(b) Suppose, as $t \to \infty$,

$$\int_0^t (1 - F(x)\, dx \sim \frac{A}{\Gamma(2-\alpha)}\, t^{1-\alpha}, \qquad 0 < \alpha \leqq 1, \qquad A > 0,$$

$$\int_0^t (1 - G(x)\, dx \sim \frac{B}{\Gamma(2-\beta)}\, t^{1-\beta}, \qquad 0 < \beta \leqq 1, \qquad B > 0.$$

It can be shown, using the Abelian theorem on p. 182 of [9], that the limit in (1) is $A/(A + B)$ (if $\alpha = \beta$), 1 (if $\alpha > \beta$), and 0 (if $\alpha < \beta$), a result also obtainable from [8].

The limit (1) could be studied from the point of view of Darling and Kac [1]. Possibly, their results would yield conditions on F and G for (1) to hold.

The behavior of $P(t)$ itself, for large t, does not seem to be ascertainable by the method given here.

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AN EXAMPLE OF AN ANCILLARY STATISTIC AND THE COMBINATION OF TWO SAMPLES BY BAYES' THEOREM

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1. Origin of the example. In [1], an example was given in which a fiducial distribution served as a distribution *a priori* to be combined with a different set of data (not capable of yielding probability statements), by Bayes' Theorem. In [2], it was shown that this procedure of combining samples, when each sample yielded a fiducial distribution, could lead to a contradiction. In [3], an attempt

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was made to show why these contradictions arise and how to eliminate them. Two conditions that all distributions *a posteriori* must fulfil, were stated. From these, the following necessary conditions were derived: the two samples to be combined by Bayes' Theorem must have sufficient statistics following:

(1) the normal distributions with means θ, $c\theta + k$, or

(2) the gamma distribution with parameters θ, $(c\theta)^k$, or

(3) the normal distribution with mean θ and the gamma distribution with parameter $c \exp k\theta$,

where c and k are known constants. Cases (1) and (2) were also shown to be sufficient conditions. It remains to show that case (3) is a sufficient condition (i.e., no contradiction arises).

2. Derivation of an ancillary statistic and the corresponding fiducial distribution. Suppose the sufficient statistics T_1 and T_2 have densities

$$L_1(T_1, \theta_1)\, dT_1 = (2\pi n)^{-\frac{1}{2}} \exp[-(T_1 - n\theta)^2/2n]\, dT_1,$$

$$L_2(T_2, \theta_2)\, dT_2 = [T_2^{m-1} c^m/\Gamma(m)] \exp[mk\theta - ce^{k\theta}T_2]\, dT_2 .$$

Thus, the simultaneous distribution of T_1 and T_2 is

$$[c^m T_2^{m-1}/(2\pi n)^{\frac{1}{2}}\Gamma(m)] \exp[mk\theta - ce^{k\theta}T_2 - (T_1 - n\theta)^2/2n]\, dT_1\, dT_2 .$$

Making the transformation

$$T_2 = \exp[-k(U_1 + U_2)], \qquad T_1 = nU_2,$$

the simultaneous distribution of U_1, U_2 is

$$[c^m nk/(2\pi n)^{\frac{1}{2}}\Gamma(m)] \exp[-mk(U_1 + U_2 - \theta) - ce^{-k(U_1+U_2-\theta)} - \tfrac{1}{2}n(U_2 - \theta)^2]\, dU_1\, dU_2 .$$

Integrating with respect to U_2, the distribution of U_1 is

$$[c^m nkI(U_1)/(2\pi n)^{\frac{1}{2}}\Gamma(m)]\, dU_1,$$

where

$$I(U_1) = \int_{-\infty}^{\infty} \exp[-mk(U_1 + w) - ce^{-k(U_1+w)} - \tfrac{1}{2}n\, w^2]\, dw,$$

and is independent of θ. Hence $U_1 = -T_1/n - (\log T_2)/k$ is an ancillary statistic.

The distribution of U_2 given U_1 is

$$(1) \quad L(U_2 \mid U_1, \theta) = \{\exp[-mk(U_1 + U_2 - \theta) - ce^{-k(U_1+U_2-\theta)} - \tfrac{1}{2}n(U_2 - \theta)^2]\}/I(U_1).$$

Using (1), the corresponding fiducial distribution is given by

$$(2) \qquad f(\theta \mid U_1, U_2) = \int_{u_2=-\infty}^{U_2} \left[\frac{\partial}{\partial\theta} L(u_2 \mid U_1, \theta)\right] du_2 .$$

3. Derivation of distribution a posteriori by Bayes' Theorem. The fiducial distribution based on T_1 is

$$h(\theta \mid T_1) = (n/2\pi)^{\frac{1}{2}} \exp [(T_1 - n\theta)^2/2n].$$

Using this as the distribution *a priori*, to be used in conjunction with T_2, gives as distribution *a posteriori*,

$$(3) \qquad b(\theta \mid T_1, T_2) = \{\exp [mk\theta - ce^{k\theta}T_2 - (T_1 - n\theta)^2/2n]\}/I(T_1, T_2),$$

where $I(T_1, T_2) = \int_{-\infty}^{\infty} \exp [mk\theta - ce^{k\theta}T_2 - (T_1 - n\theta)^2/2n]\, d\theta$. Hence

$$I(T_1, T_2) = [\exp mk(U_1 + U_2)]I(U_1),$$

and so

$$(4) \qquad b(\theta \mid U_1, U_2) = L(U_2 \mid U_1, \theta).$$

From (1) and (4) it can be seen that

$$-\frac{\partial}{\partial\theta} L(U_2 \mid U_1, \theta) = \frac{\partial}{\partial U_2} L(U_2 \mid U_1, \theta) = \frac{\partial}{\partial U_2} b(\theta \mid U_1, U_2),$$

and so from (2) $f(\theta \mid U_1, U_2) = b(\theta \mid U_1, U_2)$. Thus, the fiducial distribution based on the combined sample is the same as the *a posteriori* distribution obtained on combining the samples by Bayes' Theorem, using the fiducial distribution based on one of the samples as a distribution *a priori*. Thus all three conditions stated at the first are sufficient as well as necessary.

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