CONTRIBUTIONS TO THE THEORY OF RANK ORDER STATISTICS: THE TWO-SAMPLE CENSORED CASE[1]

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0. Summary. Rank order theory is developed for the two-sample problem in which censoring of the observations has occurred, i.e., not all of the random variables are observed. The approach is similar to [2] with the striking difference that in the present case the rank orders are not all equally likely under the null hypothesis, and thus it becomes important to work with the likelihood ratios of rank orders. In applying the results of this paper, there will be a strong analogy to sequential analysis. The censoring scheme corresponds to the stopping rule and in both cases the terminal decision should be based on the likelihood ratio. We do not give the detailed applications of the present theory either to earlier procedures or to the new ones introduced here.

1. Introduction. Consider two ordered sets of numbers $(x_1, \cdots, x_m)$ to be called the first sample, and $(y_1, \cdots, y_n)$ to be called the second sample, i.e., $x_i < x_j$ $(1 \leqq i < j \leqq m)$ and $y_i < y_j$ $(1 \leqq i < j \leqq n)$ and define $\Delta(a, b) = 0(1)$ if $a > b(a \leqq b)$. If $\Delta(x_i, y_j)$ is known for all values of i and j it is possible to combine the two sets and arrange them from smallest to largest. In that case, if the two sets of numbers correspond to observations from two random samples (with no ties), it is possible to apply the usual nonparametric procedures, e.g., Wilcoxon, Kolmogoroff-Smirnoff. In some statistical applications it may be necessary, and can be desirable, not to observe all of the order relationships between the two samples. Thus, if the measuring device is such that it is very inaccurate for small values one may learn only how many random variables occurred in each sample below some threshold and the order relationships between the observed random variables above the threshold. This same difficulty could also occur for large values or for both large and small values. In life testing, savings in experimental costs and time are often effected by stopping the experiment before all of the lives are completed. In that case one has the order relationships between the smaller random variables, between the smaller ones and larger ones, but not between the larger ones.

The examples above will not all be amenable to the present treatment. The censoring schemes that we will handle are those that depend on rules telling which order statistics of the combined sample to observe and not which values

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of the random variable. In the case of the instrument incapable of measuring small values the censoring scheme depends on the values of the random variables and therefore is not based on order statistics. That example could, however, be modified in the following manner. Wait until the first p per cent of the articles are passed (screened so that it is known that they are smaller than the remaining ones) and then measure the remainder. If p is chosen carefully there will (with high probability) be little difficulty in making the measurements. Actually, measurement would not be necessary since only the order relationships between the larger observations are required. In life testing where one waits for a fixed number of failures the censoring scheme is already the desired form.

To describe a censoring scheme more precisely we introduce the following notation. Let $z_i = 0(1)$ if the i-th smallest in the combined sample of x's and y's is from the first (second) sample. Then $z = (z_1, \cdots, z_{m+n})$ is a (uncensored) rank order. If 0 corresponds to a unit move to the right and 1 to a unit move up then each rank order can be represented by a path of horizontal and vertical unit movements on the integer lattice from (0, 0) to (m, n). Censoring schemes amenable to the present treatment can be described in terms of this lattice, e.g., the censoring scheme which continues experimentation until one of the lattice points whose coordinates add to N^* is reached is the censoring scheme which tells one to continue until the N^* smallest random variables of the combined sample have been observed.

In this paper we consider explicitly the following type of censoring scheme: Let S be a set of lattice points such that every path from (0, 0) to (m, n) has at least one point in common with S, and S does not include (0, 0). Start experimentation by observing the smaller random variables in the combined sample first and continue experimentation until a point in S is reached.

Thus, for censoring schemes explicitly considered we observe the "smaller" random variables only. The precise meaning of "smaller" depends on the particular censoring scheme. Therefore, the observed rank order is of the form $z = (z_1, \cdots, z_{N^*})$ where $z_i = 0(1)$ if the ith smallest random variable comes from the first (second) sample. Depending on the nature of the censoring scheme N^* can also be the observed value of a random variable. Note that in writing the vector z it is not necessary to know the sample sizes m and n. When computing $\Pr(Z = z)$, however, the values of m and n will always be required. The sample sizes will appear explicitly in the various formulas and be implicit in the discussion.

The following notation and assumptions are used: The random variables, $X_1, \cdots, X_m, Y_1, \cdots, Y_n$, are mutually independent. The X's (Y's) have a common continuous cumulative distribution function $F(x)[G(x)]$. The corresponding density (assumed to exist) will be denoted by $f(x)[g(x)]$.

2. Censoring Schemes. If all $m + n = N$ of the random variables are observed there are $\binom{N}{n}$ possible rank orders. It is of some interest to find the total number of rank orders in censoring schemes of the kind to be considered, i.e., the "smaller" random variables are observed. A rank order, z, is said to be "redundant" if there exists an antecedent rank order z' such that the occurrence of z' implies the occurrence of z. Thus if $m = 2$ and $n = 2$, then $z = (0101)$ is redundant since z will occur whenever $z' = (010)$ occurs. With any scheme where only order is to be used, sampling can be stopped if only redundant rank orders remain to be observed.

LEMMA 2.1.

(1) *The number of non-redundant rank orders for all censoring schemes involving the observation of the smaller random variables is* $2\binom{N}{n} - 2$.

(2) *Including redundant rank orders there are* $\binom{N+2}{n+1} - 2$ *possible rank orders.*

PROOF. The number of non-redundant rank orders consists of two parts.

(a) Those rank orders consisting of $a \leqq m - 1 (b \leqq n - 1)$ observations from the $F(x)[G(x)]$ population. The number of these is

$$\sum_{a=0}^{m-1} \sum_{b=0}^{n-1} \binom{a+b}{a} = \binom{N}{n} - 2.$$

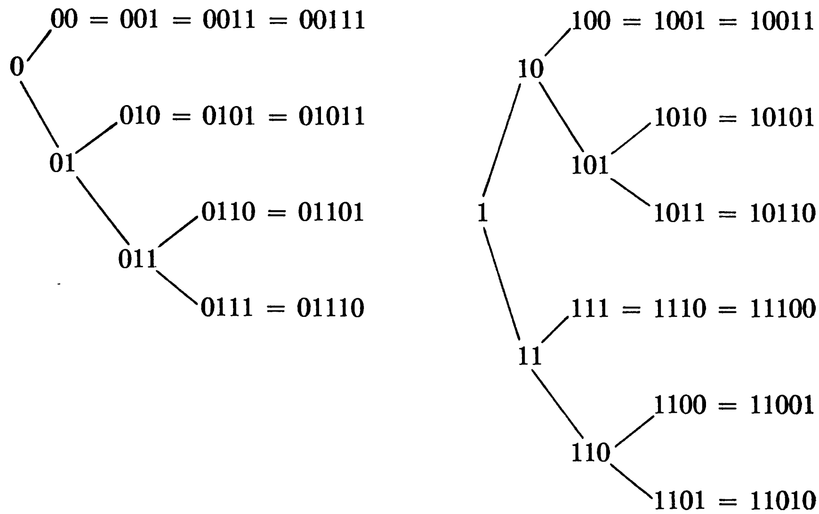
(In the summation exclude $a = b = 0$.)

(b) Those rank orders where the number of observations from $F(x)[G(x)]$ is $m(n)$. When the number of observations from $F(x)$ is m the rank order must end in 0 and have less than n from $G(x)$. Hence the total number of rank orders of this form is

$$\sum_{b=0}^{n-1} \binom{m-1+b}{m-1} + \sum_{a=0}^{m-1} \binom{n-1+a}{n-1}$$
$$= \binom{m+n-1}{m} + \binom{m+n-1}{n} = \binom{m+n}{n}.$$

The conclusion then follows by adding the results from parts (a) and (b). Part (2) of the lemma is proved in a similar manner.

As soon as one considers rank orders with differing values of N^* it is important to note that the rank orders need not correspond to disjoint events. Hence, the sum of the probabilities of all rank orders will be greater than one. A redundant rank order and its antecedent correspond to the same event. If z is a non-redundant rank order and $z_0(z_1)$ are formed from z by observing an additional random variable from $F(x)[G(x)]$ then the event z is the union of the events z_0 and z_1. By repeated use of the preceding one can compute the probabilities of all of the rank orders if all of the probabilities of rank orders with $N^* = N$ have been computed. To illustrate, the possible rank orders are listed for the case $m = 2$ and $n = 3$. The redundant rank orders are set equal to their antecedents. There are 33 rank orders, 18 of which are non-redundant.



Some specific censoring schemes follow:

(a) Continue experimentation until the N^* smallest random variables are observed, $0 < N^* \leqq N$. N^* is not a random variable.

LEMMA 2.2. *The number of possible rank orders under scheme* (*a*), *including redundant ones, is*

$$\sum_{i=\text{Max}(0,N^*-n)}^{\min(m,N^*)} \binom{N^*}{i},$$

so that when $N^* \leqq \min(n, m)$ *the number of rank orders is* 2^{N^*}.

(b) Continue experimentation until m^* random variables from $F(x)$ have been observed. If n^* is the number of random variables observed from $G(x)$ then n^* and $N^* = m^* + n^*$ are random variables.

LEMMA 2.3. *The number of non-redundant rank orders, when experimentation is continued until* m^* *random variables from* $F(x)$ *or* n *from* $G(x)$ *are observed, is* $\binom{m^* + n}{n}$.

PROOF. Some of the rank orders end with an observation from $F(x)$. The number of these is

$$\sum_{i=0}^{n-1} \binom{m^* - 1 + i}{m^* - 1} = \binom{m^* + n - 1}{n - 1}.$$

In addition, there are those rank orders where n observations are obtained from $G(x)$ before m^* are obtained from $F(x)$. The number of these is

$$\sum_{i=0}^{m^*-1} \binom{n - 1 + i}{n - 1} = \binom{m^* + n - 1}{n}.$$

LEMMA 2.4. *If observations are continued until* m^* *of the random variables from* *the first sample are observed, i.e., the possibility of observing redundant rank orders is* not *excluded, then*

$$\Pr(N^* = t) = \begin{cases} 0, & t < m^* \quad \text{or} \quad t > m^* + n \\ m^* \binom{n}{t - m^*} \binom{m}{m^*} \int_{-\infty}^{\infty} F^{m^*-1} G^{t-m^*} f (1 - F)^{m-m^*} (1 - G)^{n-t+m^*} \, dx, & m^* \leqq t \leqq m^* + n, \end{cases}$$

and if $F(x) \equiv G(x)$ *then*

$$\Pr(N^* = t) = \begin{cases} 0, & t < m^* \text{ or } t > m^* + n \\ \dfrac{\binom{t-1}{m^*-1}\binom{N-t}{m-m^*}}{\binom{N}{m}}, & m^* \leqq t \leqq m^* + n, \end{cases}$$

and

$$E(N^*) = m^*(N + 1)/(m + 1) < \mathrm{N}.$$

Proof. In the first part of the lemma the integrand is the probability of the desired event when the m^*-smallest random variable from the first sample occurs in the interval $(x, x + dx)$. The integration then gives the total probability. The second part of the lemma follows from the first by noting that when $F(x) \equiv G(x)$ the integral is a Beta integral. The second part could also be obtained by a direct combinatorial argument.

Lemma 2.5. *When* $F(x) \equiv G(x)$, *if observations are continued until either* m^* *of the random variables of the first sample or the* n *random variables of the second sample are obtained, i.e., redundant rank orders are not observed, then*

$$\Pr(N^* = t) = \begin{cases} 0, & t < \min(m^*, n) \text{ or } t \geqq m^* + n \\ \dfrac{\binom{t-1}{m^*-1}\binom{N-t}{m-m^*} + \binom{t-1}{n-1}}{\binom{N}{n}}, & \min(m^*, n) \leqq t < m^* + n. \end{cases}$$

Proof. The proof is combinatorial. When $F(x) \equiv G(x)$ all of the rank orders with $N^* = N$ are equally likely. The denominator gives the number of rank orders with $N^* = N$. The first term in the numerator gives the number of rank orders ending with the m^*-smallest random variable from the first population. The second term in the numerator gives the number of rank orders ending with the nth random variable from the second population.

(c) Continue experimentation until either the number of random variables from $F(x)$ is m^* or the number from $G(x)$ is n^*, where m^* and n^* are fixed integers.

(d) Continue experimentation until $\max_{-\infty < x \leqq x_{N^*}} [F_{m^*}(x) - G_{n^*}(x)]^2 \geqq a_{m^*,n^*}$, where $F_{m^*}(x)$ and $G_{n^*}(x)$ are the observed cumulative distribution functions based on the first m^* random variable from $F(x)$ and n^* random variables from $G(x)$, and the a_{m^*,n^*} are preassigned numbers with $a_{m,n} = 0$. (See reference [4].)

(e) Continue experimentation until $[m^* - n^*]^2 \geqq b_{m^*,n^*}$, where the b_{m^*,n^*} are preassigned numbers and $b_{m,n} = 0$.

(f) Continue experimentation until the sum of the ranks squared of the observations from $G(x)$ exceeds c_{m^*,n^*} and $c_{m,n} = 0$.

3. Theory. In this section we give formulas for the probabilities of rank orders arising under censoring. General resultsfor $F(x) \not\equiv G(x)$, special cases of $F(x) \not\equiv G(x)$, and $F(x) \equiv G(x)$ are considered. Likelihood ratios are defined and limiting values are computed for the probabilities of rank orders under censoring. Theorem 3.2 gives partial orderings of the likelihood ratios of the rank orders.

In the following we explicitly consider those rank orders involving observations on the "smaller" values of the random variables from the combined sample. When a result is given for a specific rank order or set of rank orders it is presumed that under the censoring scheme being considered these rank orders can occur. If the rank orders cannot occur for a particular scheme then the relationship would not be of interest.

The basic formula is given by

THEOREM 3.1. *Pr* $(Z = z) = \Pr((Z_1, \cdots, Z_{N^*}) = (z_1, \cdots, z_{N^*})) = [(m!n!)/((m - m^*)!(n - n^*)!)] \int W(w)\, dw$ *where*

$$\int W(w)\, dw = \int_{-\infty < w_1 < \cdots < w_{N^*} < \infty} \cdots \int \prod_{i=1}^{N^*} [f^{1-z_i}(w_i) g^{z_i}(w_i)\, dw_i] \cdot [1 - F(w_{N^*})]^{m-m^*} [1 - G(w_{N^*})]^{n-n^*},$$

and $n^* = \sum_{i=1}^{N^*} z_i$, $m^* = N^* - n^*$.

PROOF. The integrand together with outside constants is composed of the product of two multinomial probabilities—the probability that one random variable from the first (second) sample occurs in each of the intervals $(w_i, w_i + dx_i)$ where $z_i = 0(1)$ and $m - m^*(n - n^*)$ random variables from the first (second) sample occur in the interval (w_{N^*}, ∞) The integration then gives the total probability.

COROLLARY 3.1. *When* $F(x) \equiv G(x)$ *then*

$$P_0(z) = \Pr(Z = z) = \binom{N - N^*}{m - m^*} \Big/ \binom{N}{m}.$$

PROOF. Make the change of variables $F(w_i) = G(w_i) = u_i$ in the conclusion to Theorem 3.1. The integral then becomes

$$P_0(z) = \frac{m!n!}{(m - m^*)!(n - n^*)!} \int_{0 < u_1 < \cdots < u_{N^*} < 1} \cdots \int (1 - u_{N^*})^{N-N^*} \prod_{i=1}^{N^*} du_i$$

$$= \frac{m!n!}{(m - m^*)!(n - n^*)!(N^* - 1)!} \int_0^1 u^{N^*-1}(1 - u)^{N-N^*} du.$$

The last is a Beta integral.

Theorem 3.1 and Corollary 3.1 may be useful for summing finite series. We have $\sum \Pr (Z = z) = 1$ when the summation is over all possible rank orders that will terminate experimentation for a particular censoring scheme. Thus in the case of the corollary we have $\sum \binom{N - N}{m - m^*} = \binom{N}{m}$ with the same region of summation. Consider a special case: Stop experimentation on the N^*th observation, where $N^* \leqq \min (m, n)$. Now when experimentation stops there will have been observed i random variables from the first sample. And if i random variables from the first sample have been observed there can be formed $\binom{N^*}{i}$ rank orders. Thus the summation becomes

$$\sum_{i=0}^{N^*} \binom{N^*}{i}\binom{N - N^*}{m - i} = \binom{N}{m}.$$

Of primary concern in finding good tests (decision procedures) is the likelihood ratio, the probability of a rank order when $F(x) \not\equiv G(x)$ divided by the probability of the same rank order when $F(x) \equiv G(x)$. Denote this ratio by $L(z, F, G)$ or $L(z)$.

COROLLARY 3.2.

$$L(z) = L(z, F, G) = \Pr (Z = z)/P_0(z) = \frac{N!}{(N - N^*)!} \int W(w)\, dw.$$

In general good rank order test procedures of the hypothesis that the samples come from the same population against the alternative that the first sample comes from $F(x)$ and the second sample comes from $G(x)$ will be based on large values of $L(z, F, G)$, i.e., rank orders which make $L(z)$ large form the critical region.

When $F(x) = H(x, 0)$ and $G(x) = H(x, \theta)$ one can write $L(z, F, G)$ as $L(z, H, \theta)$ or $L(z, \theta)$. $H(x, \theta)$ is a cumulative distribution function with parameter θ, and $h(x, \theta)$ is the density of $H(x, \theta)$. In this case locally most powerful rank order procedures can be formed for small values of θ.

Assume that θ is real valued and that $dL(z, \theta)/d\theta = L'(z, \theta)$ exists in the neighborhood of $\theta = 0$ and denote $L'(z, 0)$ by $L'(z)$. Then

$$L(z, \theta) = L(z, 0) + \theta L'(z) + o(\theta),$$

but $L(z, 0) = 1$ so that

$$L(z, \theta) = 1 + \theta L'(z) + o(\theta).$$

Thus if the alternative is that $\theta > 0$ but near 0 the locally most powerful test will put those z's into the critical region which make $L'(z)$ largest.

COROLLARY 3.3. *If* $h(x,\theta) = (2\pi)^{-\frac{1}{2}}e^{-(x-\theta)^2/2}$, *i.e., the alternative hypothesis is that the samples come from normal distributions with the same variance, then*

$$L'(z) = \sum_{i=1}^{N^*} z_i E_{Ni} + N(n-n^*)\binom{N-1}{N^*-1} \cdot \int_{-\infty}^{\infty} H^{N^*-1}(x,0)h^2(x,0)[1-H(x,0)]^{N-N^*-1}\,dx,$$

where E_{Ni} *is the expected value of the* ith *smallest in a sample of* N *from a normal distribution with the mean zero and variance one.*

PROOF. $L'(z,\theta) = (N!/(N-N^*)!)\int W(w)\,dw[(n-n^*)g(w_{N^*})/[1-G(w_{N^*})] + \sum_{i=1}^{N^*} z_i(w_i-\theta)]$.
Hence

$$L'(z) = \frac{N!}{(N-N^*)!}\int_{-\infty<w_1<\cdots<w_{N^*}<\infty}\cdots\int \left[\prod_{i=1}^{N^*} h(w_i,0)\,dw_i\right][1-H(w_{N^*},0)]^{N-N^*}$$
$$\cdot\left\{(n-n^*)h(w_{N^*},0)/[1-H(w_{N^*},0)] + \sum_{i=1}^{N^*} z_i w_i\right\} = \frac{N!}{(N-N^*)!}$$
$$\cdot\left\{\sum_{i=1}^{N^*} z_i \frac{(N-N^*)!}{(i-1)!(N-i)!}\int_{-\infty}^{\infty} wH^{i-1}(w,0)h(w,0)[1-H(w,0)]^{N-i}\,dw + \frac{(n-n^*)}{(N^*-1)!}\int_{-\infty}^{\infty} H^{N^*-1}(w,0)h^2(w,0)[1-H(w,0)]^{N-N^*-1}\,dw\right\}.$$

The portion of the statistic depending on the E_{Ni} is the same as one proposed by Fisher and Yates for the uncensored case. The integral in the second part of the statistic has not been tabulated. When $N^* = N$ the statistic in the corollary becomes the Fisher-Yates statistic.

COROLLARY 3.4. *If* $H(x,\theta) = 1 - [1-J(x)]^{1+\theta}$, *where* $\theta > -1$ *and* $J(x)$ *is a distribution function having density* $j(x)$, *i.e., the Lehman alternative, then*

$$\text{I.}\quad L(z,\theta) = \frac{N!}{(N-N^*)!}(1+\theta)n^* \Big/ \left[\prod_{i=1}^{N^*}\left(A + i + \theta\sum_{j=0}^{i-1} z_{N^*-j}\right)\right],$$

where $A = m - m^* + (1+\theta)(n-n^*)$, *and*

$$\text{II.}\quad d(\ln L(z,\theta))/d\theta\,|_{\theta=0} = n^* - (n-n^*)\sum_{j=1}^{N^*}(A+j)^{-1} - \sum_{j=0}^{N^*-1} z_{N^*-j}\left[\sum_{i=j+1}^{N^*}(A+i)^{-1}\right].$$

PROOF.

$$L(z,\theta) = \frac{N!}{(N-N^*)!}(1+\theta)^{n^*}\int_{-\infty<w_1<\cdots<w_{N^*}<\infty}\cdots\int \left\{\prod_{i=1}^{N^*} j(w_i)[1-J(w_i)]^{z_i\theta}\,dw_i\right\} \cdot[1-J(w_{N^*})]^{m-m^*}[1-J(w_{N^*})]^{(1+\theta)(n-n^*)}.$$

This can be integrated exactly by starting with w_{N^*}.

This is similar to Corollary 7.a.1 and Equation 7.c.2 of [2].

COROLLARY 3.5. *If* $H(x, \theta) = (1 - \theta)J(x) + \theta J^2(x)$, *where* $0 \leqq \theta \leqq 1$ *and* $J(x)$ *is a distribution function with density* $j(x)$, *then* $(N + 1)L'(z) = 2\sum_{i=1}^{N^*} iz_i - n^*(N + 1) + (n - n^*)N^*$.

PROOF.

$$L'(z, \theta) = \frac{N!}{(N - N^*)!} \int W(w)\, dw \cdot \left\{ \sum_{i=1}^{N^*} \frac{z_i[2J(w_i) - 1]}{1 - \theta + 2\theta J(w_i)} + \frac{(n - n^*)[J(w_{N^*}) - J^2(w_{N^*})]}{1 - (1 - \theta)J(w_{N^*}) - \theta J^2(w_{N^*})} \right\}$$

and

$$L'(z) = \frac{N!}{(N - N^*)!} \int_{0<w_1<\cdots<w_{N^*}<1} \cdots \int \left[\prod_{i=1}^{N^*} dw_i\right] [1 - w_{N^*}]^{N-N^*} \cdot \left[\sum_{i=1}^{N^*} z_i(2w_i - 1) + (n - n^*)w_{N^*}\right].$$

The necessary integrals are of the Beta form.

When $N^* = N$ this reduces to a result of Lehmann [1]. Statistics of this form have been introduced earlier by Sobel [3].

Now assume that $f(x)$ and $g(x)$ have a *monotone likelihood ratio*, i.e., if $x < y$ then

$$\begin{vmatrix} f(x) & g(x) \\ f(y) & g(y) \end{vmatrix} \geqq 0,$$

with strict inequality for a set of positive Lebesgue measure in the (x, y) space. Two other forms of the same condition are, for $x < y$,

$$f(x)g(y) \geqq f(y)g(x),$$

$$f(x)/g(x) \geqq f(y)/g(y).$$

The monotone likelihood alternatives include many of the common situations, e.g., $f(x)$ a normal density with mean zero and variance one and $g(x)$ a normal density with positive mean and variance one, or $f(x) = e^{-x}$ for $x > 0$ and zero otherwise and $g(x) = (1 + \theta)^{-1}e^{-x/(1+\theta)}$ for $x > 0$ and zero otherwise, where $\theta > 0$.

THEOREM 3.2. *Assume* $X_1, \cdots, X_m, Y_1, \cdots, Y_n$ *are mutually independent random variables. The* X*'s have the density* $f(x)$ *and the* Y*'s have the density* $g(x)$, *where* $f(x)g(y) \geqq f(y)g(x)$ *for* $x < y$ *with strict inequality on a set of positive Lebesgue measure in the* (x, y) *space.*

a. If z and z' are identical except that $z_i = z'_j = 0$ and $z_j = z'_i = 1$ $1 \leqq i < j \leqq N^*$, then $\Pr(z) > \Pr(z')$ and $L(z) > L(z')$.

b. If z and z' are identical except that $z_{N^*} = 0$ and $z'_{N^*} = 1$, and hence $m'^* = m^* - 1$ and $n'^* = n^* + 1$, then $L(z) > L(z')$.

c. If z and z' are identical except that $N'^* = N^* + 1$ and $z'_{N^*+1} = 0(1)$, then $L(z') > L(z)$ $(L(z) > L(z'))$.

PROOF.

a. This is a simple analogue of Theorem 6.1 [2].

b. Let $D = L(z) - L(z')$. Then

$$D = \frac{N!}{(N - N^*)} \int_{-\infty < w_1 \cdots < w_{N^*} < \infty} \cdot \cdot \cdot \int \left[\prod_{i=1}^{N^*} f^{1-z_i}(w_i) g^{z_i}(w_i) \, dw_i \right]$$
$$\cdot [1 - F(w_{N^*})]^{m-m^*} [1 - G(w_{N^*})]^{n-n^*-1} \cdot q(w_{N^*}),$$

where

$$q(w_{N^*}) = \{f(w_{N^*})[1 - G(w_{N^*})] - g(w_{N^*})[1 - F(w_{N^*})]\}.$$

To show $D > 0$ it is sufficient to show $q(w_{N^*}) > 0$. Start with $f(x)g(y) - f(y)g(x) \geqq 0$, if $x < y$. Multiply this inequality by dy and integrate from x to ∞ obtaining $\int_x^\infty f(x)g(y)\, dy - \int_x^\infty g(x)f(y)\, dy > 0$ for some value of x. Now replace x by w_{N^*} and the last becomes $q(w_{N^*}) > 0$.

c. Let z'' be identical to z' except z''_{N^*+1}. Then $\Pr(Z = z) = \Pr(Z = z') + \Pr(Z = z'')$, and

$$L(z) = \Pr(Z = z)/P_0(z) = \Pr(Z = z')P_0(z')/P_0(z)$$
$$+ \Pr(Z = z'')P_0(z'')/P_0(z'')\, P_0(z) = L(z') \frac{P_0(z')}{P_0(z)} + L(z'') \frac{P_0(z'')}{P(z)}.$$

Now from Part b one has $L(z') > (L(z'')$ and thus

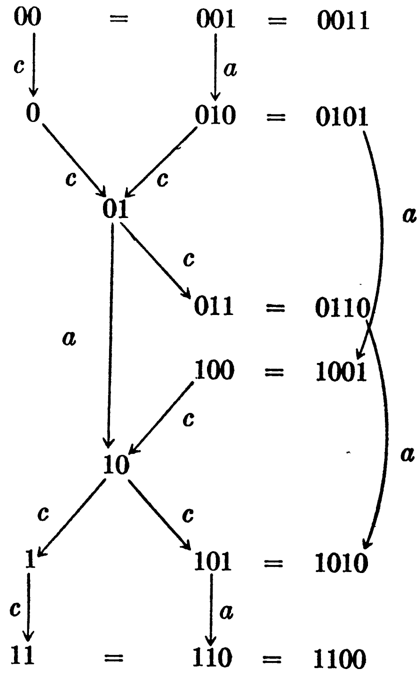
$$L(z) < L(z')[P_0(z') + P_0(z'')]/P_0(z) = L(z')$$

since $P_0(z') + P_0(z'') = P_0(z)$. This completes the proof.

When $m = n = 2$ we obtain diagram B with the aid of Theorem 3.2. An arrow leading from one rank order to the other means that the likelihood ratio of the first is greater than the second. Attached to the arrows are letters indicating the portion of Theorem 3.2 used. Antecedent and redundant rank orders are set equal.

Note that b of Theorem 3.2 is not needed in diagrams like diagram B since typically (m and $n > 3$): $L(0100) > L(0101)$ follows from a double application of c. viz., $L(0100) > L(010) > L(0101)$.

The distributions used in Corollaries 3.3, 3.4, and 3.5 have monotone likelihood ratios. Thus the locally most powerful rank order tests based on those corollaries yield simple orderings of the rank orders which are compatible with the partial orderings of Theorem 3.2. Theorem 3.2 and the resulting diagrams will be found useful in constructing good decision procedures when the monotone likelihood ratio assumption is acceptable and the sample sizes are relatively small [2].



4. Additional problems. Before applying the results of this paper several general as well as specific problems need discussion. Even the restricted class of censoring schemes discussed explicitly is large, and the class of censoring schemes amenable to treatment is very large. Hence, reasons for concentrating on specific schemes should be developed. Some possibilities are: a. Use censoring schemes that are now used, i.e., fix N^* as is done in some life testing problems. b. Use some optimality criterion, such as minimizing the expected number of observations for a fixed level of significance and power (for some alternative). c. Reason by analogy and work with procedures that continue sampling so long as $a < L(Z) < b$ and make the appropriate decision if this condition fails (a and b chosen constants). The large sample distribution theory should be developed. (The locally most powerful rank procedures are in a sense large sample procedures.) Intercomparisons of the "efficiencies" of the procedures being discussed here should be made with other procedures—parametric and nonparametric. Efficiency must include power considerations and cost of experimentation.

For each censoring scheme the distribution of the number of observations required should be investigated under the null and alternative hypotheses. At the least the first two moments should be found. Some of these distributions should be tabulated and the large sample theory developed. Tables of the integral in Theorem 3.1 are desirable. When tables exist of the uncensored rank orders this is an easy task (see paragraph following Lemma 2.1). The exact and large sample distribution of the statistic in Corollary 3.3 should be found for several censoring schemes. In particular, values of the integral need computation. For Corollaries 3.4 and 3.5 it would also be desirable to obtain the exact and large sample distributions. Presumably the results for large samples will not only show limiting normal distributions but give information regarding efficiency. Diagrams resulting from Theorem 3.2 should be prepared for several combinations of sample sizes. When a complete diagram is given it is then possible to select out the portions relevant to a particular censoring scheme. These diagrams should yield uniformly most powerful rank order procedures when the sample sizes and levels of significance are relatively small.

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