ABSTRACTS OF PAPERS

(Abstract of a paper presented at the Washington, D.C., Annual Meeting of the Institute, December 27–30, 1959.)

76. Moment Generating Functions of Quadratic Forms of Normal Order Statistics. Harold Ruben, Columbia University.

A general method is derived for obtaining the joint moment generating functions of an arbitrary set of quadratic functions, not necessarily definite positive, of order statistics in normal samples. This class of functions probably includes all or most functions of order statistics likely to be of practical interest, e.g., squared linear functions used in censored samples and other applications, squared range, squared subrange, squared deviation of extremes from the sample mean, etc. The determination of the generating functions reduces to the classic problem of the evaluation of the contents of hyperspherical simplices (the generalization of the circular arc and spherical triangle).

(Abstracts of papers presented at the Lafayette, Indiana Meeting of the Institute, April 7–9, 1960.)

1. Note on Significances of Differences for Attributes. Irving W. Burr, Purdue University.

Assuming equal sample sizes and either a Poisson or binomial population, the maximum likelihood estimate of the parameter is used. Then the exact probability of a difference in "defects" or "defectives" at least as large as was observed is obtained by double summation. This probability then gives the exact significance levels for various differences and sample sizes. A table gives these results up till when the normal curve approximation takes over accurately. A quick and accurate approximation for unequal samples is indicated.

2. A Characterization of Some Location and Scale Parameter Families. Sudhish G. Ghurye, Northwestern University. (By title)

Zinger (*Vestnik. Leningrad. Univ.*, Vol. 1 (1956), pp. 53–56) has proved the following result: Let $X_1, \cdots, X_n$, $n \geqq 6$, be independent random variables having a common distribution, which is of continuous type: let $t(X) = (1/n)\Sigma X_i$, $s(X) = [\Sigma X_i^2 - nt^2(X)]^{\frac{1}{2}}$ and $Y_i = [X_i - t(X)]/s(X)$. If the Y_i are distributed uniformly on the $(n\text{-}2)$-dimensional sphere $\{\Sigma y_i = 0, \Sigma y_i^2 = 1\}$, then the X-distribution is normal. I extend this result in an obvious way to characterize the exponential and rectangular distributions, and also the multi-variate normal and Wishart distributions. The following result is proved incidentally: Let $f(x)$ be a measurable function of real x, having the property that $x + y + z = a + b + c$ and $x^2 + y^2 + z^2 = a^2 + b^2 + c^2$ imply $f(x)f(y)f(z) = f(a)f(b)f(c)$. If $f(x) \neq 0$ for two values of x, then there exist numbers α, β, γ such that $f(x) = \alpha \exp(\beta x + \gamma x^2)$ for all x.

3. A New Class of Sequential Decision Rules for Symmetric Problems. William Jackson Hall, University of North Carolina. (By title)

A class of sequential tests is derived for choosing between two symmetric hypotheses with equal preassigned error probabilities. The class includes the Wald sequential probability ratio test (SPRT) and numerous other sequential tests. For a number of problems—

including tests on the mean of a normal distribution and a variety of two-population problems—there is one or more test available with "converging boundaries" (bounded sample size) in contrast to the "parallel boundaries" (unbounded sample size) of the SPRT. The relative merits of such tests are investigated, and some extensions to multiple-decision problems are discussed.

4. Normal Approximation to the Distribution of Two Independent Binomials, Conditional on Fixed Sum. J. Hannan, Michigan State University and W. Harkness, Pennsylvania State University. (By title)

For $i = 1, 2$, let k_i be independent binomials with parameters $(N_i - 1, p_i)$ and let

$$f_k = \Pr[k_1 = k \mid \Sigma k_i = c].$$

Theorem: With (P_1, P_2) defined by $P_2q_2/Q_2p_2 = P_1q_1/Q_1p_1$ and $\Sigma N_iP_i = c + 1$, and with $H^2 = \Sigma(N_iP_iQ_i)^{-1}$ and $X_k = H(k - N_1P_1 + \frac{1}{2}), f_k \sim H\phi(X_k)$, $\sum_\alpha^\beta f_k \sim \Phi(X_{\beta+\frac{1}{2}}) - \Phi(X_{\alpha-\frac{1}{2}})$, $\sum_\alpha^c f_k \lesssim X_\alpha^{-1}\phi(X_\alpha)$ or $\sim 1 - \Phi(X_{\alpha-\frac{1}{2}})$, as H and, respectively, HX_k^3, HX_α^3 and HX_β^3, HX_α or $HX_\alpha^3 \to 0$.

5. On the Analysis of Split-Plot Experiments. H. Leon Harter, Wright-Patterson Air Force Base, Ohio.

A crucial question in the analysis of split-plot experiments is whether or not the interaction between subplot treatments and replications should be pooled with the three-factor interaction of main plot treatments, subplot treatments, and replications, the result being called subplot error. A brief history of the controversy over this question is given, along with a rule for deciding, on the basis of a preliminary test of significance, whether or not to pool. Several numerical examples are cited, and one of these is worked out in detail.

6. An Extension of a Theoretical Gene Model to Provide for Genic-environmental Interaction Terms. Cecil L. Kaller and Virgil L. Anderson, Purdue University.

A statistical model for the study of quantitative inheritance was introduced by Anderson (1953) by utilizing the techniques of factorial experimental models. Kempthorne (1954) pursued this further by developing the general gene model in which he used the symbol $\prod_{j=1}^n A_{i_j}^j A_{i'_j}^j$ to denote the genotype of an individual from a population $\mathcal{G}$ whose members are diploid and have N loci G_j, $j = 1, 2, \cdots, N$, where locus G_j has available h_j alleles $A_{i_j}^j$, $i_j = 1, 2, \cdots, h_j$, with respective relative frequencies $p_1^j, p_2^j, \cdots, p_{h_j}^j$; where $\sum_{i_j=1}^{h_j} p_{i_j}^j = 1$. By use of algebraic identities and identification of resultant terms with genetic effects, Kempthorne provided a complete theoretical gene model. The extension of these developments to a general phenotypic model is accomplished by introducing environmental factors E_r, $r = 1, 2, \cdots, M$, where E_r has K_r "levels" $E_r^1, E_r^2, \cdots, E_r^{K_r}$, with associated occurrence frequencies $p_r^1, p_r^2, \cdots, p_r^{K_r}$, where $\sum_{i=1}^{K_r} p_r^i = 1$. Then the phenotype of any diploid individual is denoted by a symbolic product of genotypic and environmental components as $P_{i_1i_1i_2i_2\cdots i_Ni_N}^{v_1v_2\cdots v_M}$ which is expanded by use of identities and symbolic algebraic multiplications into a sum of uncorrelated terms which account for all genetic effects, all environmental effects, and all genetic-environmental interaction effects contributing to the phenotypic expression of the individual. This is the theoretical genic-environmental interaction model.

7. Comparison of Estimators for Some Generalized Poisson Distributions. S. K. Katti, Florida State University, and John Gurland, Iowa State University.

For the generalized Poisson distributions, Neyman Type A and Poisson generalized Pascal, the well-known asymptotically efficient methods of estimation yield highly cumbersome equations to solve. In view of this, certain methods have been studied for these distributions from the point of view of obtaining simple estimators and the joint asymptotic efficiency of the estimators evaluated. In the case of Neyman Type A, it is found that the estimates of the two parameters obtained by minimising the quadratic form $(t - \tau)\hat{\Omega}^{-1}(t - \tau)'$, where $t = (\hat{\kappa}_{[1]}, \hat{\kappa}_{[2]}, \log \hat{P}_0)$, $\tau = (\kappa_{[1]}, \kappa_{[2]}, \log P_0)$, and $\hat{\Omega}$ is a consistent estimate of the co-varnace matrix of t, have remarkably high efficiency in a wide region of the parameter space. For the three parameter Poisson generalized Pascal distribution, the method of using the first two moments and the ratio of the first two frequencies looks promising.

8. Generalization of Thompson's distribution III. Andre G. Laurent, Wayne State University.

Let the $p \times N$ matrix $X = (X^1, \cdots, X^i, \cdots, X^N)$ be a sample of N vectors X^i with distribution $N(BZ^i, \Sigma)$, $i = 1$ to N where B is $p \times q$ and $Z = (Z^1, \cdots, Z^i, \cdots, Z^N)$ is $q \times N$ of rank q. Let ξ be a subsample of k vectors, $q \leqq k \leqq N - p - q$, Z_ξ the corresponding $(Z^1, \cdots Z^k)$. Let $\hat{B}, \hat{\Sigma}, \hat{B}_\xi, \hat{\Sigma}_\xi$ be the M.L. estimates of B, Σ obtained with X and ξ respectively. The conditional distribution of ξ, given the sufficient statistics, $\hat{B}, \hat{\Sigma}$ is

$$C \mid I - (k \mid N)\hat{\Sigma}^{-1}\hat{\Sigma}_\xi - (1/N)\hat{\Sigma}^{-1}(\hat{B}_\xi - \hat{B})[(Z_\xi Z_\xi')^{-1} - (ZZ')^{-1}]^{-1}(\hat{B}_\xi - \hat{B})^1 \mid^{(N-k-p-q-1)/2} \mid N\hat{\Sigma} \mid^{-k/2} d\xi$$

with $C = \mid ZZ' \mid^{-p/2} \mid ZZ' - Z_\xi Z_\xi' \mid^{-p/2} \pi^{-kp/2} \prod_1^p \Gamma[(N - q + 1 - i)/2]/\Gamma[(N - k - q + 1 - i)/2]$ in the proper domain. The conditional distribution of the "studentized" variable $\eta = (N\hat{\Sigma})^{-\frac{1}{2}}(\xi - \hat{B}Z_\xi)$ is, in the proper domain

$$C' \mid I - k\hat{\Sigma}_\eta - \eta w_\xi'[w_\xi(I - w_\xi' w_\xi)w_\xi']^{-1} w_\xi \eta' \mid^{(N-k-q-p-1)/2} d\eta$$

where $w_\xi = (ZZ')^{-\frac{1}{2}}Z_\xi$; η is independent from $\hat{\Sigma}, \hat{B}$. Formulae simplify when at least one of p, q, k, is unity. Applications to estimation problems are given.

9. An Expansion for the Quadrivariate Normal Integral when $\rho_{13} = \rho_{14} = \rho_{24} = 0$. J. A. McFadden, Purdue University. (By title) (Introduced by J. H. Abbott)

Let ξ_1, ξ_2, ξ_3, and ξ_4 obey a quadrivariate normal distribution with all mean values equal to zero. Let the correlation coefficient between ξ_i and ξ_j be ρ_{ij}, and let $\rho_{ij} = 0$ when $\mid i - j \mid > 1$. The value of the quadrivariate normal integral, i.e., the probability that ξ_1, ξ_2, ξ_2, and ξ_4 are simultaneously positive, is equal to $(\frac{1}{16})\{1 + (2/\pi)[\sin^{-1} \rho_{12} + \sin^{-1} \rho_{23} + \sin^{-1} \rho_{34}] + W(\rho_{12}, \rho_{23}, \rho_{34})\}$, where

$$W(\rho_{12}, \rho_{23}, \rho_{34}) = (4/\pi^2)\rho_{12}\rho_{34}\sum_0^\infty (\tfrac{1}{2})_m \rho_{23}{}^{2m}(m!)^{-1}G_m(\rho_{12})G_m(\rho_{34});$$

$G_m(x) = F(\frac{1}{2}, \frac{1}{2} + m; \frac{3}{2}; x^2)$; $(a)_m = a(a + 1) \cdots (a + m - 1)$; $(a)_0 = 1$. $G_0(x)$ is expressible in terms of an arcsine function, and the other $G_m(x)$ can be written as products of $(1 - x^2)^{\frac{1}{2}-m}$ and polynomials of degree m in x^2; thus the series is well suited for computation. Numerical values from the first four terms compare well with known, exact values of the quadrivariate normal integral.

10. An Expansion for the Quadrivariate Normal Integral for a Stationary Markov Process. J. A. McFadden, Purdue University. (Introduced by J. H. Abbott)

Let ξ_1, ξ_2, ξ_3, and ξ_4 be successive measurements from a stationary Gaussian Markov process, with the mean value equal to zero. Let the correlation coefficient between ξ_i and ξ_j be ρ_{ij}. The value of the quadrivariate normal integral, i.e., the probability that ξ_1, ξ_2, ξ_3, and ξ_4 are simultaneously positive, is equal to $(\frac{1}{16})\{1 + (2/\pi)[\sin^{-1} \rho_{12} + \sin^{-1} \rho_{23} + \sin^{-1} \rho_{34} + \sin^{-1} (\rho_{12}\rho_{23}) + \sin^{-1} (\rho_{23}\rho_{34}) + \sin^{-1} (\rho_{12}\rho_{23}\rho_{34})] + W(\rho_{12}, \rho_{23}, \rho_{34})\}$, where $W(\rho_{12}, \rho_{23}, \rho_{34}) = (4/\pi^2)\rho_{12}\rho_{34} \sum_0^\infty (-\frac{1}{2})_m(-\frac{1}{2})_m[(\frac{1}{2})_m]^{-1}\rho_{23}^{2m}(m!)^{-1}F_m(\rho_{12})F_m(\rho_{34})$; $F_m(x) = F(\frac{1}{2}, \frac{1}{2} - m; \frac{3}{2} - m; x^2)$; $(a)_m = a(a + 1) \cdots (a + m - 1)$; $(a)_0 = 1$. $F_0(x)$ is expressible in terms of an arcsine function, and the other $F_m(x)$ can be written as products of $(1 - x^2)^{\frac{1}{2}}$ and polynomials of degree m in x^2; thus the series is well suited for computation. Numerical values from the first four terms compare well with known results obtained by numerical integration.

11. On Evaluation of Negative Binomial Distribution Function. G. P. Patil, University of Michigan. (By title)

In this paper, we show that in order to evaluate the negative binomial distribution function $Y(r, p, k) = \sum_{x=0}^{r} \binom{k + n - 1}{x} p^k(1 - p)^x$ where $0 \leqq p \leqq 1$, $0 < k < \infty$, we can use (positive) binomial distribution function tables, when k is a positive integer. To be more general, we show that we can use the incomplete beta function tables for any general k. Thus, we indicate that there is no necessity as such of having numerical tables for the negative binomial distribution function, since extensive tables are available for binomial distribution function and incomplete beta function. To be precise we establish *Theorem* 1: $Y(r, p, k) = 1 - B(k - 1, p, r + k)$, $k = 1, 2, 3, \cdots$, where

$$B(c, p, n) = \sum_{x=0}^{c} \binom{n}{x} p^x(1 - p)^{n-x}.$$

Also *Theorem* 2: $Y(r, p, k) = I_p(k, r + 1)$, $0 < k < \infty$, where

$$I_p(m, n) = 1/B(m, n) \int_0^p u^{m-1}(1 - u)^{n-1}\, du.$$

Incidentally, one gets from the above the well-established identity between the binomial distribution function and the incomplete beta function, namely $B(k - 1, p, r + k) = 1 - I_p(k, r + 1) = I_{1-p}(k, r'' 1)$.

12. On Some Extensions of Sampling with Probability Proportional to Size. D. K. Ray-Chaudhuri, Case Institute of Technology.

Consider a finite population Π consisting of N units U_1, U_2, $\cdots$ U_N. Let Y denote the variate under inquiry and X denote an auxiliary variate related to Y. Let X_i $(X_i > 0)$ denote the value of X for U_i which is assumed to be known, $i = 1, 2, \cdots N$. Sampling with probability proportional to size (PPS) is an efficient method of utilizing the supplementary information provided by X for the purpose of estimating $\bar{Y}$, the population mean of Y only if Y is approximately proportional to X in a certain sense. Several extensions of PPS sampling have been obtained which give efficient ways of utilizing the supplementary information provided by X even when Y is approximately any linear function of X. A derived unit W_{ij} is defined to be a pair of original units (U_i, U_j) where $X_i > \bar{X}$ and $X_j < \bar{X}$ and $\bar{X}$ denotes the mean of X. In one of the sampling schemes considered a number of derived units is selected with probability proportional to $| X_i - \bar{X} | + | X_j - \bar{X} |$. These extensions of PPS sampling are compared with other sampling schemes and the method is generalized to the case when U is approximately a quadratic function of X.

13. Application to Stochastic Processes of a Uniqueness Property of the Rectangular Distribution. HERMAN RUBIN, Michigan State University.

If the random variable X has a tth moment $\mu(t)$ for all $t \varepsilon (-1, 1)$, and for all $t \varepsilon (-1, 0)$, we have $(t + 1)\ \mu(t) = -t\mu(t + 1)$, then $X/(1 + X)$ is rectangular (0, 1). This can be shown by observing that $\mu(t) \sin \pi t/\pi t$ is periodic of period 1 and bounded in the strip $| R(t) | \leqq \frac{1}{2}$ by $A + B | \sin \pi t |$, and hence is constant. Let Y be a process with independent increments, stationary on both sides of a value α. If $Z(\lambda)$ is the likelihood ratio for $\alpha = \lambda$ against $\alpha = 0$ and is positive almost surely, and $\alpha = 0$, then $\int_0^\infty Z(\lambda)\, d\lambda / \int_{-\infty}^\infty Z(\lambda)\, d\lambda$ is rectangular (0, 1). This follows from the recursion formulas for the moments (of all orders less than 1) of $\int_0^\infty Z(\lambda)\, d\lambda$ and $\int_{-\infty}^0 Z(\lambda)\, d\lambda$, and an application of the preceding theorem.

14. Test for Regression Coefficients when Errors are Correlated. M. M. SIDDIQUI, National Bureau of Standards, Boulder, Colorado.

In a previous paper (*Ann. Math. Stat.*, Vol. 29 (1958), pp. 1251–56) the variances and covariances of least-squares estimates of regression coefficients were obtained when the errors are assumed to be correlated. In this paper it is shown that the usual test statistic for a regression coefficient is approximately distributed as ct, where c is a constant and t is a Student variate with h degrees of freedom. h is a number determined by the covariance matrix of errors.

15. Joint Distribution of Medians in Samples from a Bivariate Population, M. M. SIDDIQUI, National Bureau of Standards, Boulder, Colorado. (By title)

Let $F(x, y)$ be the joint distribution function of (X, Y), possessing a pdf $f(x, y)$. A random sample (X_i, Y_i), $i = 1, 2, \cdots, n$ is drawn, n odd. Let X_0 and Y_0 denote the medians of sets $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$, respectively. The joint distribution of (X_0, Y_0) is obtained and it is shown that it tends to $N(\xi, \Sigma)$ as $n \to \infty$ where $\xi = (\xi_1, \xi_2)$, $\Sigma = \begin{pmatrix} \sigma_1^2 & \rho^*\sigma_1\sigma_2 \\ \rho^*\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$. Here ξ_1 and ξ_2 are the medians of the marginal pdf's $f_1(x)$ and $f_2(y)$ of X and Y respectively, $4nf_1^2(\xi_1)\sigma_1^2 = 1$, $4nf_2^2(\xi_2)\sigma_2^2 = 1$, and $\rho^* = 4\theta - 1$, where $\theta = F(\xi_1, \xi_2)$. As a corollary it is shown that $(F(X_0, Y_0)$ is asymptotically normally distributed with mean θ and variance c/n where c is constant depending on the parameters of F. Generalization to the distribution of the median vector in samples from multivariate populations is obvious.

16. A Characterization of the Uniform Distribution in Compact Topological Groups. JAMES H. STAPLETON, Michigan State University.

Let Γ be a connected compact topological group with a countable basis. Let $X_1, X_2, \cdots, X_n$ be independently and uniformly distributed (I.U.D.) in Γ (the distribution of the n-tuple is the Haar measure in $\Gamma^{(n)}$). Define $Y_i = \sum_{j=1}^n a_{ij} X_j$ $(i = 1, \cdots, n)$ for integers a_{ij}. Then the Y_i are I.U.D. if and only if (a_{ij}) is non-singular. In a sense this characterizes the uniform distribution in Γ. Let $X_1, \cdots, X_n$ be independent, and suppose that for no j

does X_j take all its values in a fixed coset of a proper compact subgroup of Γ. Let $Y_1, \cdots, Y_n$ be as before, assume at least two a_{ij} are non-zero for each i, and let $\det(a_{ij}) = \pm 1$. Then, if $Y_1, \cdots, Y_n$ are independent, each X_j which has an absolutely continuous part with respect to the Haar measure is uniformly distributed in Γ. The proofs make use of the theory of characteristic functions for compact topological groups.

17. **Some Results in the Analysis of Variance I.** (Preliminary Report) SELIG STARR, George Washington University. (By title)

Using the finite model (Model III) for the nested case, it is shown that the expected values of certain quadratic forms in the observations can be expressed simply in terms of the same quadratic forms in the population values. The usual expected mean squares are obtained as an immediate consequence. Using Model III for a complete n-factor asymmetrical factorial, without replication, it is shown that the usual mean squares based on observations can be expressed in terms of the sums of squares of all possible 2^n factorials that can be formed from the array. This result is used to develop the usual expected mean squares. The factorial with replication is then derived by a combination of the two foregoing results. The approach in both cases uses only the simplest combinatorial considerations and does not involve the expectations of cross-products usually encountered. Matrix algebra simplifies the presentation and, in the case of the factorial, leads to the Kronecker product of n simple 2×2 matrices. The results are proved rigorously, by induction, for the general case. It is then shown that the development by the usual linear models is a natural consequence.

18. **Power of Some Two Sample Distribution Free Tests.** B. V. SUKHATME, Michigan State University. (By title)

A two sample distribution free test based on the number of observations of one sample lying outside the extreme values of the other sample was first proposed by Wilks (1942) and its probability distribution was later tabulated by Rosenbaum (1953). Kamat (1956) proposed another two sample distribution free test based on the numbers of observations of each of the two samples lying outside the extreme values of the other sample. This paper gives the exact distributions of the two test statistics both under the hypothesis and the alternative. These results are used to compare the power of these two tests against scalar alternatives for small samples from normal population for different levels of significance. A discussion is also given concerning the relative efficiency of these tests with respect to the variance ratio F test.

19. **Nonparametric Tests for Location and Scale Parameters in a Mixed Model with Discrete and Continuous Variables.** SHASHIKALA B. SUKHATME, Michigan State University. (By title) (Introduced by B. V. Sukhatme)

Let $Z_1, Z_2, \cdots, Z_N$ with $Z_i = (X_\epsilon, Y_\epsilon)$ be independent observations from a bivariate population. Let the random variable X assume two values 1 and 0 with probabilities p and $1 - p$ respectively. Let $P(Y \leqq y \mid X = j) = F_j(y)$, $(j = 0, 1)$. This paper considers the problem of testing the hypothesis $H: F_1 = F_0$ against the alternative $A: F_1 \neq F_0$. Several nonparametric tests for location (e.g. two sample median and Wilcoxon tests, etc.) and for dispersion (e.g. rank test) have been proposed and their asymptotic properties investigated in the case when p is known. In the case when p is unknown, the test statistics are modified by replacing p by its usual estimator and it has been proved that some of the tests based on the modified statistics are asymptotically distribution free. The generalisation to the case when the random variable X has a multinomial distribution is also considered.

20. Efficiencies of Estimators of Scale and Location Parameters Constructed From Order Statistics of Censored Samples. J. A. Tischendorf, Bell Telephone Laboratories. (Invited Paper)

Estimators of the location and/or scale parameters of distribution functions with p.d.f.'s of the form $g(x) = \sigma^{-1}[f(x - m)/\sigma]$ are constructed from k order statistics where the sample size n is large. The order statistics are the sample quantiles corresponding to the specified constants $0 < \lambda_1 < \cdots < \lambda_k < 1$. The estimators are unbiased, linear and of minimum variance for the particular set of λ_i's, $i = 1, \cdots, k$. Necessary conditions for an optimum spacing of $\lambda_1, \cdots, \lambda_k$ are given for distributions satisfying certain continuity and differentiability conditions. This optimum spacing may be approximated by a relatively simple, graphical procedure in each of the three cases, estimating the location parameter, the scale parameter, or both parameters. Upper bounds on the efficiencies of these estimators are obtained. These bounds may be interpreted with respect to the ordered sample in such a way as to also yield upper bounds on the efficiencies of such estimators when the large sample is a censored one. Interesting comparisons of estimation situations can be made for the case where time is the random variable, i.e., censoring is on the right.

21. Some New Single Level Continuous Sampling Plans. (Preliminary Report) John S. White, Aero Division, Minneapolis-Honeywell Reg. Co. (By title)

Generalizing the methods of Dodge (*Ann. Math. Stat.* Vol. 14 (1943) pp. 264–279 and *Ind. Qual. Cont.* Vol. 7, pp. 7–11) some new single level continuous sampling plans are given. The procedure for these plans is as follows: (a) At the outset inspect, in succession, 100% of the units produced until i units in succession are found clear of defects. (b) When i successive units are found clear of defects, discontinue 100% inspection, and inspect only a fraction f of the units. (c) When a defect is found, revert to 100% inspection until either a second defect is found or until m successive units have been found clear of defects. (1) If a defect is found before m successive units have passed inspection, revert to 100% inspection as per (a). (2) If no defect is found, revert to sampling inspection at rate f. (i) If a defect is found in the next k sample units inspected, revert to 100% inspection as per (a). (ii) If no defect is found in the next k sample units, continue sampling until a defect is found and then proceed as in (c). Tables have been computed giving AOQL, i and f values corresponding to various values of k and m.

22. Existence of Wald's Sequential Test in the General Case. Robert A. Wijsman, University of Illinois.

A sequential probability ratio test (SPRT) for choosing between two hypotheses H_i, $i = 1, 2$, is defined by the acceptance intervals I_i. Let $u = \sup I_1$, $v = \inf I_2$, $u \leqq v$. In order to cope with discrete distributions, define a randomized SPRT $R(s, t)$, with error probability vector $\alpha(s, t)$, $s = (u, \lambda)$, $t = (v, \mu)$, $0 \leqq \lambda, \mu \leqq 1$, as follows: If $u < v$, u is included in or excluded from I_1 with probabilities λ and $1 - \lambda$; v is included in or excluded from I_2 with probabilities $1 - \mu$ and μ. If $u = v$, then $\mu \geqq \lambda$ and u is included in I_1, in I_2, or in neither, with probabilities $1 - \mu$, λ, and $\mu - \lambda$. The existence proof in the continuous case (*Ann. Math. Stat.* Vol. 29 (1958) p. 938) remains formally valid in the general case if s and t are considered elements of a space Z of points $z = (x, y)$, $0 < x < \infty$, $0 \leqq y \leqq 1$, with the points $(0, 1)$ and $(\infty, 0)$ added. Define a linear ordering: $z_1 < z_2$ if $x_1 < x_2$ or $x_1 = x_2$ and $y_1 < y_2$. A topology for Z is generated by sets of form $z < z_1$, or $z > z_2$. If f is continuous on Z, if $a, b \varepsilon Z$ and c is a number between $f(a)$ and $f(b)$, then $f(z) = c$ for some z,

$a \leqq z \leqq b$. This is applied to the functions $\alpha_i(s, t)$ for fixed s or t, and $\alpha_i(s, s)$, which are continuous and monotonic on Z. Let $C = \{\alpha(s, s): s \varepsilon Z\}$ and let A be the closed set in the α-plane bounded by C and the coordinate axes. Then, if $\alpha^* \not\varepsilon A$, there is no solution to $\alpha(s, t) = \alpha^*$, and if $\alpha^* \varepsilon A$ there is an essentially unique solution. This solution has optimum property, hence is admissible, provided $u \leqq 1 \leqq v$. In any case, the solution has optimum property among all solutions which take at least one observation. For any α^* with $\alpha_1^* + \alpha_2^* \leqq 1$ there is a test with error probability vector α^*, possessing optimum property, in the form of a mixture of R_1, R_2 and $R(s, t)$ for some s, t with $u \leqq 1 \leqq v$, where R_i accepts H_i without any observation.

(Abstracts of papers presented at the Eastern Regional Meeting, Columbia University, April 21–23, 1960.)

1. **Transition Probabilities for Telephone Traffic.** V. E. Beneš, Bell Telephone Laboratories and Dartmouth College. (By title)

A stochastic process $N(t)$, representing the number (out of a total N) of telephone trunks that are in use, is defined by the conditions that arrivals form a renewal process, and that holding-times of calls have a negative exponential distribution. The transition probabilities of the (not necessarily Markov) process $N(t)$ are determined in terms of their Laplace transforms (i) by augmenting $N(t)$ to be a suitable Markov process, and (ii) directly by using the regeneration points of $N(t)$. The practical relevance of the transition probabilities to traffic measurement are described.

2. **Efficient Sequential Estimators With High Precision Only in a Small Interval.** Allan Birnbaum, New York University.

The requirement that an estimator $\theta^* = \theta^*(x)$ of a real-valued parameter θ have high precision in a small interval $[\theta_1, \theta_2]$ can be formulated in part thus: The probability that θ^* be closer to θ_1 than to θ_2 when θ_1 is true, and the probability that θ^* be closer to θ_2 than to θ_1 when θ_2 is true, should equal or exceed specified lower bounds $1 - \alpha$, $1 - \beta$ respectively. In many problems such specifications cannot be met by an estimator based on a single observation. If sequential sampling is allowed, these requirements can be met most efficiently, in the sense of minimizing the expected sample size under *all* values of θ, by use of the sampling rule of Wald's sequential probability ratio test of θ_1 against θ_2 at strength (α, β) under general conditions met in common examples. On the resulting sample space $\{x\}$, the stated requirements are met efficiently by every estimator which takes values exceeding $\theta' = (\theta_1 + \theta_2)/2$ on points x where the corresponding sequential test would reject θ_1, and values less than θ' on other points. The definition of the estimator can be completed to make it admissible. The description of such estimators is simple when there is no "excess at termination" (or when excess is ignored): let $t = t(x) = 1/n$ if x is a "rejection" point based on n observations, let $t = -1/n$ if x is an "acceptance" point based on n observations, and let θ^* be any monotone function of t meeting the above condition. In problems with suitable symmetry, θ^* can be determined thus so as to be a median-unbiased admissible estimator. Admissibility is proved by noting that the class of estimators first described are (sequential) Bayes solutions, and by determining within this class a unique Bayes solution for another *a priori* distribution.

3. **Partially Balanced Arrays.** I. M. Chakravarti, University of North Carolina and Indian Statistical Institute. (Introduced by David B. Duncan)

Earlier (1956) this author had defined partially balanced arrays as follows: An array involving n factors $F_1, F_2, \cdots, F_n$, each at s levels such that for any group of d factors

($d \leqq n$), a combination of levels of d factors, F_{1i_1}, F_{2i_2}, $\cdots$, F_{di_d}, occurs $\lambda_{i_1 i_2 \cdots i_d}$ times where $\lambda_{i_1 i_2 \cdots i_d}$ remains the same for all permutations of a given set (i_1, i_2, $\cdots$ i_d) of levels and for all groups of d factors chosen out of n, i_j ranging from 0 to ($s - 1$) for all j. Then it can be easily shown that this property also holds for any $k \leqq d$. Examples of partially balanced arrays are given. These arrays require less number of assemblies than the corresponding orthogonal arrays for estimating the effects of interest; but the estimates are not mutually uncorrelated. For $s = 2$, it is shown that a class of partially balanced arrays are derivable from the well-known (λ-μ-ν) configurations.

4. **Extensions of the Poisson and the Negative Binomial Distribution.** A. CLIFFORD COHEN, JR., The University of Georgia.

In biological studies which involve fitting the Poisson or the negative binomial distribution to counts of organisms, considerable disparity is often encountered between observed and expected frequencies in the zero class. This paper concerns the addition of a selection parameter to these distributions in order to alleviate this difficulty. Maximum likelihood estimators of the original and the added selection parameters are derived. Asymptotic variances and covariances are given and illustrative examples are included.

5. **Asymptotic Variance as an Approximation to Expected Loss for Maximum Likelihood Estimates.** WILLIAM D. COMMINS, JR., Alexandria, Va.

From bounded estimation loss functions which are approximately parabolic when the estimate is near the parameter θ, Chernoff (*Ann. Math. Stat.*, Vol. 27, pp. 1–22) defines a normalized loss function. For an estimate based on a large number n of observations, the normalized expected loss is generally sandwiched between the variance $\sigma^2(\theta)$ of the asymptotic distribution of the estimate (the asymptotic variance) and the expected squared error (normalized). This paper is a proof under suitable restrictions that, for the maximum likelihood estimate T_n the normalized expected loss converges to the asymptotic variance, which can be smaller than the limit of expected squared error (normalized). The proof of the conjecture resolves into a proof that $\lim_{n\to\infty} nP(| T_n - \theta | > K) = 0$ for any $K > 0$ and $\lim_{n\to\infty} \int_{|T_n-\theta| \leqq K} n(T_n - \theta)^2 \, dP = \sigma^2(\theta)$ for small $K > 0$. The proof that the first limit holds is a modification of Wald's proof (*Ann. Math. Stat.*, Vol. 20, pp. 595–601) that the maximum likelihood estimate is consistent. The analysis of the integral involves a modification of the standard proof that the maximum likelihood estimate is asymptotically normal. The multiparameter case is treated separately but analogously.

6. **Multi-Stage Bayesian Lot-by-lot Sampling Inspection.** HERBERT B. EISENBERG, System Development Corp, (Introduced by Herbert T. David)

Based on the work of Arrow, Blackwell, and Girshick (*Econometrica*, 1949), this paper develops the theory for constructing Bayesian multi-stage (that is, single, double, multiple, and sequential) attribute sampling plans for finite lot size, arbitrary profit function, and arbitrary *a priori* lot quality distribution. Using a linear profit function, this theory is applied to the following *a priori* lot quality distributions: binomial, two-point, degenerate one-point, discrete mixed binomial, and continuous mixed binomial. Parametric and distributional conditions under which sampling never pays are discussed. Profit efficiencies of single, double, and multiple sampling plans relative to sequential plans can be computed. Effect of optimizing with respect to the wrong lot quality distribution is considered.

7. A Representation of the Bivariate Cauchy Distribution. THOMAS S. FERGUSON, U. C. L. A. and Princeton University. (By title)

A pair of random variables, (X, Y), is said to have a bivariate Cauchy distribution if every linear combination, $aX + bY$, has a (one-dimensional) Cauchy distribution (possibly degenerate). The main *theorem* proved is the following: A function, $\psi(u, v)$, is the logarithm of the characteristic function of a bivariate Cauchy distribution if, and only if, $\psi(u, v) = iau + ibv - g(u, v)$ where, (1) a and b are real numbers, (2) $g(u, v)$ is a real, non-negative, and positive-homogenous function of degree one (i.e. $g(tu, tv) = |t| g(u, v)$ for all real values of t, u and v), and (3) the set $\{(u, v): g(u, v) \leqq 1\}$ is convex. The relation between this representation and that found in Levy's book, *Theorie de l'addition des variables aleatoires*, 1937, is discussed. It is shown that this theorem does not extend directly to higher dimensions: namely, that in three dimensions, there are convex sets symmetric with respect to the origin which cannot be obtained as a set, $\{(u, v, w): g(u, v, w) \leqq 1\}$.

8. A Noiseless Comma-free Coding Theorem. THOMAS S. FERGUSON, UCLA and Princeton University.

A unique feature of *noiseless* coding theorems is unique decipherability of the code. However, this decipherability is unique only when one knows on which of the members of the infinite sequence of incoming symbols new words start. Ordinarily, wrong guesses as to the starting position may be detected only through the "nonsense" of the decoded sequence. This drawback may be avoided through the use of so-called comma-free codes (Golomb, Gordon, and Welch, *Can. Jour. Math.* 1958). It is shown that one can achieve the same asymptotic average transmission rate under the stronger restriction that the code be comma-free and uniquely decipherable.

9. Inference About Non-Stationary Markov Chains, RUTH Z. GOLD, Columbia University.

Extending results of Anderson and Goodman (*Ann. Math. Stat.*, Vol. 28 (1957), pp. 89–110), we consider N(large) observations taken at times $0, 1, 2, \cdots, T$ on a finite non-stationary Markov chain in which the transition probabilities are specified functions of a set of unknown parameters. By methods analogous to those of Neyman ("Contribution to the theory of the χ^2-test," *Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability*, University of California Press, Berkeley, 1949, pp. 239–274), best asymptotically normal estimates and tests of hypotheses are derived for these parameters. We also show that certain χ^2 expressions arising in Markov chains with arbitrary transition probabilities can be decomposed into a sum of squares of asymptotically independent normal variables with 0 means and unit variances after the manner of "partitioning" proposed by Lancaster (*Biometrika*, Vol. 36 (1949), pp. 117–129) despite the fact that in Markov chains the number corresponding to the number of observations in a contingency table is a random variable. A method of finding joint asymptotic confidence intervals for linear combinations of transition probabilities as well as of probabilities in independent sequences of multinomial trials analogous to that used in the analysis of variance is suggested.

10. A Central Limit Theorem for Systems of Regressions. E. J. HANNAN, University of North Carolina. (Introduced by David B. Duncan)

The theory of regression on fixed variables, when the residuals are generated by a stationary process, has been illuminated by the introduction of certain restrictions on the regressor vectors by Grenander. It is the purpose of the paper to show that, for a reasonably wide class of stationary residuals, these conditions are sufficient to ensure that the estimates of the regression coefficients are asymptotically normal. The case of a multiple system of regressions is considered.

11. Power Functions for the Test of Independence in 2 × 2 Contingency Tables. William Harkness, Pennsylvania State University.

A unified treatment for testing for independence in 2×2 tables is given. Using the uniformly most powerful test for independence in each of the three 2×2 tables, as determined by the number of restrictions on the marginal totals, a comparison of the exact power function for each test is made. Using an asymptotic normal approximation to the distribution of two independent binomials, conditional on fixed sum, asymptotic power is examined. The adequacy of the non-central chi-square approximation to power for small sample sizes ($n = 10, 20$, and 30) is considered, with exact values of power having been calculated. The availability of these exact values makes it possible to evaluate the adequacy of other approximations, particularly Patnaik's [*Biometrika*, Vol. 35, pp. 157–175] and Sillitto's [*Biometrika*, Vol. 36, pp. 347–352] approximations to the power for the test of equality of two binomial parameters. The normal approximation theorem shows Patnaik's results are based on erroneous considerations. The asymptotic results are similar to those of Mitra [*Ann. Math. Stat.*, Vol. 29, pp. 1121–1234].

12. The Partition of Phenotypic Variance Based on the Genic-environmental Interaction Model. Cecil L. Kaller and Virgil L. Anderson, Purdue University.

The genic-environmental interaction model in population genetics is developed by a direct extension of Kempthorne's (1954) theoretical gene model to include environmental factors. Both the genetic and the environmental factors are considered as the *main factors* in a factorical experimental design. This permits the separation of all orders of interaction terms. By employing algebraic identities and extensive algebraic manipulations, a complete phenotypic model is developed which is the sum of terms to account for all environmental and all genetic main effects influencing phenotypic expression as well as terms for all possible orders of genic-environmental interactions. Each effect is accounted for by a different term in the model, and all terms are shown to be uncorrelated. Hence for a population described by this model, the total phenotypic variance, σ_P^2, may be readily partitioned into a sum of components due to the various effects. Any changes made in the assumptions or the existence of interactions in the genic-environmental interaction model are reflected immediately in the variance partition by merely adding or dropping components of the sum.

13. A Robust Approximate Confidence Interval for Components of Variance. Howard Levene, Columbia University.

Let $x_{ij} = \mu + y_i + z_{ij}$, $i = 1, \cdots, I$; $j = 1, \cdots, J$, with $E(y_i) = E(z_{ij}) = 0$, $\text{Var}(y_i) = \sigma_y^2$, $\text{Var}(z_{ij}) = \sigma^2$. The classical F test for testing H_0: $\sigma_y^2 = 0$ is exact for normality of the y and z, and is robust. Previously suggested confidence intervals for σ_y^2 are approximate, and strongly affected by non-normality. (See e.g. Scheffé, *The Analysis of Variance*.) I give a robust method for testing equality of variances in *Contributions to Probability and Statistics: Essays in Honor of Harold Hotelling*. In the same spirit let $V_i = I(x_i - x_{..})^2 (I - 1)^{-1} - \Sigma_j(x_{ij} - x_{i\cdot})^2 (J - 1)^{-1}$. Then $E(V_i) = \sigma_y^2$, the V_i have a common variance and they have a positive correlation of order I^{-2}. An ordinary Student's t test may be used on the

V_i to test $H_1: E(V_i) = \sigma_v^2 = a_1$ for any a_1 and hence to obtain confidence intervals. However for $a_1 = 0$ the F test should be used. If $I \geqq 10$, and probably for even smaller values, the V test is generally satisfactory, while for very small I any confidence interval is unsatisfactorily long. The above method can be extended to r-way classifications, and, less satisfactorily, to unbalanced 1-way classifications.

14. An Inequality for Balanced Incomplete Block Designs. WADIE F. MIKHAIL, University of North Carolina. (By title)

Consider a Balanced Incomplete Block Design (B.I.B.D.) with parameters v, b, r, k, λ where b is the number of blocks, r is the number of replications of each treatment, and λ is the number of times a pair of treatments occur together in a block. It was proved by Bose (*Sankhya*, Vol. 6, 1942) that if the B.I.B.D. is resolvable, then $b \geqq v + r - 1$. The present paper shows that the condition of resolvability is not necessary and that the above inequality holds under the weaker condition $v = nk$ where n is an integer greater than 1.

15. Markov Renewal Processes of Zero Order. RONALD PYKE, Columbia University. (Invited Paper)

A Markov Renewal process (M.R.P.) determined by (m, A, Q), where m is the number of states, A is the $1 \times m$ vector of initial probabilities and Q is the matrix of transition distributions $Q_{ij}(t)$, is said to be of zero order if for every i, $Q_{ij}(t) = Q_{ik}(t)$ for all j, k and $t > 0$. The general theory of M.R.P.'s simplifies considerably in this case, and the author is able to give more explicit results pertaining to first passage times, stationary distributions, and limit theorems of the number of visits to specified states. An example of a zero order M.R.P. which arises in counter theory is worked out in detail.

16. On Centering Infinitely Divisible Processes. RONALD PYKE, Columbia University.

The concept of centering stochastic processes having independent increments, introduced by Lévy, is applied to processes having both stationary and independent increments (i.e. to Infinitely Divisible (I.D.) processes). The question of what centering functions preserve the stationarity of the increments is studied. It is shown that for an I.D. process, there exists a unique centering function c satisfying $c(s + t) = c(s) + c(t)$ for all $s, t \geqq 0$ and $c(1) = 0$, such that the resulting centered process is also an I.D. process. A proof of this result which does not use the Lévy-Khintchine representation of the characteristic function of an infinitely divisible random variable is given.

17. The Asymptotic Power of the Kolmogorov Tests of Goodness of Fit. DANA QUADE, University of North Carolina.

Let $F_n(x)$ be the empirical distribution function of a random sample from some continuous distribution function $G_n(x)$. Then the (two-sided) Kolmogorov test of the hypothesis that $G_n(x) = H(x)$ rejects if $\sup_x n^2 \mid F_n(x) - H(x) \mid \geqq Q_n$. Let $Z_n(t) = (n)^2(F_n\, [G_n^{-1}(t)] - t)$ and $S_n(t) = (n)^2(H[G_n^{-1}(t)] - t)$. Then the power of the test is

$$P_n = 1 - \text{pr}\,\{\sup_t \mid Z_n(t) - S_n(t) \mid < Q_n\}\,.$$

As n increases, and α is kept fixed, Q_n approaches a limit Q, and $Z_n(t)$ becomes a certain Wiener process $Z(t)$. Suppose that $S_n(t)$ also approaches a limit $S(t)$. Then, extending Donsker's justification of Doob's "heuristic procedure", some sufficient conditions that

$\lim_{n\to\infty} P_n = 1 - \text{pr}\{\sup_t \mid Z(t) - S(t) \mid < Q\}$ are given. The one-sided test can be treated similarly. Upper and lower bounds on the asymptotic power of both tests against the class of all possible sequences $\{G_n(x)\}$ such that $\lim_{n\to\infty} \sup_t \mid S_n(t) \mid = \Delta$, and various subclasses of this class, are exhibited. Finally, some numerical examples for the case where $G_n(x)$ consists in a translation of $H(x)$ are provided: in particular, it is shown that for detecting shifts in the mean of a normal population the one-sided test has an asymptotic efficiency of roughly .6 to .7.

18. Some Results on Error-Correcting Non-Binary Codes. D. K. Ray-Chaudhuri, Case Institute of Technology.

Consider a communication channel which can transmit p symbols where p is a prime number. A group code for such a channel with n places of which k are information places, is called an (n, k) p-ary code. A matrix M with elements in a field is said to possess the property (P_t) if no t rows of the matrix are dependent. If there is a $(n \times \overline{n-k})$ matrix M with elements in $GF(p)$, the Galois field containing p elements, which possesses (P_{2t})-property, then there exists a t error-correcting (n, k) p-ary code. Let n be the least integer such that for some integer c, $cn + 1 = p^m$. Let $r(j)$ denote the number of distinct residue classes mod n among the integers $j, pj, p^2j, \cdots p^{m-1}j, (j + 1), p(j + 1), p^2(j + 1) \cdots p^{m-1}(j + 1), \cdots (j + 2t - 1), p(j + 2t - 1), p^2(j + 2t - 1), \cdots p^{m-1}(j + 2t - 1)$. Using the theorem on (P_{2t})-property, a t-error correcting (n, k) p-ary code is constructed with $k = n - r(j)$. The result is extended to the case when p is a prime power.

19. Concerning Achievement of the Lower Bound for the Power of the Kolmogorov-Smirnov Test of Fit. Judah Rosenblatt, Purdue University.

A lower bound for the power of the Kolmogorov-Smirnov test, as a function of distance from the null hypothesis, F_0, is easy to compute, using the fact that for fixed x, $nF_n(x)$ has the Binomial distribution with parameters n and $p = F(x)$. A natural question which arises is whether there is any distribution function F, with $\sup_x \mid F(x) - F_0(x) \mid = l$ for which the computed lower bound for power is achieved. It is shown that if the asymptotic theory of these tests is conservative, then for those alternatives satisfying the condition $l \leqq .1$ and for which the computed lower bound for power exceeds .95, there is a distribution function F which comes close to achieving this lower bound, when the asymptotic probability of Type I error does not exceed .05.

20. On the Admissibility of a Class of Tests in Normal Multivariate Analysis. S. N. Roy and W. F. Mikhail, University of North Carolina.

This paper proves the admissibility of (i) the largest root test, under a normal multivariate linear model, for a linear hypothesis against the general linear alternative, (ii) the largest root test for independence between a p-set and a q-set of variates (having a $(p + q)$-variate normal distribution) against $\Sigma_{12} \neq 0$, where Σ_{12} is the covariance matrix between the p-set and the q-set, and (iii) the largest or smallest root test for the equality of two dispersion matrices against certain types of one-sided alternatives. In each of the cases (i), (ii) or (iii), the test has an acceptance region which is the intersection of a class of regions, and the proof depends upon showing that if the acceptance region is to be proved inadmissible by a rival, then that rival region must be contained in every member of the class of regions just mentioned, or, in other words, the rival itself must coincide with the acceptance region of the proposed test. Hotelling's T^2 is a very special case of (i) and the usual multiple correlation test is a special case of (ii). The same kind of proof, with some modifications, would go through for the corresponding λ-criteria.

21. On Dependent Tests in Analysis of Variance. S. N. Roy and P. R. Krishnaiah, University of North Carolina.

Let $F_i = S_i^2/S^2$ for $i = 1, \cdots\cdots, k$ be k F-statistics to test the null hypotheses H_{0i} where S_i^2 is the mean square due to H_{0i} and S^2 is the error mean square in ANOVA. Ramachandran (these *Annals*, 1956) solved the distribution problems connected with the simultaneous test of $H_{01} \cdots\cdots\cdots, H_{0k}$ when $S_1^2, \cdots\cdots, S_k^2, S^2$ are independently distributed. The present paper extends the above results to the situations where $S_1^2, \cdots\cdots\cdots, S_k^2$ are *not* independently distributed.

22. Lower Bounds on the Probability Associated with Certain Confidence Regions for the Multivariate Median. (Preliminary Report) Ernest M. Scheuer, Space Technology Laboratories. (Invited paper)

Consider a k-dimensional random variable $(X_1, \cdots, X_k)$ having unique median $(\nu_1, \cdots \nu_k)$. (DEF.: ν_i = med X_i.) Take a sample of size n of this random variable: $(x_{11}, \cdots, x_{k1}), \cdots, (x_{1n}, \cdots, x_{kn})$ and order the values $x_{i1}, \cdots, x_{in}$ to yield $x_i(1) \leqq x_i(2) \leqq \cdots \leqq x_i(n)$, $i = 1, \cdots, k$. Select positive integers r_i such that $2r_i < n$ $(i = 1, \cdots, k)$ and form the set

$$R = \{(x_1, \cdots, x_n): x_i(r_i) < x_i < x_i\,(n - r_i + 1),\, i = 1, \cdots, k\}.$$

We ask (*) "what is the probability $\mathcal{P}$ that R covers $(\nu_1, \cdots, \nu_k)$?" The answer (for $k > 1$) depends on the joint distribution of $(X_1, \cdots, X_k)$, but sharp lower bounds over all distributions having unique medians have been obtained for $\mathcal{P}$ (a) by Dunn (*Ann. Math. Stat.*, Vol. 30 (1959), pp. 192–197) for the case $k = 2$ and $r_1 = r_2 = r$ (say); and (b) by the present author in this paper for the cases $3 \leqq k \leqq 7$ and $r_1 = \cdots = r_k = 1$. The results under (b) can be summarized in the inequality $\mathcal{P} \geqq 1 - 2k(\frac{1}{2})^n + 4\,(3k - 7)\,(\frac{1}{4})^n - 16\,(k - 3)\,(\frac{1}{8})^n$. It is conjectured that this result is true for all $k > 3$. A result on the general problem (*) which, while not sharp, may prove to be quite satisfactory is given by the simple result $\mathcal{P} \geqq 1 - \sum_{i=1}^{k} [1 - P\,\{x_i(r_i) < \nu_i < x_i(n - r_i + 1)\}]$. This formula is useful in that the terms in square brackets are readily obtainable from tables of the incomplete beta distribution or of the cumulative binomial distribution.

23. Asymptotic Shapes of Optimal Stopping Regions for Sequential Testing. Gideon Schwarz, Columbia University. (Introduced by T. W. Anderson)

A hypothesis $\theta \leqq a$ is to be tested sequentially against an alternative $\theta \geqq b$ $(a < b)$ on the basis of independent observations on a random variable X whose distribution depends on a single parameter θ and is of the Koopman-Darmois type. If $a < \theta < b$ neither decision is penalized. A given *a priori* distribution of θ is assumed. The cost per observation is c. *Theorem*: If the optimal stopping region in the $(n, \sum_1^n X_i)$-plane is transformed by dividing both coordinates by log $(1/c)$, the transformed region approaches a finite limiting region as $c \to 0$. An explicit formula for the limiting region ("asymptotic shape") for arbitrary *a priori* distribution of θ is given. By transforming the asymptotic shape back to original scale, approximations to the optimal regions for small c are obtained. The theorem is proved by showing that the optimal boundary lies between two curves of constant Bayes risk, and finding the asymptotic shape of such curves. Finally the theorem is extended to two cases of two-parameter families. One is the case of testing the mean of a normal distribution with unknown bounded variance. In the other case H_0, H_1 and the indifference region consist of three arbitrary specified mutually dominating distributions.

24. Invariant Bayes Rules. (Preliminary Report) MORRIS SKIBINSKY, Purdue University.

Given a sequence of independent random variables with a common distribution function known *a priori* to be one of K specified distribution functions, let $\mathcal{S}$ be a class of rules for deciding which one of the K is correct. Let $\mathfrak{J}(g, \lambda)$ denote the set of all Bayes rules in $\mathcal{S}$ relative to *a priori* probabilities $g = (g_1, g_2, \cdots, g_k)$, loss matrix $\lambda = (\lambda_{ij})$, and cost per observation unity. Let $G^0 = \{g: \sum_1^k g_i = 1, g_j > 0, j = 1, \cdots, K\}$; Λ be the space of $K \times K$ loss matrices, with zero diagonal and non-negative off-diagonal elements; M be a mapping from G^0 into Λ. The class $\mathfrak{J}(M)$ of invariant Bayes rules relative to M, is defined to be $\bigcap (\mathfrak{J}(g, M(g)) \mid g \varepsilon G^0)$. The class of invariant Bayes rules is then, $\mathfrak{J} = \bigcup (\mathfrak{J}(M) \mid \text{all mappings } M)$. The importance of this class follows from the fact that if $T \varepsilon \mathfrak{J}(M)$, then T minimizes the expected number of observations required uniformly over the K hypotheses, among all rules whose error probabilities are bounded above by its own. Necessary and sufficient conditions for a rule to be an invariant Bayes rule are given. Several examples are considered for $K = 3$. For $K = 2$ (and $\mathcal{S}$ the class of all rules) it has been shown (See Wald and Wolfowitz, "Optimum Character of Sequential Probability Ratio Test", *Ann. Math. Stat.*, Vol. 19 (1948)) that $\mathfrak{J}$ is equivalent to the sequential probability ratio tests.

25. On a Generalization of Balanced Incomplete Block Designs. J. N. SRIVASTAVA AND S. N. ROY, University of North Carolina.

A generalized BIBD (GBIBD) is defined as follows: Let the total number of treatments $v = v_1 + v_2 + \cdots + v_s$ be divided into S sets, the ith set containing v_i treatments. Then the GBIBD is such that any treatment belonging to the ith set occurs (once only) in r_i blocks, a pair from the ith set occurs in λ_i blocks, and a pair consisting of one treatment the ith set and another from the jth set occurs (once only) in exactly μ_{ij} blocks. The design may be arranged in equal or unequal block sizes. The motivation behind the use of such designs is that it permits treatment contrasts corresponding to the comparisons within a set or for comparison of different sets among themselves, to be estimated with any given precision, provided the design with the corresponding values of λ's and μ's exist. Since with fixed d.f. for hypothesis and error, the power of a test depends on the noncentrality parameter only, these designs allow the total hypothesis H_0: $t_1 = t_2 = \cdots = t_v$ to be split up into subhypotheses corresponding to comparisons between or within sets, the subhypotheses being tested with given powers. The multivariate or multiresponse generalization has also been considered. In this paper, the analysis of this design (for the case of equal block sizes) has been presented both for fixed and mixed models. Several basic relations and inequalities among the parameters have been defined. Some studies on the structure of BIBD's and GBIBD's have been made. Several methods of construction of different kinds of GBIBD's using known BIBD's, factorial and other designs, have been presented.

26. Maximum Likelihood Characterization of the Normal Distribution. HENRY TEICHER, Purdue University.

Let $\{F(x - \theta)\}$, θ real, be a translation parameter family of absolutely continuous distributions on the real line. If, for all (random) samples of size two and three, a maximum likelihood estimator of θ is the sample arithmetic mean, then $F(x)$ is the normal distribution. Analogous results are demonstrated in the case of a scale parameter family.

27. On Two Methods of Unbiased Ratio and Regression Estimation. W. H. WILLIAMS, McMaster University. (By title)

R. M. Mickey (*J. Amer. Stat. Assoc.*, Vol. 54, pp. 594–612) introduced a procedure for generating unbiased ratio and regression estimators. Another was used by Williams (*Ann. Math. Stat.*, Vol. 29, p. 618). The two procedures have different appearances, but it is shown that a slight modification of Mickey's method will lead to estimators which have the same form as those presented by Williams. The combinatorial equivalence is demonstrated also.

28. On Linear Estimation of a Single Parameter of a Mean Function Under Second Order Disturbance. (Preliminary Report) N. DONALD YLVISAKER, Columbia University.

Let $\{P_\lambda, \lambda \varepsilon \Lambda \subset R_1\}$ be a family of probability measures on (Ω, A). Let $\{Y(t), t \varepsilon T\}$ be a family of random variables on (Ω, A) satisfying $E_\beta Y(t) = m(t; \beta)$, $t \varepsilon T$, $E_\beta[Y(s) - m(s; \beta)] [Y(t) - m(t; \beta)] = K(s, t)$, $s, t \varepsilon T$, T an abstract set. The problem of linear estimation of the parameter β is considered. Let $H(K, T)$ denote the reproducing kernel space of functions on T associated with K. Then the span of $\{Y(t), t \varepsilon T\}$ in $L_2(dP\beta)$, written $V_\beta[Y(t), t \varepsilon T]$ is operationally independent of β if $m(\cdot, \beta) \varepsilon H(K, T)$ for all $\beta \varepsilon \Lambda$ (operational independence here means a sequence of random variables of the form $\{\sum_j c_{jn} Y(t_{jn})\}$ is a Cauchy sequence in $L_2(dP_\beta)$ for all $\beta \varepsilon \Lambda$ or no $\beta \varepsilon \Lambda$). The precise lower bound

$$E_\beta[Z - \beta]^2 \geqq \beta^2/(1 + \|m(\cdot, \beta)\|^2_{H(K)})$$

is obtained for $Z \varepsilon V_\beta[Y(t), t \varepsilon T]$ with the bound interpreted as zero if $m(\cdot, \beta) \not\varepsilon H(K, T)$. Bayes estimates of β are considered in special cases of the above model.

29. A Generalization of a Theorem of Balakrishnan. N. DONALD YLVISAKER, Columbia University. (By title)

Let T be an abstract set and let K be a covariance kernel defined on $T \times T$. A function m defined on T is said to be an admissible mean function for the covariance kernel K if and only if there exists a family $\{X(t), t \varepsilon T\}$ of random variables on some probability space (Ω, A, P) with $E[X(t)] = m(t)$, $t \varepsilon T$, $E[X(s) X(t)] = K(s, t)$, $s, t \varepsilon T$. Let $H(K, T)$ denote the reproducing kernel space of functions on T associated with K. *Theorem*: m is an admissible mean function for the covariance kernel K if and only if $m \varepsilon H(K, T)$ and $\| m \|^2_{H(K)} \leqq 1$. This is a generalization of a result due to Balakrishnan (*Ann. Math. Stat.*, September, 1959) and provides an alternative proof of that result.