ON THE UNIQUENESS OF THE TRIANGULAR ASSOCIATION SCHEME

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1. Summary. Connor [3] has shown that the relations among the parameters of the triangular association scheme themselves imply the scheme if $n \geqq 9$. This result was shown by Shrikhande [6] to hold also if $n \leqq 6$. (The problem has no meaning for $n < 4$.) This paper shows that the result holds if $n = 7$, but that it is false if $n = 8$.

2. Introduction. A partially balanced incomplete block design with two associate classes [1] is said to be triangular [2], [3] if the number of treatments, v, is $n(n - 1)/2$ for some integer n, and the association scheme is obtainable as follows:

Let the v treatments be regarded as all possible arcs of the graph determined by n points; let the first associates of any arc (= treatment) be all arcs each of which share exactly one end point with the given arc; let the second associates of any arc be all arcs each of which does not share an end point with the given arc and does not coincide with the given arc.

Then the following relations hold:

(2.1) The number of first associates for any treatment is $2(n - 2)$.

(2.2) If θ_1 and θ_2 are two treatments which are first associates, the number of treatments which are first associates of both θ_1 and θ_2 is $n - 2$.

(2.3) If θ_1 and θ_2 are second associates, the number of treatments which are first associates of both θ_1 and θ_2 is 4.

It is natural to inquire if conditions (2.1)–(2.3) imply that the $v = n(n - 1)/2$ treatments can be represented as arcs on the graph determined by n points in the manner described above; i.e., if (2.1)–(2.3) imply the triangular association scheme. This is known ([3], [6]) to be so if $n \neq 7, 8$.

We prove the result for 7. Actually we will prove the result for all n except 8. For $n = 8$, the theorem is false, as we shall demonstrate by exhibiting a counter-example. The derivation of this counter-example and a procedure for finding all counter-examples are given in [4]. They are based on an elaboration of the devices used in Sections 3 and 4 of this paper. Other illustrations of the use of these devices are contained in [5].

Henceforth, we assume (2.1)–(2.3).

3. The Association Matrix. Number the treatments from 1 to v in any order. Define the square matrix A of order v by

$$(3.1) \qquad A = (a_{ij}) = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } i \text{ and } j \text{ are first associates} \\ 0 & \text{if } i \text{ and } j \text{ are second associates} \end{cases}$$

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Note that $a_{ij} = a_{ji}$. Next let $B = AA^T = A^2$, since A is symmetric. From (2.1), we have $b_{ii} = 2(n - 2)$. From (2.2), we have $b_{ij} = (n - 2)$ if i and j are first associates. From (2.3), we have $b_{ij} = 4$ if i and j are second associates. If we let J be the square matrix of order v, with every entry unity, and I the identity matrix of order v, then the foregoing may be summarized by

$$(3.2) \quad A^2 = 2(n - 2)\, I + (n - 2)A + 4(J - I - A) = (2n - 8)I + (n - 6)A + 4J.$$

All the matrices appearing in (3.2) can be simultaneously diagonalized. Imagine (3.2) in diagonal form, and one sees that the diagonal entries relate the eigenvalues of the matrices.

Now J has the eigenvalue v, corresponding to the eigenvector $(1, 1, \cdots, 1)$; all other eigenvalues of J are zero. The eigenvector $(1, 1, \cdots, 1)$ clearly corresponds to the eigenvalue $2(n - 2)$ of A. Any other eigenvalue, α, of A corresponds to a zero eigenvalue of J; hence (3.2) implies that α satisfies the equation $\alpha^2 = (2n - 8) + (n - 6)\alpha$, so that $\alpha = -2$, or $\alpha = n - 4$.

The trace of A is zero, since $a_{ii} = 0$ for all i; hence the sum of the eigenvalues of A is 0. If k is the multiplicity of $n - 4$, it follows that $0 = 2n - 4 + k(n - 4) + (v - k - 1)(-2)$. So the eigenvalues of A are

(3.3)
(a) $2n - 4$ with multiplicity 1, eigenvector $(1, 1, \cdots, 1)$
(b) $n - 4$ with multiplicity $n - 1$
(c) -2 with multiplicity $v - n$.

Note that $v > n$, so -2 is the least eigenvalue of A.

This is the only use we shall make of (3.3) (c) in the present paper, although it plays a major role in the analysis of the exceptional cases for $n = 8$. We shall make no use of (3.3) (b).

In what follows, we shall use two well-known properties of eigenvalues and eigenvectors of symmetric matrices, and for ease of reference, we now list them explicitly.

Let M be a (real) symmetric matrix whose least eigenvalue is β, and whose maximum eigenvalue is $\alpha > \beta$, with x an eigenvector corresponding to α. Let K be a principal submatrix of M, δ the least eigenvalue of K and y an eigenvector of K corresponding to δ. Then

$$(3.4) \qquad \delta \geqq \beta;$$

and

(3.5) if $\delta = \beta$, then y is orthogonal to the projection of x on the subspace corresponding to K.

From (3.4) and (3.3) (c) follow the fact that a principal submatrix of A cannot have an eigenvalue less than -2. From (3.5) and (3.3) (a) and (c), if -2 is an eigenvalue of a principal submatrix of A, then the corresponding eigenvector has zero as the sum of its co-ordinates.

4. The Case $n \neq 8$.

LEMMA 1. *A does not contain*

$$\begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix} \tag{4.1}$$

as a principal submatrix.

This was proved by Connor [3] for $n \geqq 9$. We now prove it for all $n \neq 8$. We contend that A cannot contain any of the following three square matrices of order 5, each of which contains (4.1) as a principal submatrix:

$$\underset{(4.2)}{\begin{matrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{matrix}} \qquad \underset{(4.3)}{\begin{matrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{matrix}} \qquad \underset{(4.4)}{\begin{matrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{matrix}}$$

The impossibility of (4.2) and (4.4) follows from (3.4), since each has an eigenvalue smaller than -2. Matrix (4.3) has -2 as an eigenvalue, with $(1, 1, 1, -1, -1)$ as corresponding eigenvector, violating (3.5).

Let us denote by 1, 2, 3, 4 respectively the rows and columns of A that produced submatrix (4.1). Because (4.2) and (4.3) are impossible, it follows that 4 is the only treatment that is a first associate of 1, 2, and 3. Hence, by (2.3), there are exactly nine additional treatments, each of which is a first associate of two of the set 1, 2, 3. Since (4.4) is impossible, it follows that each of the nine is a first associate of four. Together with 1, 2, 3, this yields twelve treatments, each of which is a first associate of 4. From (2.1), we must have $12 \leqq 2n - 4$, which is impossible if $n \leqq 7$.

Now suppose $n \geqq 9$. Treatments 1 and 4 are first associates, and, by (2.2), there are $n - 2$ first associates of each. We have previously encountered 6, three of which are first associates also of 2, and three of which are also first associates of 3. Hence there are $n - 8$ additional ones. Similary, there are $n - 8$ additional first associates of 2 and 4, and $n - 8$ additional first associates of 3 and 4. Hence, from (2.1), $2(n - 2) \geqq 12 + 3(n - 8)$, which is impossible for $n \geqq 9$.

Next, we prove

LEMMA 2. *If* 1 *and* 2 *are second associates,* 3, 4, 5, 6 *first associates of both* 1 *and* 2, *then* (*after renumbering, if necessary*) *the principal submatrix of A corresponding to rows and columns* 1–6 *is*

$$\begin{matrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{matrix} \tag{4.5}$$

PROOF: Consider the $2(n - 2)$ treatments which are first associates of 3. None of them can be second associates of both 1 and 2, for this would violate Lemma 1. Hence, if we let t be the number of first associates of 3 which are first associates of 1 and 2, we have from (2.1) and (2.2), $t + (n - 2 - t) + (n - 2 - t) = 2(n - 2) - 2$, or $t = 2$. These two must be some two of 4, 5, 6, say 5 and 6. It follows that 3 and 4 are second associates, while 3 is a first associate of both 5 and 6. The inevitability of (4.5) is now clear.

LEMMA 3. *Any matrix of form*

$$\begin{matrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & x \\ 1 & 0 & 1 & 0 & 0 & 1 & x & 0 \end{matrix} \tag{4.6}$$

is not a principal submatrix of A.

PROOF: If (4.6) were to exist, then $x \neq 1$. For 6 and 7 would be second associates, and, if $x = 1$, then 1, 3, and 8 would mutually be first associates, but this contradicts Lemma 2. So we must take $x = 0$. But then 2, 7, and 8 are pairwise second associates; 3 is a first associate of each of 2, 7, 8, and this violates Lemma 1.

LEMMA 4. *The matrix*

$$\begin{matrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{matrix} \tag{4.7}$$

is not a principal submatrix of A.

PROOF: All we want to show is that the other entries in (4.7) imply that 7 and 8 are first associates, not second associates as (4.7) alleges. If 7 and 8 are second associates, then using the same reasoning as in the first part of Lemma 3, some two of 1, 3, 5 are by Lemma 2 second associates. But this is not so in (4.7).

LEMMA 5. *The $2(n - 2)$ first associates of any treatment can be split into two classes so that the $n - 2$ treatments of one class are mutually first associates of each other; the $n - 2$ treatments of the other class are mutually first associates.*

PROOF: Let 1 be the treatment. Let 3 be a first associate of 1, 2 a second associate of 1 and a first associate of 3, and 4, 5, 6 chosen so that we have the submatrix of Lemma 2. In addition to 5 and 6, there are $n - 4$ other first associates of both 1 and 3. Each of these must be a first associate of at least one of

5 and 6. Otherwise it, 5 and 6, would be mutually second associates, and 1 would be a first associate of each of the three, violating Lemma 1. Further, by Lemma 3, each of these $n - 4$ treatments is a first associate of 5 or each is a first associate of 6. Without loss of generality, say it is 5. By Lemma 4, these $n - 4$ treatments are mutually first associates. Further, each is a first associate of 3 and 5, which are themselves first associates, and thus 3, 5, and these $n - 4$ treatments are altogether $n - 2$ first associates of 1, which are mutually first associates.

Of the $n - 2$ first associates of 1 and 4, 5 is in the class already described, 6 is not, and there are $n - 4$ others. These $n - 4$ are mutually first associates by the same reasoning as above; they are entirely different from the previous $n - 4$ of the first class, since each of those was a second associate of 4; each is obviously a first associate of 6 as well as 4; so 4, 6, and these $n - 4$ treatments constitute our second class.

THEOREM 1. *If $n \neq 8$, then condition (2.1)–(2.3) characterize the triangular association scheme.*

PROOF: It has been shown by Shrikhande [6] that Lemma 5 implies Theorem 1.

5. The Case $n = 8$.

THEOREM 2. *If $n = 8$, then conditions (2.1)–(2.3) do not necessarily imply the triangular association scheme.*

PROOF: Here is a counter-example. Notice that the first principal submatrix of order 5 violates the triangular association scheme.

0	0	1	1	1	1	0	0	0	0	0	0	1	1	1	0	1	0	1	0	1	0	1	1	0	0	0	0
0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	0	0	1	1	1	1
1	1	0	1	1	1	1	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	1	0	1	1	0	0
1	1	1	0	1	1	0	1	0	0	0	0	0	1	0	0	1	1	0	0	0	0	1	0	0	0	1	1
1	1	1	1	0	1	0	0	1	0	0	0	0	1	0	0	0	0	1	1	0	0	0	1	1	0	1	0
1	1	1	1	1	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	1	1	0	1	0	1	0	1
0	0	1	0	0	0	0	1	1	1	1	1	0	1	1	1	0	0	0	0	0	0	1	0	1	1	0	0
0	0	0	1	0	0	1	0	1	1	1	1	0	1	0	0	1	1	0	0	0	0	1	0	0	0	1	1
0	0	0	0	1	0	1	1	0	1	1	1	0	1	0	0	0	0	1	1	0	0	0	1	1	0	1	0
0	0	0	0	0	1	1	1	1	0	1	1	0	1	0	0	0	0	0	0	1	1	0	1	0	1	0	1
0	0	0	0	0	0	1	1	1	1	0	0	0	0	1	0	1	0	1	0	1	0	0	0	1	1	1	1
0	0	0	0	0	0	1	1	1	1	0	0	1	1	0	1	0	1	0	1	0	1	1	1	0	0	0	0
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1	0	0	0	1	0	0	0	1	0	1	0	1	0	1	0	1	0	0	1	1	0	0	1	1	0	1	0
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1	0	0	0	1	1	0	0	1	1	0	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	0	0
0	1	1	0	1	0	1	0	1	0	1	0	0	0	1	1	0	0	1	1	0	0	0	0	0	1	1	0
0	1	1	0	0	1	1	0	0	1	1	0	0	0	1	1	0	0	0	0	1	1	0	0	1	0	0	1
0	1	0	1	1	0	0	1	1	0	1	0	0	0	0	0	1	1	1	1	0	0	0	0	1	0	0	1
0	1	0	1	0	1	0	1	0	1	1	0	0	0	0	0	1	1	0	0	1	1	0	0	0	1	1	0

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NOTE

The results of this paper have also been obtained, using different methods, by Chang, L. C., "The Uniqueness and Nonuniqueness of the Triangular Association Schemes," *Science Record*, Vol. III, New Series, 1959, pp. 604-613. Chang has also shown that there are exactly three counterexamples when $n = 8$ ("Association Schemes of Partially Balanced Designs with Parameters $v = 28$, $n_1 = 12$, $n_2 = 15$ and $p_{11}^2 = 4$," *Science Record*, Vol. IV, New Series, 1960, pp. 12-18).