A MONOTONICITY PROPERTY OF THE SEQUENTIAL PROBABILITY RATIO TEST[1]

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0. Summary. Using the basic inequalities (1) it is shown that, if, in a sequential probability ratio test, the upper stopping bound is increased and the lower stopping bound decreased, and if the new test is not equivalent to the old one, then at least one of the error probabilities is decreased. This implies the monotonicity result of Weiss [5] in the continuous case, and the uniqueness result of Anderson and Friedman [1] in the general case. The relation of the monotonicity property to the optimum property and the uniqueness of sequential probability ratio tests is discussed.

The monotonicity property is a consequence of the following stronger result. Let the old and new tests be given by the stopping bounds (B', A') and (B, A), respectively, with $B < B' < A' < A$; let (α_1', α_2') and (α_1, α_2) be the error probabilities and $\Delta\alpha_i = \alpha_i - \alpha_i'$ the changes in the error probabilities; then the vector $(\Delta\alpha_1, \Delta\alpha_2)$ is restricted to a cone consisting of the 3rd quadrant, plus the part of the 2nd quadrant where $-\Delta\alpha_2/\Delta\alpha_1 < B$, plus the part of the 4th quadrant where $-\Delta\alpha_2/\Delta\alpha_1 > A$. Another consequence of this result is that (α_1, α_2) cannot lie in the closed triangle with vertices (α_1', α_2'), $(0, 1)$ and $(1, 0)$. Finally, the following monotonicity property follows: If the lower stopping bound is fixed and the upper stopping bound increased, then $\alpha_1/(1 - \alpha_2)$ decreases monotonically. The same holds for $\alpha_2/(1 - \alpha_1)$ if the upper stopping bound is held fixed and the lower stopping bound decreased.

1. Introduction and discussion. We consider Wald's sequential probability ratio test [3] with upper stopping bound A and lower stopping bound B. It is usually assumed that $B < 1 < A$, but no such restriction will be made in this paper. Weiss [5] has shown, under certain continuity assumptions, that, if A and B are separated in such a way that one of the error probabilities remains constant, then the other error probability decreases monotonically. This is a very useful result, since it not only provides a uniqueness proof, but also it shows that there exists a test of given strength if and only if the error probability vector lies in a certain set [6]. In this paper a monotonicity property will be proved which makes no assumptions as regards to the probability distributions (other than that they be non-degenerate) and which include Weiss' result as a special case. The monotonicity property, stated and proved in Section 2, can be described as follows: if the upper stopping bound of a sequential probability ratio test is increased and the lower stopping bound decreased, then at least one of the error probabilities decreases, unless the new test is equivalent to the old one, in

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which case the error probabilities are, of course, unchanged. (Two tests will be called *equivalent*, more or less following [1], if their sample sequences differ on a set of probability 0 under both distributions.) Weiss' result is obtained as a particular case by specifying the distributions to be continuous, with positive probabilities in non-degenerate intervals, and by reading the conclusion: then if one of the error probabilities is fixed, the other decreases.

Before proving the indicated monotonicity property, its relation to the uniqueness and to the optimum property [4] of sequential probability ratio tests will be discussed. In [1] it is shown how the optimum property can be used to prove uniqueness, i.e. the fact that two sequential probability ratio tests with the same error probabilities are equivalent. The restriction $B < 1 < A$ had to be made, though, since the optimum property had been proved only under this condition. Actually, this restriction is unnecessary. It will be indicated in a future paper [2] that any sequential probability ratio test has the optimum property among all tests which take at least one observation. In particular, then, every sequential probability ratio test has the optimum property among all sequential probability ratio tests, which is all that is needed in the uniqueness proof in [1]. This kind of optimum property will be labeled *restricted* in the following.

First of all it will be shown now that the restricted optimum property and the monotonicity property are equivalent. The following notation and terminology will be used: the error probabilities corresponding to the two distributions under consideration are denoted by α_i, $i = 1, 2$; the expected sample sizes are ν_i; in passing from one test to another, $\Delta\alpha_i$ and $\Delta\nu_i$ denote the changes in the α_i and ν_i; a test will be called *inadmissible* if there exists another test such that $\Delta\alpha_i \leqq 0$, $\Delta\nu_i \leqq 0$, $i = 1, 2$, with strict inequality in at least one of the four. Obviously, the optimum property implies admissibility, and the restricted optimum property implies restricted admissibility, i.e. admissibility within the class of sequential probability ratio tests. Consider a sequential probability ratio test (B, A) and another, (B^*, A^*), with $B^* \leqq B < A \leqq A^*$. Unless the two tests are equivalent, we have $\Delta\nu_i > 0$ for both i (see Section 2 for support of this statement, and similar ones to follow). Assume the restricted optimum property. This implies restricted admissibility, and this implies that $\Delta\alpha_i < 0$ for at least one i. In other words, one of the α_i has to decrease, which is the monotonicity property. Conversely, assume the monotonicity property, and compare tests (B, A) and (B^*, A^*), which are supposed to be not equivalent and for which $\Delta\alpha_i \leqq 0$ for both i. Then we cannot have $B^* \leqq B$, $A^* \leqq A$, for in that case $\Delta\alpha_1 > 0$ and $\Delta\alpha_2 < 0$. Similarly, $B^* \geqq B$, $A^* \geqq A$ is excluded. Also $B < B^* < A^* < A$ is excluded since otherwise, by the monotonicity property, one of the $\Delta\alpha_i$ would be positive. Hence the only remaining possibility is $B^* < B < A < A^*$, which implies $\Delta\nu_i > 0$ for both i, i.e. the optimum property.

Secondly, the monotonicity property implies uniqueness. For, if the stopping bounds are changed in the same direction, then both error probabilities change (in opposite directions), whereas, if the stopping bounds are changed in opposite directions, then according to the monotonicity property at least one of the error probabilities changes; unless, of course, the two tests are equivalent.

It is true that a separate proof of the monotonicity property is not strictly necessary, since this property is a consequence of the optimum property. However, the optimum property is a rather deep theorem, requiring a sizable machinery for its proof, whereas the monotonicity property follows in an elementary way from the basic inequalities (1). Since two interesting properties of sequential probability ratio tests within their own class—the uniqueness and the restricted optimum property—are immediate consequences of the monotonicity property, it seems worth-while to prove the latter independently. Moreover, the methods used yield a stronger result, which does not follow from the optimum property and which has, besides the monotonicity property, some other interesting consequences. These further results are obtained in Section 3.

2. Statement and proof of the monotonicity property. Let $X_1, X_2, \cdots$ be a sequence of independent and identically distributed random variables (or vectors) with common density p_i with respect to some sigma-finite measure. Here and in the following, i runs over 1 and 2, corresponding to the two hypotheses under consideration. The trivial case $p_1 = p_2$ a.e. will be excluded. Let Y_n be the probability ratio at the nth observation, i.e. $Y_n = \prod_{j=1}^{n} p_2(X_j)/p_1(X_j)$. If some stopping rule is defined, let N be the random number of observations. Of fundamental importance in what follows is the basic double inequality

$$aP_1(a < Y_N < b) \leqq P_2(a < Y_N < b) \leqq bP_1(a < Y_N < b) \tag{1}$$

for any real numbers a and b, including ∞. The strict inequality signs within the parentheses in (1) may be replaced by less-or-equal signs, and we will do so whenever this is convenient. For instance, the following inequalities will be considered special cases of (1):

$$P_2(Y_N \geqq a) \geqq aP_1(Y_N \geqq a) \tag{2}$$

$$P_2(Y_N \leqq b) \leqq bP_1(Y_N \leqq b). \tag{3}$$

These basic inequalities have been used already by Wald ([3], Section 3.2) and are briefly discussed there. Also Weiss [5] makes use of (3). An important consequence of (1) is that either

$$P_1(a < Y_N < b) = P_2(a < Y_N < b) = 0 \tag{4}$$

or

$$P_1(a < Y_N < b) > 0 \quad \text{and} \quad P_2(a < Y_N < b) > 0. \tag{5}$$

As an application, compare the sequential probability ratio tests (B, A) and (B, A^*), with $B < A < A^*$. In (4) and (5) identify a with A, b with A^*, and N with the random number of observations if test (B, A) is used. If (4) prevails, the two tests are clearly equivalent. If (5) prevails we can conclude $\Delta\alpha_1 < 0$, $\Delta\alpha_2 > 0$, $\Delta\nu_i > 0$ for both i. Similar conclusions can be drawn if both stopping bounds are changed, and these facts have already been used in the discussion in Section 1.

In a sequential probability ratio test with stopping bounds s and $t(s < t)$[2] and random number of observations N, the error probabilities α_i are functions of s and t,

$$\alpha_1(s, t) = P_1(Y_N \geqq t) = 1 - P_1(Y_N \leqq s), \tag{6}$$

$$\alpha_2(s, t) = P_2(Y_N \leqq s) = 1 - P_2(Y_N \geqq t). \tag{7}$$

It is convenient to introduce the functions U_i and V_i defined by

$$U_i(s, t) = P_i(Y_N \leqq s) \tag{8}$$

$$V_i(s, t) = P_i(Y_N \geqq t) \tag{9}$$

if s and t are the stopping bounds. We have

$$U_i(s, t) + V_i(s, t) = 1, \tag{10}$$

and the relation between the α_i, U_i and V_i is simply

$$\alpha_1(s, t) = V_1(s, t) \qquad \alpha_2(s, t) = U_2(s, t). \tag{11}$$

THEOREM 1. *Let* (u, v) *and* (u', v') *define two non-equivalent sequential probability ratio tests, with* $0 < u \leqq u' < v' \leqq v < \infty$, *and let* $\Delta\alpha_i = \alpha_i(u, v) - \alpha_i(u', v')$, $i = 1, 2$. *Then at least one of the* $\Delta\alpha_i$ *must be* <0.

PROOF. Let N be the random number of observations in the test (u', v'), and define $F_i(y) = P_i(Y_N \leqq y)$. Then, using (11), we have

$$\Delta\alpha_1 = V_1(u, v) - V_1(u', v') \tag{12}$$

$$\Delta\alpha_2 = U_2(u, v) - U_2(u', v'). \tag{13}$$

We compute[3]

$$V_1(u', v') = \int_v^\infty dF_1(y) + \int_{v'}^v dF_1(y) \tag{14}$$

$$V_1(u, v) = \int_v^\infty dF_1(y) + \int_{v'}^v V_1\left(\frac{u}{y}, \frac{v}{y}\right) dF_1(y) + \int_u^{u'} V_1\left(\frac{u}{y}, \frac{v}{y}\right) dF_1(y). \tag{15}$$

In (15) we used the fact that the Y_{n+1}/Y_n are independent and identically distributed. Substitution into (12) and using (10) gives

$$\Delta\alpha_1 = \int_u^{u'} V_1\left(\frac{u}{y}, \frac{v}{y}\right) dF_1(y) - \int_{v'}^v U_1\left(\frac{u}{y}, \frac{v}{y}\right) dF_1(y). \tag{16}$$

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[2] For notational convenience we shall henceforth use lower case symbols instead of A and B for the stopping bounds.

[3] In (14) the lower limits on the integrals should, and the upper limits should not be included in the integrations. On the other hand, in (15) in the third integral on the right the lower limit u should not be included and the upper limit u' should. These facts have not been made explicit in the formulas, since they are inessential for the proof.

Similarly,

$$\Delta\alpha_2 = \int_{v'}^{v} U_2\left(\frac{u}{y}, \frac{v}{y}\right) dF_2(y) - \int_{u}^{u'} V_2\left(\frac{u}{y}, \frac{v}{y}\right) dF_2(y). \tag{17}$$

Suppose temporarily that $v' < v$. Then for y in the interval $[v', v)$ we have, using (8) and (3),

$$U_2\left(\frac{u}{y}, \frac{v}{y}\right) \leqq \frac{u}{y} U_1\left(\frac{u}{y}, \frac{v}{y}\right) \leqq \frac{u}{v'} U_1\left(\frac{u}{y}, \frac{v}{y}\right), \tag{18}$$

and, using (1),

$$dF_2(y) \leqq v\, dF_1(y) \tag{19}$$

so that

$$\int_{v'}^{v} U_2\left(\frac{u}{y}, \frac{v}{y}\right) dF_2(y) \leqq \frac{uv}{v'} \int_{v'}^{v} U_1\left(\frac{u}{y}, \frac{v}{y}\right) dF_1(y). \tag{20}$$

In (19), and therefore in (20), there is strict inequality unless

$$\int_{v'}^{v} dF_i(y) = 0 \qquad \text{for both } i. \tag{21}$$

If $v' = v$, (20) remains trivially true. Similarly,

$$\int_{u}^{u'} V_2\left(\frac{u}{y}, \frac{v}{y}\right) dF_2(y) \geqq \frac{uv}{u'} \int_{u}^{u'} V_1\left(\frac{u}{y}, \frac{v}{y}\right) dF_1(y), \tag{22}$$

and again this inequality is strict unless

$$\int_{u}^{u'} dF_i(y) = 0 \qquad \text{for both } i. \tag{23}$$

The tests are equivalent if and only if both (21) and (23) hold. Therefore, if the tests are not equivalent, then at least one of the inequalities in (20) and (22) is strict. Using (16), (17), (20) and (22), it is now easy to verify the following two inequalities:

$$uv\Delta\alpha_1 + v'\Delta\alpha_2 < 0 \tag{24}$$

$$uv\Delta\alpha_1 + u'\Delta\alpha_2 < 0. \tag{25}$$

The conclusion of Theorem 1 is, of course, an immediate consequence of either of the inequalities (24) and (25).

3. Strengthening of the result. Let $\Delta\alpha$ be the 2-vector whose components $\Delta\alpha_i$ are defined in Theorem 1. If the two tests are equivalent, then, of course, $\Delta\alpha = 0$. Otherwise, the conclusion of Theorem 1 states that $\Delta\alpha$ cannot lie in the set defined by $\Delta\alpha_i \geqq 0$ for both i, i.e. the (closed) 1st quadrant. In other words, $\Delta\alpha$ has to lie in the 2nd, 3rd or 4th quadrant. However, the inequalities (24) and (25) already claim something more: $\Delta\alpha$ is not only excluded from the 1st quadrant but also from the part of the 2nd quadrant where $-\Delta\alpha_2/\Delta\alpha_1 \geqq uv/v'$ (using (24)) and from the part of the 4th quadrant where $-\Delta\alpha_2/\Delta\alpha_1 \leqq uv/u'$ (using (25)). What remains is a cone of angle $<\pi$. We shall show now that we can sharpen the bounds uv/v' and uv/u' for $-\Delta\alpha_2/\Delta\alpha_1$ to u and v, respectively. This will be the content of

THEOREM 2. *Under the same conditions as in Theorem* 1 *we have*

$$u\Delta\alpha_1 + \Delta\alpha_2 < 0 \tag{26}$$

$$v\Delta\alpha_1 + \Delta\alpha_2 < 0. \tag{27}$$

Before proving Theorem 2 we will indicate some of its consequences. Consider u', v' fixed and u, v varying, subject to $u < u' < v' < v$. Consider all possible $\Delta\alpha$. The cone given by (26) and (27) to which $\Delta\alpha$ is restricted depends on u and v. To obtain a fixed cone we remark that $-\Delta\alpha_2/\Delta\alpha_1 < u$ implies $-\Delta\alpha_2/\Delta\alpha_1 < u'$ and $-\Delta\alpha_2/\Delta\alpha_1 > v$ implies $-\Delta\alpha_2/\Delta\alpha_1 > v'$. Therefore, (26) and (27) imply

$$u'\Delta\alpha_1 + \Delta\alpha_2 < 0 \tag{28}$$

$$v'\Delta\alpha_1 + \Delta\alpha_2 < 0. \tag{29}$$

The inequalities (28) and (29) are less sharp than (26) and (27), but they do represent a fixed cone within which $\Delta\alpha$ is restricted as u and v vary. In fact, this cone is the union of all cones given by (26) and (27) as u and v vary.

We can also consider the $\alpha_1 - \alpha_2$ plane and see what happens to the vector of error probabilities as u', v' is fixed and u, v vary. The only portion of the plane which needs to be considered is the triangle $\alpha_i \geqq 0$, $\alpha_1 + \alpha_2 \leqq 1$. Let $\alpha_i = \alpha_i(u, v)$ $\alpha_i' = \alpha_i(u', v')$, and let $\alpha = (\alpha_1, \alpha_2)$, $\alpha' = (\alpha_1', \alpha_2')$. The inequalities (28) and (29) say that α lies in a cone with vertex α', containing the point (0, 0), and bounded by two lines with slopes $-u'$ and $-v'$. This cone does not contain any point of the triangle with vertices α', (0, 1) and (1, 0). To see this we only have to look at the slopes of the lines connecting α' with (0, 1) and (1, 0). The first is $-(1 - \alpha_2')/\alpha_1'$, the second $-\alpha_2'/(1 - \alpha_1')$. Now, using (2), we have $(1 - \alpha_2')/\alpha_1' \geqq v' > u'$, and, using (3), $\alpha_2'/(1 - \alpha_1') \leqq u' < v'$, which establishes the fact mentioned. Thus, α cannot lie in the closed triangle with vertices α', (0, 1) and (1, 0).

There is another consequence which is of enough interest in itself to state separately. We introduce the quantities

$$\beta_1 = \alpha_1/(1 - \alpha_2) \tag{30}$$

$$\beta_2 = \alpha_2/(1 - \alpha_1) \tag{31}$$

The quantities β_i' are defined in the same manner in terms of the α_i', and $\Delta\beta_i = \beta_i - \beta_i'$. Then β_1 is the tangent of the angle that the line through α and (0, 1) makes with the α_2-axis; β_2 has a similar interpretation. The result of the preceding paragraph, namely that α is excluded from the closed triangle with vertices α', (0, 1) and (1, 0), is then seen to be equivalent to

$$\Delta\alpha_1 < 0 \Rightarrow \Delta\beta_1 < 0 \tag{32}$$

$$\Delta\alpha_2 < 0 \Rightarrow \Delta\beta_2 < 0. \tag{33}$$

This result can also be stated as

COROLLARY 1. *Under the same conditions as in Theorem* 1, *at least one of the* $\Delta\beta_i$ *must be* < 0.

Now $\Delta\alpha_1 < 0$ is in particular satisfied if $u = u'$, and $\Delta\alpha_2 < 0$ if $v = v'$. Using this, and referring to (32) and (33), we have

COROLLARY 2. *Let the* β_i *be defined by* (30) *and* (31). *If the lower stopping bound* u *of a sequential probability ratio test is fixed,* β_1 *is a monotonic non-increasing function of the upper stopping bound* v. *The function is strictly monotonic except in any point* v *for which there is a* $v^* > v$ *such that the tests* (u, v) *and* (u, v^*) *are equivalent. A completely analogous statement for* β_2 *is obtained by fixing* v *and decreasing* u.

Finally, we remark that Theorem 2 can be generalized slightly. So far we have considered only sequential probability ratio tests whose continuation region is an open interval. We can also consider a sequential probability ratio test whose continuation interval contains one or both of its endpoints. In Theorems 1 and 2 we shall then consider tests with continuation intervals I and I', where I has endpoints u, v, and I' has u', v', and such that $I' \subset I$. With this generalization the conclusion (26) and (27) remains valid, except that one of the inequalities may be an equality.

We proceed now with the proof of Theorem 2, which starts with (24) and (25). Notice that for very small changes from u' to u and v' to v we have almost $u/u' = 1$ and $v/v' = 1$ so that then (26) and (27) follow approximately from (24) and (25), respectively. The idea of the proof is to link the tests (u', v') and (u, v) by a chain of intermediate tests, each of which is close to the next one.

PROOF OF THEOREM 2. Consider the chain of tests (u', v'), (u, v') (u, v_1), $\cdots$, (u, v_n) in which $v_n = v$ and $v_1, \cdots, v_{n-1}$ is a sequence to be specified later. Put $(\Delta\alpha_i)_0 = \alpha_i(u, v') - \alpha_i(u', v')$ and $(\Delta\alpha_i)_k = \alpha_i(u, v_k) - \alpha_i(u, v_{k-1})$, $k = 1, \cdots, n$, where we identify v_0 with v'. In passing from (u', v') to (u, v') we have $(\Delta\alpha_1)_0 \geqq 0$, with strict inequality unless (u', v') and (u, v') are equivalent. Consequently, using (25) in the second inequality, $u(\Delta\alpha_1)_0 + (\Delta\alpha_2)_0 \leqq u(v/u')(\Delta\alpha_1)_0 + (\Delta\alpha_2)_0 \leqq 0$, with equality if and only if (u', v') and (u, v') are equivalent. Then there exists $\epsilon_0 > 0$ such that for all ϵ with $0 < \epsilon < \epsilon_0$ we have

$$u(\Delta\alpha_1)_0 + (1 - \epsilon)(\Delta\alpha_2)_0 \leqq 0 \tag{34}$$

with the same remark about equality as before. For fixed ϵ, $0 < \epsilon < \epsilon_0$, choose $v_1, \cdots, v_{n-1}$ in such a way that $1 - \epsilon < v_{k-1}/v_k < 1$, $k = 1, \cdots, n$. In passing from (u, v_{k-1}) to (u, v_k) we have $(\Delta\alpha_2)_k \geqq 0$ so that $u(\Delta\alpha_1)_k + (1 - \epsilon)(\Delta\alpha_2)_k \leqq u(\Delta\alpha_1)_k + (v_{k-1}/v_k)(\Delta\alpha_2)_k$. The right hand side of the last inequality is $\leqq 0$, by (24), with equality if and only if (u, v_{k-1}) and (u, v_k) are equivalent. We have established now

$$u(\Delta\alpha_1)_k + (1 - \epsilon)(\Delta\alpha_2)_k \leqq 0, \qquad k = 0, 1, \cdots, n \tag{35}$$

(for $k = 0$ this was established as (34)). In (35) there is strict inequality for at least one k, otherwise (u', v') and (u, v) would be equivalent. Adding the inequalities (35) for $k = 0, 1, \cdots, n$ yields

$$u\Delta\alpha_1 + (1 - \epsilon)\Delta\alpha_2 < 0. \tag{36}$$

Letting $\epsilon \to 0$ then gives the desired result (26). Inequality (27) is proved analogously, using a chain of tests (u', v'), (u', v), (u_1, v), $\cdots$, (u_n, v), with $u_n = u$.

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