ON THE ESTIMATION OF THE SPECTRUM OF A STATIONARY STOCHASTIC PROCESS

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1. Introduction. Recently many authors have been interested in the problem of estimating the spectral density function of a weakly stationary process. Under assumptions of linearity of the process and existence of derivatives of the spectral density, U. Grenander and M. Rosenblatt [1] have investigated the asymptotic behaviour of various estimates. E. Parzen [2] has investigated the asymptotic behaviour of different types of errors of the estimates under assumptions of fourth order stationarity and exponential or algebraic decrease of the covariance sequence.

In this paper, the problem of estimating the spectral distribution as well as the spectral density (if it exists) of a weakly stationary process is solved under the sole assumption that the sample covariances converge almost surely and in mean to the true covariances. The relevance of Bochner's work on Fourier analysis [3], in obtaining more exact expressions for the bias of estimates, is pointed out. The existence of estimates which converge uniformly strongly to the spectral density of the process is proved under the assumption that the density has an absolutely convergent Fourier series. It should be added that only questions of consistency are discussed here and, no attempt is made to derive the asymptotic distribution of the estimates.

2. Estimates of the Spectral Distribution Function.

Definitions: We suppose that $x_1, x_2, \cdots x_N$ are observations at N consecutive time points on a discrete weakly stationary stochastic process

$$\{x_t\}(t = \cdots, -1, 0, 1, \cdots),$$

with the well-known spectral representation (cf. [1])

$$(2.1)\quad x_t = \int_{-\pi}^{\pi} e^{it\lambda}\, dZ(\lambda); \qquad Ex_t = 0; \qquad \rho_\nu = \rho_{-\nu} = Ex_t\, x_{t+\nu} = \int_{-\pi}^{\pi} e^{i\nu\lambda}\, dF(\lambda),$$

where $Z(\lambda)$ is an orthogonal stochastic set function (cf. [1]) and $F(\lambda)$ is a monotonic right continuous function in $[-\pi, \pi]$. It is easily seen that

$$(2.2)\qquad \hat{\rho}_\nu = \hat{\rho}_{-\nu} = (x_1 x_{1+|\nu|} + \cdots + x_{N-|\nu|} x_N)/(N - |\nu|)$$

is an unbiased estimate of ρ_ν. We shall consider the following estimate of the spectral distribution:

$$(2.3)\qquad \hat{F}_N(\lambda) = 1/2\pi \sum_{k=-R(N)}^{+R(N)} a_{k,N} \cdot (\hat{\rho}_k/ik) e^{ik\lambda},$$

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[1] $\rightarrow$ implies the usual weak convergence of distributions.

where the term corresponding to $k = 0$ is $a_{0,N}(\lambda + \pi)$, and the $a_{k,N}$ are constants chosen such that the following conditions are satisfied:

1) $a_{k,N} \to 1$ as $N \to \infty$ for each fixed k,

2) $a_{k,N} = a_{-k,N}$,

3) $\hat{F}_N(\lambda)$ is a distribution function.

As is known from previous work [1], [2], it is advantageous to choose $R(N) = o(N)$. We shall now state, without proof, a theorem concerning the convergence of the estimates $\hat{F}_N(\lambda)$.

THEOREM 2.1: *If $\{x_t\}$ is a weakly stationary process with a spectral distribution $F(\lambda)$, and the sample covariances converge almost surely to the true convariances, then*[1] $P[\hat{F}_N \to F] = 1$. *If, however, $F(\lambda)$ is continuous, then*

$$P[\sup_{|\lambda| \leq 2\pi} | \hat{F}_N(\lambda) - F(\lambda) | \to 0 \text{ as } N \to \infty] = 1.$$

If, further, the sample covariances converge in mean to the true covariances, then

$$\lim_{N\to\infty} E \sup_{|\lambda| \leq 2\pi} | \hat{F}_N(\lambda) - F(\lambda) | = 0.$$

The first part of the theorem is contained in Doob [4]; the second part follows by an application of a theorem of Pólya to the effect that the weak convergence of a sequence of distributions to a continuous distribution implies uniform convergence; the last part follows from an easy computation.

The choice of the constants $a_{k,N}$: Our main object is to make a suitable choice of the constants $a_{k,N}$, and to examine the order of the bias, convergence, etc., of the estimates thus obtained. The method we use for this purpose is simply a Fourier analysis. It is based almost entirely on the work of Bochner [3]. We now state the main result of Bochner, in the form required here.

Let $f(x)$ be a continuous periodic function with period 2π and let

$$K(t) = \left(\frac{\sin t/2}{t/2}\right)^2, \qquad M(r) = \int_{-\infty}^{+\infty} \left(\frac{\sin t/2}{t/2}\right)^{2r} dt,$$

$$K_r(t) = \frac{[K(t)]^r}{M(r)},$$

$$S_R^r(x) = \int_{-\infty}^{+\infty} f(x + t)\, R/r\, K_r(Rt/r)\, dt.$$

THEOREM (BOCHNER): *For any continuous periodic function, $f(x)$,*

$$| S_R^r(x) - f(x) | = O[w(4r/R) + 4^{-r}],$$

where

$$w(x) = \max_{|x_1 - x_2| < x} | f(x_1) - f(x_2) | .$$

Write

$$f_N^*(\lambda) = \frac{1}{2\pi N} \int_{-\infty}^{+\infty} \left| \sum_{t=1}^{N} x_t e^{it(\lambda+u)} \right|^2 \frac{R}{r} K_r \left(\frac{Ru}{r}\right) du \tag{2.4}$$

and

$$F_N^*(\lambda) = \int_{-\pi}^{\lambda} f_N^*(\lambda)\, d\lambda. \tag{2.5}$$

Then it is possible to write $F_N^*(\lambda)$ as given in (2.3) and to show that all the required conditions are satisfied. Thus, by Theorem 2.1, F_N^* is a consistent estimator of F under very mild conditions. We now state a theorem concerning the bias of F_N^* as an estimate of the spectral distribution.

THEOREM 2.2. *For a weakly stationary process, $\{x_t\}$, with a continuous spectral distribution, F, we have*

$$\sup_{|\lambda/<\pi} | EF_N^*(\lambda) - F(\lambda) | = O[w(4r/R) + 4^{-r} + R/Nw(R^{-\frac{1}{2}})],$$

where
$$w(x) = \max_{|\lambda_1-\lambda_2|<x} | G(\lambda_1) - G(\lambda_2) |, \qquad x \leqq 2\pi,$$

and

$$G(\lambda) = F(\lambda) - [(\lambda + \pi)/2\pi]\rho_0 .$$

Since the above is an easy consequence of Bochner's theorem, the proof is omitted.

COROLLARY: *If $F(\lambda)$ satisfies Lipschitz's condition, i.e.*

$$| F(\lambda_1) - F(\lambda_2) | < c | \lambda_1 - \lambda_2 |,$$

where c is a constant, then $w(x) < cx$ for any $x > 0$, and hence

$$\sup | EF_N^*(\lambda) - F(\lambda) | = O[r/R + 4^{-r} + ((R)^{\frac{1}{2}}/N)].$$

Thus, in order to obtain an asymptotically unbiased and consistent estimator of F, we have only to choose r and R such that $r \to \infty$, $R \to \infty$, $r/R \to 0$ and $R/N \to 0$ as $N \to \infty$ in $F_N^*(\lambda)$.

For Gaussian processes the following theorem can be easily proved.

THEOREM 2.3. *For a Gaussian process with a square integrable spectral density we have*

$$E \sup_{|\lambda| \leqq 2\pi} | F_N^*(\lambda) - F(\lambda) | = O[(\log R/(N)^{\frac{1}{2}}) + w(4r/R) + 4^{-r}].$$

3. Convergence of the Spectral Density. In this section we shall discuss the choice of r and R so that the estimate $f_N^*(\lambda)$ given in (2.4) converges (almost surely) uniformly to the spectral density of the process. Our choice will be such that r and R are not only functions of N but of the observations themselves. It should be noted that, even if r and R depend on the observations, Theorem 2.1 remains valid provided that r and R diverge to infinity with probability one.

We require the following

LEMMA 3.1. *For any weakly stationary process x_t, if $\sum_1^N x_t^2/N$ is convergent with probability one as $N \to \infty$, then, for $\epsilon > 0$,*

$$P[\sup_N \sup_{0 \leq k \leq N^{1-\epsilon}} |\hat{\rho}_k| < \infty] = 1,$$

where $\hat{\rho}_k$ is as in (2.2).

PROOF:

$$|\hat{\rho}_k| = (|\sum_{t=1}^{N-k} x_t x_{t+k}|)/(N-k) \leqq 1/(N-k)[(\sum_1^{N-k} x_t^2)(\sum_{k+1}^{N} x_t^2)]^{\frac{1}{2}},$$

so that

$$(3.1) \quad \sup_{0 \leq k \leq N^{1-\epsilon}} |\hat{\rho}_k| \leqq 1/(N - N^{1-\epsilon}) \sum_1^N x_t^2 < [(\sum_1^N x_t^2)/N(1 - 2^{1-\epsilon})]$$

for $N \geqq 2$. Since by assumption $(\sum_1^N x_t^2)/N$ converges, the expression on the right side of 3.1 is bounded with probability one. This completes the proof.

Our estimate of the spectral density function is

$$f_N^*(\lambda) = 1/2\pi N \int_{-\infty}^{+\infty} \left| \sum_{t=1}^N x_t e^{it(\lambda+u)} \right|^2 R/r\, K_r\, (Ru/r)\, du,$$

which can also be rewritten as

$$(3.2) \qquad f_N^*(\lambda) = 1/2\pi \sum_{m=-R}^{+R} \varphi^r(rm/R)(1 - |m|/N)\hat{\rho}_m e^{im\lambda},$$

where

$$\varphi^r(t) = \int e^{itx} K_r(x)\, dx.$$

We now prove the following

THEOREM 3.1. *Let $\{x_t\}$ be a weakly stationary process, with spectral density function $f(\lambda)$ and covariance sequence $\{\rho_k\}$, which has the property that $\sum_{-\infty}^{+\infty} |\rho_k|$ is convergent. Suppose, further, that the sample variance and covariances converge almost surely, and in the L_1 mean, to the true variance and covariances respectively. Then there exist $R(N, x_1, x_2 \cdots x_N)$ and $r(N, x_1, x_2, \cdots x_N)$ such that*

$$\sup_{|\lambda| < \pi} |f_N^*(\lambda) - f(\lambda)| \to 0$$

almost surely as $N \to \infty$.

PROOF:

$$(3.3) \quad \begin{aligned} f_N^*(\lambda) &= 1/2\pi \sum_{-R}^{+R} \varphi^r(rm/R)(1 - (|m|/N))(\hat{\rho}_m - \rho_m)e^{im\lambda} \\ &+ 1/2\pi \sum_{-R}^{+R} \varphi^r(rm/R)(1 - (|m|/N))\rho_m e^{im\lambda} = S_1 + S_2, \quad \text{say.} \end{aligned}$$

For S_1 we have

$$\sup_\lambda | S_1 | \leqq 1/2\pi \sum_{-R}^{+R} | \hat{\rho}_m - \rho_m | \leqq 1/\pi \sum_{0}^{R} | \hat{\rho}_m - \rho_m | \tag{3.4}$$
$$\leqq 1/\pi R^{1+\delta} \sum_{1}^{[N^{1-\epsilon}]} [(| \hat{\rho}_m - \rho_m |)/m^{1+\delta}]$$

if $R < [N^{1-\epsilon}]$, $\epsilon > 0$, $\delta > 0$. Since for each m, $| \hat{\rho}_m - \rho_m | \rightarrow 0$ with probability one, and by Lemma 3.1, $\sum_1^{[N^{1-\epsilon}]} [(| \hat{\rho}_m - \rho_m |)/m^{1-\delta/2}]$ is bounded, we get by Toeplitz's lemma [5] the following:

$$P[\lim_{N\to\infty} \sum_{1}^{[N^{1-\epsilon}]} [(| \hat{\rho}_m - \rho_m |)/m^{1+\delta/2}]\cdot 1/m^{\delta/2} = 0] = 1. \tag{3.5}$$

We choose R such that $R \rightarrow \infty$, with probability one, $R < [N^{1-\epsilon}]$, and

$$R = o\left[\sum_{1}^{[N^{1-\epsilon}]} [(| \hat{\rho}_m - \rho_m |)/m^{1+\delta}] \right]^{-1/(1+\delta)}. \tag{3.6}$$

Then

$$P[\sup_\lambda | S_1 | \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty] = 1.$$

Turning to S_2, we have

$$S_2(\lambda) - f(\lambda) = \int_{-\infty}^{+\infty} [f_N(\lambda + t) - f(\lambda)] R/r\, K_r\, (Rt/r)\, dt, \tag{3.7}$$

where

$$f_N(\lambda) = 1/2\pi \sum_{-N}^{+N} (1 - (| m |/N))\rho_m e^{im\lambda} \tag{3.8}$$

is the Nth Fejer mean of $f(\lambda)$. Since $\sum_{-\infty}^{+\infty} | \rho_k |$ is convergent, $f(\lambda)$ is bounded and continuous. Since $f(\lambda)$ is symmetric in λ, $f(\pi) = f(-\pi)$. Hence, by Fejer's theorem,

$$\lim_{N\to\infty} \sup_{|\lambda| \leqq \pi} | f_N(\lambda) - f(\lambda) | = 0. \tag{3.9}$$

From (3.7) we have

$$\sup_\lambda | S_2(\lambda) - f(\lambda) | \leqq \sup_\lambda \int_{-\infty}^{+\infty} | f_N(\lambda + t) - f(\lambda + t) | \tag{3.10}$$
$$\cdot (R/r) K_r(Rt/r)\, dt + \sup_\lambda \left| \int_{-\infty}^{+\infty} [f(\lambda + t) - f(\lambda)]\, (R/r) K_r(Rt/r)\, dt \right|.$$

Since $\int_{-\infty}^{+\infty} (R/r)K_r(Rt/r)\, dt = 1$, the first term on the right of (3.10) goes to zero as $N \rightarrow \infty$. If we choose r such that $r \rightarrow \infty$ and $(r/R) \rightarrow 0$ as $N \rightarrow \infty$, it is easily seen from Bochner's theorem, that the second term also goes to zero with probability one.

We remark that, if we choose $rR = o(N)$ and $r = o(R)$, the theorems of U. Grenander and M. Rosenblatt [1] on the consistency of the spectral estimates for linear processes become applicable.

Finally, let us consider the behaviour of the periodogram of a stationary Gaussian process. It is well-known that the periodogram does not converge to any random variable as the sample size increases to infinity. However, the following theorem holds.

THEOREM 3.2. *For a stationary Gaussian process with a spectral density $f(\lambda)$ satisfying Lipschitz's condition,*

$$P\left[\limsup_{N\to\infty} \frac{1}{2\pi}\cdot\frac{\left|\sum_{t=1}^{N} x_t \cos t\lambda\right|^2}{2N\log\log N} + \limsup_{N\to\infty}\frac{1}{2\pi}\frac{\left|\sum_{t=1}^{N} x_t \sin t\lambda\right|^2}{2N\log\log N} = f(\lambda)\right] = 1.$$

The proof follows from the analyses of W. Feller [6] and G. Maruyama [7].

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