thus $G_1G = \alpha G = \alpha^* G_1$. If $\alpha = 0$ then in the representation of G_1 by monomials we have $\sum\alpha_i = 0$ and therefore $G_1^2 = G_1'G_1 = 0$ whence $G_1 = 0 = 0G$. If $\alpha^* \neq 0$, $\alpha \neq 0$ we have $G_1 = \alpha/\alpha^* \, G$. This proves Theorem 5.

If $G^2 = \lambda G$ we may replace G by G/λ. Hence we may assume that G is idempotent and decompose $\mathfrak{A}$ into

$$\mathfrak{A} = (\mathfrak{A} - \mathfrak{A}G) + \mathfrak{A}G = (\mathfrak{A} - \mathfrak{A}G) + \{\alpha G\}. \tag{13}$$

The one dimensional two sided ideal $\{\alpha G\}$ is a principal component of $\mathfrak{A}$ and in $\mathfrak{A} - \mathfrak{A}G$ the element 0 is the only element G_1 for which $p_ip_i'G_1 = G_1$.

2. Algebras generated by two idempotent generators. If p_1p_1' has an inverse in $\mathfrak{A}$ then $p_1p_1' = P_1A_1$ and the algebra becomes trivial. Hence we may assume that p_1p_1' and therefore also $p_1p_1'\,p_2p_2'\,p_1p_1'$ are singular.

THEOREM 6. *Let the algebra $\mathfrak{A}$ be generated by two idempotent generators p_1p_1', p_2p_2'. Let $T_1 = p_1p_1'\,p_2p_2'\,p_1p_1'$, $T_2 = p_2p_2'\,p_1p_1'\,p_2p_2'$ and let $M(x) = x(x - \lambda_1) \cdots (x - \lambda_n)$ be the minimal polynomial of T_1. Put*

$$\begin{aligned}
F_1 &= p_1p_1' \frac{(T_1 - \lambda_1)\cdots(T_1 - \lambda_n)}{(-1)^n\lambda_1\cdots\lambda_n}, \\
F_2 &= p_2p_2' \frac{(T_2 - \lambda_1)\cdots(T_2 - \lambda_n)}{(-1)^n\lambda_1\cdots\lambda_n}, \\
\epsilon_1^{(\alpha)} &= \frac{T_1(T_1 - \lambda_1)\cdots(T_1 - \lambda_{\alpha-1})(T_1 - \lambda_{\alpha+1})\cdots(T_1 - \lambda_n)}{\lambda_\alpha(\lambda_\alpha - \lambda_1)\cdots(\lambda_\alpha - \lambda_{\alpha-1})(\lambda_\alpha - \lambda_{\alpha+1})\cdots(\lambda_\alpha - \lambda_n)}, \\
\epsilon_2^{(\alpha)} &= \frac{T_2(T_2 - \lambda_1)\cdots(T_2 - \lambda_{\alpha-1})(T_2 - \lambda_{\alpha+1})\cdots(T_2 - \lambda_n)}{\lambda_\alpha(\lambda_\alpha - \lambda_1)\cdots(\lambda_\alpha - \lambda_{\alpha-1})(\lambda_\alpha - \lambda_{\alpha+1})\cdots(\lambda_\alpha - \lambda_n)}, \\
f_\alpha &= \epsilon_1^{(\alpha)} = G \qquad \text{if } \lambda_\alpha = 1, \\
f_\alpha &= \frac{(\epsilon_1^{(\alpha)} - \epsilon_2^{(\alpha)})^2}{(1 - \lambda_\alpha)} \qquad \text{if } \lambda_\alpha \neq 1.
\end{aligned} \tag{14}$$

Then

(i) $\mathfrak{A} = \mathfrak{A}F_1 + \mathfrak{A}F_2 + \sum_{\alpha=1}^{n} \mathfrak{A}f_\alpha$ *is the decomposition of $\mathfrak{A}$ into principal components. (One or both of the components $\mathfrak{A}F_1$, $\mathfrak{A}F_2$ may reduce to* 0.)

(ii) $p_1p_1'F_1 = F_1$, $p_1p_1'F_2 = 0$, $p_2p_2'F_1 = 0$, $p_2p_2'F_2 = F_2$, $p_1p_1'G = p_2p_2'G = G$. *The algebras $\mathfrak{A}F_1$, $\mathfrak{A}F_2$, $\mathfrak{A}G$ are complete 1×1 matrix algebras or zero.*

(iii) *For $\lambda_\alpha \neq 1$ the algebras $\mathfrak{B}_\alpha = \mathfrak{A}f_\alpha$ are complete 2×2 matrix algebras.*

PROOF: It is clear from (9) that F_1, F_2, $\epsilon_1^{(\alpha)}$, $\epsilon_2^{(\alpha)}$ are idempotents.

Furthermore $F_1p_1p_1' = p_1p_1'F_1 = F_1$, $F_1p_1p_1'p_2p_2'p_1p_1' = 0$. By Theorem 1 we have from this $F_1p_1p_1'p_2 = 0$, and so

$$F_1p_1p_1'p_2p_2' = F_1p_2p_2' = 0, \tag{15}$$

and by transposing $p_2p_2'F_1 = 0$. Hence F_1 and similarly F_2 are in the center of $\mathfrak{A}$.

That F_1, F_2 and the f_α are orthogonal follows from the following Lemma.

LEMMA 1: *For any polynomial $H(x) = x\psi(x)$ we have*

$$p_1p_1'H(p_2p_2'p_1p_1'p_2p_2') = H(p_1p_1'p_2p_2'p_1p_1')p_2p_2'.$$

This follows easily since the relation

$$p_1p_1'(p_2p_2'p_1p_1'p_2p_2')^m = (p_1p_1'p_2p_2'p_1p_1')^m p_2p_2'.$$

holds for every $m > 0$.

If $\lambda_\alpha = 1$ put $G = G_1$ and $G_2 = \epsilon_2^{(\alpha)}$.

We have $(T_1 - 1)G_1 = (T_2 - 1)G_2 = 0$.

Hence

$$T_1G_1 = G_1, \qquad T_2G_2 = G_2. \tag{16}$$

Putting $c = G_1 - G_1p_2p_2'$ we find from (16) $cc' = 0$, hence $c = 0$ and $G_1 = G_1p_1p_1' = G_1p_2p_2'$ and similarly $G_2 = G_2p_1p_1' = G_2p_2p_2'$. Hence by Theorem 5 $G_1 = G_2 = G$, where G satisfies the relations of Theorems 5.

Now let $\lambda_\alpha \neq 1$. We have

$$\begin{aligned} p_1p_1'\,\epsilon_1^{(\alpha)} &= \epsilon_1^{(\alpha)}, & p_2p_2'\,\epsilon_1^{(\alpha)} &= \epsilon_2^{(\alpha)}p_1p_1'. \\ p_2p_2'\,\epsilon_2^{(\alpha)} &= \epsilon_2^{(\alpha)}, & p_1p_1'\,\epsilon_2^{(\alpha)} &= \epsilon_1^{(\alpha)}p_2p_2', \\ T_1\,\epsilon_1^{(\alpha)} &= \lambda_\alpha\,\epsilon_1^{(\alpha)}, & T_2\,\epsilon_2^{(\alpha)} &= \lambda_\alpha\,\epsilon_2^{(\alpha)}. \end{aligned} \tag{17}$$

Thus

$$\epsilon_1^{(\alpha)}\,\epsilon_2^{(\alpha)}\,\epsilon_1^{(\alpha)} = \lambda_\alpha\,\epsilon_1^{(\alpha)}$$

and

$$f_\alpha p_1p_1' = p_1p_1'f_\alpha = \epsilon_1^{(\alpha)}, \qquad f_\alpha p_2p_2' = p_2p_2'f_\alpha = \epsilon_2^{(\alpha)}.$$

This shows that f_α is in the center of $\mathfrak{A}$. Moreover $T_1f_{(\alpha)} = \lambda_\alpha\,\epsilon_1^{(\alpha)}$ $T_2f_\alpha = \lambda_\alpha\epsilon_2^{(\alpha)}$.

Using these relations one easily finds $f_\alpha^2 = f_\alpha$. We show next that the direct sum $\mathfrak{A}F_1 + \mathfrak{A}F_2 + \sum\mathfrak{A}f_\alpha$ contains the algebra $\mathfrak{A}$. From Lagrange's interpolation formula we have setting $\lambda_0 = 0$,

$$\sum\frac{M(x)}{(x - \lambda_\alpha)M'(\lambda_\alpha)} = 1.$$

Substituting in this identity T_1 for x and P_1A_1 for the unit element and multiplying by p_1p_1' we get

$$\sum f_\alpha p_1p_1' + F_1 = p_1p_1'$$

and similarly

$$\sum f_\alpha p_2p_2' + F_2 = p_2p_2'.$$

Hence $\mathfrak{A}F_1 + \mathfrak{A}F_2 + \sum\mathfrak{A}f_\alpha$ contains both generators of $\mathfrak{A}$ and therefore $\mathfrak{A}$ itself.

Every element of the algebra $\mathfrak{A}f_\alpha = \mathfrak{B}_\alpha$ may be written in the form $\alpha_1\epsilon_1^{(\alpha)} + \alpha_2\epsilon_2^{(\alpha)} + \alpha_{12}\epsilon_1^{(\alpha)}\epsilon_2^{(\alpha)} + \alpha_{21}\epsilon_2^{(\alpha)}\epsilon_1^{(\alpha)}$. The elements

$$f_{11} = \frac{\epsilon_1^{(\alpha)} - \epsilon_2^{(\alpha)}\epsilon_1^{(\alpha)}}{1 - \lambda_\alpha}, \qquad f_{22} = \frac{\epsilon_2^{(\alpha)} - \epsilon_1^{(\alpha)}\epsilon_2^{(\alpha)}}{1 - \lambda_\alpha},$$

$$f_{12} = \frac{\epsilon_1^{(\alpha)}\epsilon_2^{(\alpha)} - \lambda_\alpha\epsilon_2^{(\alpha)}}{1 - \lambda_\alpha}, \qquad f_{21} = \frac{\epsilon_2^{(\alpha)}\epsilon_1^{(\alpha)} - \lambda_\alpha\epsilon_1^{(\alpha)}}{1 - \lambda_\alpha}$$

satisfy the condition $f_{ij}f_{jk} = f_{ik}$, $f_{ij}f_{kl} = 0$ for $j \neq k$.

Hence we have the isomorphism from $\mathfrak{B}_\alpha$ onto a complete two dimensional matrix algebra

$$f_{11} \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad f_{22} \leftrightarrow \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$f_{12} \leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f_{21} \leftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

This completes the proof of Theorem 6.

Corollary: *If the scalar field of $\mathfrak{A}$ is real then the principal components of $\mathfrak{A}$ are real.*

This follows since T_1 is a symmetric (even positive semi definite) matrix. Hence its characteristic roots are real (even non-negative).

In applying Theorem 6 to concrete situations it is often of advantage to replace the matrices p_ip_i' by $p_ip_i' - G$. In the algebra $\mathfrak{A} - G$ obtained in this way one then has $\lambda_\alpha \neq 1$ for all α.

3. Relations to tests of hypotheses. If f is a principal idempotent of the algebra $\mathfrak{A}$ generated by $p_1p_1', \cdots, p_rp_r'$ then f' is also an idempotent of the center of $\mathfrak{A}$. Since $f'f \neq 0$ we must have $f'f = f$, hence f is symmetric. Since every idempotent of the center is a sum of principal idempotents it follows that all idempotents of the center are symmetric.

The significance of the decomposition into principal components is pointed up by the following theorem.

Theorem 7. *Let $P_1A_1 = I_1 + I_2$ where I_1, I_2 are orthogonal symmetric idempotents of $\mathfrak{A}$. The idempotents I_1, I_2 belong to the center of $\mathfrak{A}$ if and only if for every matrix P such that $PP' \varepsilon \mathfrak{A}$, the relations*

$$(18) \qquad P'PA = P', \qquad P'I_1PB_1 = P'I_1, \qquad P'I_2PB_2 = P'I_2$$

imply

$$(19) \qquad PA = I_1PB_1 + I_2PB_2.$$

Proof: By Theorem 3 we have PA ε $\mathfrak{A}$. If I_1 is in the center of $\mathfrak{A}$ we get from (18)

$$(20) \qquad P'I_1 = P'PAI_1 = P'I_1PA.$$

Because of the uniqueness of the regression values (Theorem 3) this implies

$$I_1PA = I_1PB_1 \tag{21}$$

and similarly $I_2PA = I_2PB_2$ if I_2 is in the center of $\mathfrak{A}$. Hence

$$I_1PA + I_2PA = P_1A_1PA = PA = I_1PB_1 + I_2PB_2$$

if I_1, I_2 are in the center of $\mathfrak{A}$.

To prove the sufficiency of (19) put $P = p_i$. Multiplying (19) from the left by I_1 gives $I_1p_iA = I_1p_iB_1$. The matrices p_iA and $I_1p_iB_1$ are symmetric by (18) (see Theorem 1). Hence $I_1p_iA = p_iAI_1$. Moreover $p_iA = p_ip_i'$ (see the corollary to Theorem 1), and therefore $I_1p_ip_i' = p_ip_i'I_1$. Since this relation holds for every value of i, the matrix I_1 and similarly I_2 are in the center of $\mathfrak{A}$.

Theorem 7 shows that it is sufficient to study the tests of linear hypotheses for each of the principal components separately.

We return now to the case of two idempotent generators.

Every vector $a = (a_1, \cdots, a_n)$ can be decomposed uniquely into two parts

$$a = a(I - P_1A_1) + aP_1A_1. \tag{22}$$

THEOREM 8. *If $aQ = a$ for some element Q of the algebra $\mathfrak{A}$ generated by p_1p_1', p_2p_2' then $a = aP_1A_1$ and hence $a(I - P_1A_1) = 0$.*

For we have $a(I - P_1A_1) = aQ(I - P_1A_1) = 0$.

DEFINITION: *A form ax, $a = a_1, \cdots, a_N$, $x = x_1, \cdots, x_N$ is called totally confounded or confounded with coefficient 1 in $\mathfrak{A}$ if $a \neq (0, \cdots, 0)$ and*

$$ap_1p_1' = ap_2p_2' = a, \tag{23}$$

it is called confounded with coefficient $\lambda \neq 1$ if

$$a(p_1p_1' - p_2p_2')^2 = (1 - \lambda)a. \tag{24}$$

If $\lambda = 0$ then a is called unconfounded. The rows of G are all totally confounded. The rows of F_1 and F_2 are unconfounded. The rows of f_α are confounded with coefficient λ_α.

Multiplying (24) by p_1p_1' from the right we get

$$ap_1p_1'p_2p_2'p_1p_1' = \lambda ap_1p_1' \tag{25}$$

and similarly

$$ap_2p_2'p_1p_1'p_2p_2' = \lambda ap_2p_2'. \tag{25a}$$

THEOREM 9. *Let a be any vector then*

$$a = a_0 + a_{10} + a_{20} + a_1 + \sum_{\alpha=2}^{n} a_\alpha \tag{26}$$

where

(i) $a_0p_1p_1' = a_0p_2p_2' = 0$,

(ii) $a_{10}p_1p_1' = a_{10}$, $a_{10}p_2p_2' = 0$,
$a_{20}p_2p_2' = a_{20}$, $a_{20}p_1p_1' = 0$

(iii) a_α *is confounded with coefficient* λ_α, $a = 1, \cdots, n$.
where the λ_α *are the distinct characteristic roots of* $p_1p_1'p_2p_2'p_1p_1'$ *and* $\lambda_1 = 1$ if 1 *is a c.r. The decomposition* (26) *is unique.*

PROOF: We have the decomposition

$$I = I - P_1A_1 + F_1 + F_2 + f_1 + \sum_{\alpha=2}^{n} f_\alpha . \tag{27}$$

Multiplying (27) by a we obtain (26).

That the decomposition (26) is unique follows from Theorem 8 and from the following two lemmas.

LEMMA 1: *If* a *is confounded with coefficient* λ *then* $\lambda = \lambda_\alpha$ *for some* α *and* $a = af_\alpha$ *where we set* $f_0 = F_1 + F_2$.

LEMMA 2: *If* $ap_1p_1' = a$, $ap_2p_2' = 0$ *then* $a = aF_1$.

PROOF OF LEMMA 1: Since a is confounded it follows from Theorem 8 that $a(I - P_1A_1) = 0$. For $\lambda_\alpha \neq 1$ we have

$$af_\alpha = a\frac{(p_1p_1' - p_2p_2')^2}{1 - \lambda_\alpha} f_\alpha = a\frac{1 - \lambda}{1 - \lambda_\alpha} f_\alpha .$$

Hence $af_\alpha = 0$ for $\lambda \neq \lambda_\alpha$. For $\lambda \neq 1$ we have

$$aG = a\frac{(p_1p_1' - p_2p_2')^2}{1 - \lambda} G = 0.$$

Hence since $a \neq 0$ we must have $\lambda = \lambda_\alpha$ for some α and multiplying (27) by a we find $a = af_\alpha$.

PROOF OF LEMMA 2: From Lemma 1 we have $a = af_0 = a(F_1 + F_2)$. Multiplying from the right by p_1p_1' we have $ap_1p_1' = a = aF_1$.

Theorem 9 shows that the rows of f_α span the space of all those linear forms which are confounded with coefficient λ_α. The rows of $F_i (i = 1, 2)$ form the space of all those forms ax which are unconfounded and for which $ap_ip_i'x = ax$.

We shall now consider the power of the tests of our linear hypotheses and it will be necessary to assume that the reader is familiar with the theory of testing linear hypotheses and with the power functions associated with these tests. For the concepts and results that will be used in the following the reader may be referred to [2] Chapter IV, pp. 22–30 and Chapter VI. It will be seen that the power of the tests is closely related to the confounding coefficients.

Suppose that we have observed a set of linear forms Qy where Q is an idempotent matrix of the center of $\mathfrak{A}$ and $y' = (y_1, \cdots, y_N)$. In testing the hypothesis $H_1 : \beta_1 = \cdots = \beta_{s_1} = 0$ under the assumption $E(y) = p_1\beta^{(1)} + p_2\beta^{(2)}$, where $\beta^{(1)\prime} = (\beta_1, \cdots, \beta_{s_1}, 0, \cdots, 0)$, $\beta^{(2)\prime} = (0, \cdots, 0, \beta_{s_1+1}, \cdots, \beta_{s_1+s_2})$ and the other assumptions of a linear hypothesis as stated on page 23 of [2]; using the forms Qy we first have to solve the equation

$$(p_1' + p_2')Q = (p_1' + p_2')Q(p_1 + p_2)B_1 . \tag{28}$$

The quadratic form

$$y'(Q - Q(p_1 + p_2)B_1)y = Q_a \tag{29}$$

divided by its rank h_2 forms the denominator of the statistic F. We then compute the regression value of Q under the assumption and the hypothesis $H_1 : \beta_1 = \cdots = \beta_{s_1} = 0$. That is to say we have to solve the equation

$$p_2'Q = p_2'Qp_2B_2 .$$

Since Q is in the center of $\mathfrak{A}$ this equation can be solved by putting $B_2 = p_2'$ (see the corollary to Theorem 1). Hence

$$Qp_2B_2 = Qp_2p_2' = Qp_2p_2'Q.$$

We then put

$$y'(Q - Qp_2B_2)y = Q_r , \qquad y'Q_{r-a}y = Q_r - Q_a . \tag{30}$$

The matrix Q_{r-a} is orthogonal to $Qp_2p_2'Q$ (see the paragraph following equation (8)) and so by Theorem 1

$$p_2'QQ_{r-a} = 0. \tag{31}$$

Hence if instead of the forms Qy we substitute in $Q_r - Q_a$ their expectations $Q(p_1\beta^{(1)} + p_2\beta^{(2)})$, under some alternative hypothesis H_1^* we obtain

$$\beta^{(1)\prime}p_1'Q_{r-a}p_1\beta^{(1)} = 2\sigma^2\delta \tag{32}$$

where σ^2 is the variance of one observation and δ is the quantity denoted by λ in formula 6.37 of [2]. If Q_{r-a} has the rank h_1 then the power of the F test is a monotonically increasing function of δ/h_1 and of h_2. (See formula 6.37 of [2] and the paragraph following it. To avoid confusion with the confounding coefficients we have written δ instead of λ.) Moreover, if h_2 is fairly large the increase in power obtained by increasing h_2 is negligibly small. We shall therefore call $2\delta/h_1 = \rho$ the power index with respect to H_1^*.

If Q is not orthogonal to p_2p_2' a certain amount of power is lost in eliminating the parameters $\beta_{s_1+1}, \cdots, \beta_{s_1+s_2}$. To measure this loss we consider the power index of the test of the same hypothesis H_1 but under the assumption $\beta_{s_1+1}, \cdots, \beta_{s_1+s_2} = 0$. This will result in another power index ρ^*. The ratio

$$e = \frac{\rho}{\rho^*} \tag{33}$$

is called the efficiency factor of Q with respect to H_1^*.

Now let f_α be an idempotent of the center of $\mathfrak{A}$ with confounding coefficient λ_α. (If $\lambda_\alpha = 0$ let $f_\alpha = F_1$). Testing the hypothesis H_1 under the assumption $E(y) = p_1\beta^{(1)} + p_2\beta^{(2)}$ gives $Q_{r-a} = f_\alpha - f_\alpha p_2p_2'$.

Hence $2\sigma^2\delta = \beta^{(1)\prime}(p_1'f_\alpha p_1 - p_1'f_\alpha p_2p_2'p_1)\beta^{(1)}$. Now $p_1p_1'f_\alpha p_2p_2'p_1p_1' = \lambda_\alpha p_1p_1'f_\alpha p_1p_1'$ and on account of Theorem 1 we obtain

$$p_1'f_\alpha p_2p_2'p_1 = \lambda_\alpha p_1'f_\alpha p_1 \tag{34}$$

so that

$$2\sigma^2\delta = (1 - \lambda_\alpha)\beta^{(1)\prime}p_1'f_\alpha p_1\beta^{(1)}. \tag{35}$$

On the other hand if the assumption is changed to A & H_2 then the matrix of Q_a becomes $f_\alpha - f_\alpha p_1 p_1'$ and the matrix of Q_r is f_α and hence

$$2\sigma^2\delta^* = \beta^{(1)'}p_1'f_\alpha p_1\beta^{(1)}. \tag{36}$$

For $\lambda = 1$ we have $f_\alpha - f_\alpha p_2 p_2' = 0$ so that no test is possible. For $\lambda \neq 1$ we have

$$p_1p_1'(f_\alpha - f_\alpha p_2 p_2')p_1p_1' = (1 - \lambda_\alpha)f_\alpha p_1 p_1' \tag{37}$$

which shows that rank $(f_\alpha - f_\alpha p_2 p_2') =$ rank $(f_\alpha p_1 p_1')$ so the efficiency of the matrix f_α is $1 - \lambda_\alpha$. Hence

THEOREM 10. *If f_α is a principal component with confounding coefficient $\lambda_\alpha \neq 1$ and if for $\lambda_\alpha = 0$, $f_\alpha p_1 p_1' = f_\alpha$ then the efficiency of f_α with respect to every alternative hypothesis H_1^* is $1 - \lambda_\alpha$.*

From (37) we also have

THEOREM 11. *If λ is any confounding coefficient then $0 \leqq \lambda \leqq 1$.*

PROOF: $f_\alpha - f_\alpha p_2 p_2'$ as well as $f_\alpha p_1 p_1'$ are symmetric idempotent matrices and therefore positive semi definite. Also $f_\alpha p_1 p_1' \neq 0$. Hence $(1 - \lambda_\alpha) \geqq 0$. Similarly (34) implies $\lambda_\alpha \geqq 0$.

If we increase the size of the sample by replicating the experiments, then the quantity $2\sigma^2\delta/h_1$ will be increased in direct proportion to the increase in sample size. If we neglect the increase in power do to a corresponding increase in h_2 we can interpret Theorem 10 as stating that λ_α is proportional to the amount of money spent in eliminating the parameters $\beta_{s_1+1}, \cdots, \beta_{s_1+s_2}$. In a situation where the inhomogeneity of the second parameter set could be eliminated at a given expense the confounding coefficients λ_α could therefore be used to decide whether the elimination of inhomogeneity is really worthwhile.

4. Applications. A. T. James [1] has considered the important case in which the coefficients $p_{\alpha j}$ are either 0 or 1 and where with $S_l = s_1 + \cdots + s_l$ we have

$$\sum_{j=S_{l-1}+1}^{S_l} p_{\alpha j} = 1, \qquad \alpha = 1, \cdots, N, \qquad l = 1, \cdots, r.$$

The matrix $p_l p_l' = T_l = (T_{\alpha\beta}^{(l)})$ consists in this case of ones and zeros only. We have $T_{\alpha\beta}^{(l)} = 1$ if for some j we have $p_{\alpha j} = p_{\beta j} = 1$ otherwise $T_{\alpha\beta}^{(l)} = 0$. Such matrices T_l are called relationship matrices since $T_{\alpha\beta}^{(l)} = 1$ if and only if the αth and βth plot (experimental unit) receive the same treatment from the lth set of treatments. Applying the matrix T_l to the vector $y = (y_1, \cdots, y_N)'$ will replace every y_α by the total of those observations which receive the same treatment of the lth set as y_α. If every treatment of the lth set is repeated the same number say k_l of times then applying the foregoing remark to the columns of T_l itself we get $T_l^2 = k_l T_l$ so that $T_l/k_l = t_l$ will be idempotent. The matrix t_l applied to y replaces every observation y_α by the mean of those observations which receive the same treatment as y_α.

A. T. James has given the decomposition for balanced incomplete block design. If the design is asymmetric, $r > k$, then one obtains three one dimensional and one 2×2 complete matrix algebras as principal components of the algebra $(\mathfrak{A} \cup I)$. If the design is symmetric then one of the one dimensional algebras (the algebra $\mathfrak{A}E_2$ of [1] p. 1000) reduces to 0 since in this case $BTB \equiv (r - \lambda)B$ (Mod. G). It may be left to the reader to obtain this decomposition from Theorem 6.

In the following we shall decompose the algebra of an s dimensional cubic lattice design into its principal components. This example exhibits all the features of the general case and at the same time does not present any computational difficulties.

5. The principal components of an s dimensional cubic lattice design. In an s dimensional cubic lattice design m^s treatments are arranged into s sets of blocks each containing a complete replication. The blocks are formed in the following way. The treatments are distinguished by a set of s indices and are written $t_{i_1 \cdots i_s}$, $1 \leqq i_1 \leqq m$, $1 \leqq i_s \leqq m$. In the first replication the blocks are formed by keeping the indices $i_2, \cdots, i_s$ fixed and varying the first index. In the αth replication the blocks are formed from all treatments with indices $a_1, \cdots, a_{\alpha-1}, a_{\alpha+1} \cdots a_s$ fixed. Thus every replication contains m^{s-1} blocks of m treatments each. For instance for $s = 2, m = 3$ we have the blocks

$$\begin{array}{ll} (t_{11}, t_{21}, t_{31}), & (t_{11}, t_{12}, t_{13}), \\ (t_{12}, t_{22}, t_{32}), & (t_{21}, t_{22}, t_{23}), \\ (t_{13}, t_{23}, t_{33}), & (t_{31}, t_{32}, t_{33}). \end{array}$$

The values observed for the treatment $t_{a_1 \cdots a_s}$ in the αth replication will be denoted by $(\alpha)x_{a_1 \cdots a_s}$. By $(\alpha)x_{a_1 \cdots a_u}^{i_1 \cdots i_u}$ we shall denote the sum of all observations with i_1st index a_1, i_2nd index a_2, $\cdots$, i_uth index a_u and we shall call such a quantity a class total. The assumption reads

$$E((\alpha)x_{a_1 \cdots a_s}) = t_{a_1 \cdots a_s} + (\alpha)b_{a_1 \cdots a_{\alpha-1} a_{\alpha+1} \cdots a_s}.$$

(Usually the restriction $\sum_{a_1, \cdots, a_s} t_{a_1 \cdots a_s} = 0$ is imposed and a general mean introduced, but since by Theorem 4.4 of [2] the Lagrange operator for this restriction is 0 we may ignore it and add the general mean to the block effects. By Theorem 3 this does not affect the regression values.)

We form according to Section 3 the matrices T relating two plots with the same treatment and B relating two plots from the same block.

From Section 3 we have

$$B((1)x_{a_1 \cdots a_s}) = (1)x_{a_2 \cdots a_s}^{2 \cdots s} = \sum_{a_1} (1)x_{a_1 \cdots a_s}^{1 \cdots s},$$

$$B((\alpha)x_{a_1 \cdots a_s}) = (\alpha)x_{a_1 \cdots a_{\alpha-1} a_{\alpha+1} \cdots a_s}^{1 \cdots \alpha-1\, \alpha+1 \cdots s},$$

$$TB((\alpha)x_{a_1 \cdots a_s}) = \sum_{\alpha} (\alpha)x_{a_1 \cdots a_{\alpha-1} a_{\alpha+1} \cdots a_s}^{1 \cdots \alpha-1\, \alpha+1 \cdots s}.$$

Thus we have

PROPOSITION 1: *If* $(1)x^{1\cdots s}_{a_1\cdots a_s} = (\alpha)x^{1\cdots s}_{a_1\cdots a_s} = x^{1\cdots s}_{a_1\cdots a_s}$ *then*

$$TB((\alpha)x^{1\cdots s}_{a_1\cdots a_s}) = \sum_\alpha x^{1\cdots\alpha-1\alpha+1\cdots s}_{a_1\cdots a_{\alpha-1}a_{\alpha+1}\cdots a_s}. \tag{38}$$

A class total $(\alpha)x^{i_1\cdots i_u}_{a_{i_1}\cdots a_{i_u}}$ *is called confounded if* $\alpha \neq i_j$, $j = 1 \cdots u$. Let $\sum_k^{a_1\cdots a_s}$ denote the sum of all confounded class totals with $s - k$ indices chosen out of $a_1, \cdots, a_s$. For instance

$$\sum\nolimits_2^{213} = (1)x_1^2 + (1)x_3^3 + (2)x_2^1 + (2)x_3^3 + (3)x_2^1 + (3)x_1^2.$$

PROPOSITION 2: *For* $k < s$

$$TB\sum\nolimits_k^{a_1\cdots a_s} = mk\sum\nolimits_k^{a_1\cdots a_s} + k\sum\nolimits_{k+1}^{a_1\cdots a_s}. \tag{39}$$

PROOF: We put $(\alpha)x^{1\cdots s}_{a_1\cdots a_s} = \sum_k^{a_1\cdots a_s}$ and apply proposition 1. We obtain

$$TB\sum\nolimits_k^{a_1\cdots a_s} = \sum\nolimits_{b_1}\sum\nolimits_k^{b_1a_2\cdots a_s} + \sum\nolimits_{b_2}\sum\nolimits_k^{a_1b_2a_3\cdots a_s\cdots +} + \sum\nolimits_{b_s}\sum\nolimits_k^{a_1a_2\cdots a_{s-1}b_s}. \tag{40}$$

In the lth sum every class total not containing the upper index l occurs m times. Therefore since there are k upper indices missing in every class total occurring in any sum $\sum_k^{b_1\cdots b_s}$ every class total with k upper indices missing and $s - k$ lower indices chosen out of $a_1, \cdots, a_s$ will occur mk times on the right of (40) giving rise to the first term on the right of (39). The class totals with $k + 1$ upper indices missing arise from those terms of the lth sum which contain the upper index l but do not have the prefix l (since terms with prefix and upper index l are not confounded). Hence each such term arises from exactly k of the terms in the right of (40). This proves (39).

Let Σ_k denote the transformation which replaces $(\alpha)x^{1\cdots s}_{a_1\cdots a_s}$ by $\Sigma_k^{a_1\cdots a_s}$. We have

$$TB = \Sigma_1,$$

$$(TB)^2 = TB\Sigma_1 = m\Sigma_1 + \Sigma_2 = mTB + \Sigma_2.$$

Suppose we have shown that for $k < s - 1$

$$TB(TB - m) \cdots (TB - km) = k!\,\Sigma_{k+1}. \tag{41}$$

We multiply (41) by TB and get

$$(TB - (k + 1)m)k!\Sigma_{k+1} = (k + 1)!\Sigma_{k+2}.$$

Hence we have proved

$$TB(TB - m) \cdots (TB - km) = k!\Sigma_{k+1} \quad \text{for} \quad k \leqq s - 1. \tag{42}$$

Since $\sum_s^{a_1\cdots a_s} = x =$ sum of all observation we get

$$\Sigma_s = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix} = G \quad \text{say.}$$

and from (42) for $k = s - 1$

$$(43) \qquad TB(TB - m) \cdots (TB - (s-1)m) = (s-1)!G.$$

Dividing (43) by $(ms)^s$ we get

$$(44) \qquad (tb)\left(tb - \frac{1}{s}\right) \cdots \left(tb - \frac{s-1}{s}\right) = \frac{s!}{s^s} g$$

where t, b, g are the idempotents corresponding to T, B and G.

Remembering the effect of TB we see that one application of TB can delete only at most one upper index in a class total. Hence s applications of TB are needed to produce a term with all indices deleted. On the other hand $(TB)^k x_{a_1 \cdots a_s}^{1, \cdots, s}$ for $k \leqq s$ involves terms which are not involved in $(TB)^{k-1} x_{a_1 \cdots a_s}^{1 \cdots s}$. Hence a polynomial in TB of degree less than s cannot vanish nor be a multiple of G. Thus if we put $t - g = t_1$, $b - g = b_1$ then

$$(45) \qquad t_1 b_1 \left(t_1 b_1 - \frac{1}{s}\right) \cdots \left(t_1 b_1 - \frac{s-1}{s}\right) = 0$$

is the minimal equation of $t_1 b_1$.

From (45) and Theorem 6 the decomposition of the algebra of the s dimensional cubic lattice can be obtained without any effort.

6. The case $r > 2$.

A part of Theorem 6 carries over easily to the case $r > 2$. If there is a matrix $G \varepsilon \mathfrak{A}$ satisfying the conditions of Theorem 5 we may write

$$\mathfrak{A} = \mathfrak{A} - \mathfrak{A}G + \mathfrak{A}G.$$

If there is no $G \neq 0$ satisfying Theorem 5 we shall put $G = 0$. To exclude trivialities we also assume that $p_i p_i'$ is singular. Using these conventions we can state

THEOREM 12. *Let*

$$(46) \qquad \begin{aligned} Q_i &= P_1 - p_i, \\ T_i &= p_i p_i' Q_i Q_i' p_i p_i'. \end{aligned}$$

Let $\lambda_\alpha^{(i)}$, $i = 1, \cdots, r$, $\alpha = 1 \cdots n_i$ *be the distinct non* 0 *characteristic roots of* T_i. *Let*

$$(47) \qquad \begin{aligned} F_i &= \frac{(T_i - \lambda_1^{(i)}) \cdots (T_i - \lambda_{n_i}^{(i)})}{(-1)^{n_i} \lambda_1^{(i)} \cdots \lambda_n^{(i)}} p_i p_i', \\ e_i &= p_i p_i' - F_i - G \end{aligned}$$

and let $\mathfrak{B}$ *be the algebra generated by* $e_1, \cdots, e_r$. *Then*

(i) $p_i p_i' F_j = \begin{cases} F_i & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}, \qquad F_i F_j = \begin{cases} F_i & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}.$

(ii) $\mathfrak{A} = \mathfrak{A}G + \mathfrak{A}F_1 + \cdots + \mathfrak{A}F_r + \mathfrak{B}$.

(iii) *G, $F_1, \cdots, F_r$ are annulled by $\mathfrak{B}$ and are principal one dimensional components of $\mathfrak{A}$ or are equal to* 0.

(iv) *The equation*

$$e_i \sum_{j \neq i} e_j B_i = e_i \tag{48}$$

has a solution $B_i \varepsilon \mathfrak{B}$.

PROOF: From

$$T_i F_i = 0 \tag{49}$$

we get multiplying by G, $(r - 1)GF_i = 0$. Hence $GF_i = 0$.

From (49) we get on account of Theorem 1

$$Q_i' F_i = 0. \tag{50}$$

From the definition (1) of p_j we find

$$p_j Q_i' = p_j p_j' \quad \text{for} \quad j \neq i, \tag{51}$$

and so from (50), $p_j p_j' F_i = 0$ for $i \neq j$.

From (9) we see that F_i is idempotent, so that (i) is proved.

The statements (ii) and (iii) are immediate consequences of (i). By (47) we may write mod. G

$$e_i = T_i(a_0 P_1 A_1 + a_1 T_i + \cdots a_{n-1} T_i^{n-1}) = e_i \left(\sum_{i \neq j} e_j \right) B_i$$

with $B_i \varepsilon \mathfrak{B}$. This completes the proof of Theorem 12 since $e_i G = 0$.

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