THE FIRST-PASSAGE MOMENTS AND THE INVARIANT MEASURE OF A MARKOV CHAIN

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We consider an irreducible, recurrent Markov chain with transition probability matrix $P = [p_{ij}]$. The random variables constituting the chain are $\{X_n\}$; let $N > 0$ be the smallest positive time n at which $X_n = 0$. Then the quantities

$$E\{N(N-1)\cdots(N-k+1) \mid X_0 = i \neq 0\} = \mu_{i0}^{(k)}$$

are the factorial first-passage time moments. In case $i = 0$, we will let $\mu_{00}^{(k)} = \delta_{0k}$. However, it is also convenient to introduce the actual recurrence-time moments for state 0:

$$E\{N(N-1)\cdots(N-k+1) \mid X_0 = 0\} = \mu_0^{*(k)}.$$

Let $\{\pi_i\}$ be the unique positive solution of the equation

$$\pi_j = \sum_i \pi_i p_{ij}, \tag{1}$$

often called the "invariant measure" of the chain. Then this measure and the first-passage moments are related by the

Theorem. *The equation*

$$\pi_0 \mu_0^{*(k+1)} = (k+1) \sum_i \pi_i \mu_{i0}^{(k)}, \qquad k = 0, 1, 2, \cdots, \tag{2}$$

is always valid. (Both sides may be $+\infty$.)

Remarks. If $k = 0$, (2) reduces to the familiar assertion that the mean recurrence time of state 0 is $\pi_0^{-1}\Sigma\pi_i$. If $k = 1$, (2) is equivalent to a "remarkable formula" discovered by Chung [1], who gave a proof rather different from that which follows.

Proof of the Theorem. We shall use generating functions; let

$$f_{i0}^{(n)} = \Pr\{X_n = 0, X_l \neq 0 \text{ for } l < n \mid X_0 = i \neq 0\}$$
$$= \Pr\{N = n \mid X_0 = i \neq 0\}; \qquad f_{00}^{(n)} = \delta_{n0}; \qquad F_{i0}(x) = \sum_{n=0}^{\infty} f_{i0}^{(n)} x^n.$$

Thus $F_{00}(x) = 1$, and $F_{i0}^{(k)}(1) = \mu_{i0}^{\prime(k)}$ for all i including 0. Similarly we put $g_0^{(n)} = \Pr\{X_n = 0, X_l \neq 0 \text{ for } 1 \leqq l < n \mid X_0 = 0\} = \Pr\{N = n \mid X_0 = 0\}$, and $G_0(x) = \sum_{n=1}^{\infty} g_0^{(n)} x^n$. Notice that $g_0^{(n)} = \sum_i p_{0i} f_{i0}^{(n-1)}$, so that

$$G_0(x) = x \sum_i p_{0i} F_{i0}(x). \tag{3}$$

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Similarly, $f_{i0}^{(n)} = \sum_j p_{ij} f_{j0}^{(n-1)}$ provided that $i \neq 0$; this gives

$$F_{i0}(x) = x \sum_j p_{ij} F_{j0}(x), \qquad i \neq 0. \tag{4}$$

The next step is to multiply by π_i in (4) and sum, which yields

$$\sum_{i \neq 0} \pi_i F_{i0}(x) = x \sum_j \sum_{i \neq 0} \pi_i p_{ij} F_{j0}(x) = x \sum_j (\pi_j - \pi_0 p_{0j}) F_{j0}(x)$$

from (1). In view of (3) we have $\sum_i \pi_i F_{i0}(x) - \pi_0 = x \sum_i \pi_i F_{i0}(x) - \pi_0 G_0(x)$, which, upon differentiating $k + 1$ times, becomes (for $|x| < 1$)

$$(1 - x) \sum_i \pi_i F_{i0}^{(k+1)}(x) + \pi_0 G_0^{(k+1)}(x) = (k + 1) \sum_i \pi_i F_{i0}^{(k)}(x). \tag{5}$$

Relation (2) with "$\leqq$" is immediate from (5), for, letting $x \to 1$,

$$\pi_0 \mu_0^{*(k+1)} = \pi_0 G_0^{(k+1)}(1) \leqq (k + 1) \sum_i \pi_i F_{i0}^{(k)}(1) = (k + 1) \sum_i \pi_i \mu_{i0}^{(k)},$$

since the first term in (5) is non-negative. Now suppose that $G_0^{(k+1)}(1) < \infty$, but that $\sum_i \pi_i \mu_{i0}^{(k)} = \infty$. Then from (5) we obtain

$$\lim_{x \to 1-} \frac{(1 - x) \sum_i \pi_i F_{i0}^{(k+1)}(x)}{\sum_i \pi_i F_{i0}^{(k)}(x)} = k + 1.$$

It follows by Theorem 2 of [3] that

$$\sum_i \pi_i F_{i0}^{(k)}(x) = \frac{1}{(1 - x)^{k+1}} L\left(\frac{1}{1 - x}\right),$$

where $L(y)$ is a slowly varying function.[2] Integrating k times, we would then have

$$\sum_i \pi_i F_{i0}(x) = \frac{1}{1 - x} L_1\left(\frac{1}{1 - x}\right),$$

with L_1 again slowly varying.[3] This, however, is inconsistent with the fact that $\sum \pi_i F_{i0}(x)$ is bounded as $x \to 1$. (The assumption $G_0^{(k+1)}(1) < \infty$ excludes the null-recurrent case, so that $\sum \pi_i = \sum \pi_i F_{i0}(1) < \infty$.)

We have thus established that $\mu_0^{*(k+1)}$ and $\sum \pi_i \mu_{i0}^{(k)}$ are both finite or both infinite; assume the former is the case. We are through if it can be shown that the first term on the left in (5) tends to 0 as $x \to 1-$. This follows at once upon applying to the function $h(x) = \sum \pi_i F_{i0}^{(k)}(x)$ the following simple

LEMMA. *Let $h(x)$, $0 < x < 1$, be a positive, monotone increasing, convex function with a finite limit as $x \to 1-$. Then* $\lim_{x \to 1-} (1 - x) h'(x) = 0$. This fact is obvious upon drawing a diagram, and this completes the proof of the theorem.

S. Karlin has pointed out (in conversation) that the theorem can be proved

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[2] That is, $L(cy)/L(y) \to 1$ as $y \to \infty$ for every $c > 0$.

[3] This well-known fact may be easily deduced from the canonical form of the slowly varying function $L(y)$ [2].

in a somewhat different manner, which avoids the use of slowly varying functions. This has its advantages, but the author confesses to a mild proprietary pleasure in the argument given above.

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