DISTRIBUTION OF A DEFINITE QUADRATIC FORM FOR NON-CENTRAL NORMAL VARIATES

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1. Summary. In this paper we generalize the result of James Pachares [1] on the distribution of a definite quadratic form to the case of non-central normal variates.

2. The problem. Suppose we have a quadratic form $Q = (\frac{1}{2})\mathbf{y}'\mathbf{A}\mathbf{y}$ where $\mathbf{A}$ is a $p \times p$ symmetric positive semi-definite matrix of rank n $(\leqq p)$ and $\mathbf{y}' = (y_1, \cdots, y_p)$. The y_i's are independently distributed normal variates with means ν_i and variance one, $i = 1, 2, \cdots, p$. It is well-known that we can make an orthogonal transformation reducing Q to its canonical form, i.e.,

$$Q = \tfrac{1}{2}\sum_{i=1}^{n} a_i x_i^2 ,$$

where the coefficients $a_1, \cdots, a_n$ are the non-zero latent roots of the matrix $\mathbf{A}$, (all a_i's positive). Under such a transformation $x_1, \cdots, x_p$ remain independent normal variates with means μ_i and variance one; (μ_i is obtained from the ν_i's in the same manner as x_i is obtained from the y_i's.) The problem is to find $F(t) = \Pr(Q \leqq t)$.

3. The solution.

Theorem. *Let* $Q = \frac{1}{2}\sum_{i=1}^{n} a_i x_i^2$, *where the* x_i *are independent* $N(\mu_i, 1)$ *variates and where* $a_i > 0$ *for* $i = 1, 2, \cdots, n$. *Let*

$$Q^* = \tfrac{1}{2}\sum_{i=1}^{n} a_i^{-1}(x_i - \mu_i)^2, \qquad L^* = \sum_{i=1}^{n} (2a_i)^{-\frac{1}{2}}(x_i - \mu_i)\mu_i ,$$

and $D = a_1 \cdot a_2 \cdots a_n$. *Then*

a) $$F(t) = D^{-\frac{1}{2}} \exp\left(-\tfrac{1}{2}\sum_{i=1}^{n} \mu_i^2\right) t^{\frac{1}{2}n} \sum_{j,k=0}^{\infty} \frac{(-1)^j 2^{2k} t^{j+k} E(Q^{*j} L^{*2k})}{j!(2k)!\Gamma(j + k + 1 + \frac{1}{2}n)},$$

b) *the series in* (a) *is absolutely convergent,*

c) *for any two non-negative integers* r *and* s *and every* $t > 0$,

$$S_{2s} = \sum_{j=0}^{2s} d_j > F(t) > \sum_{j=0}^{2r+1} d_j = S_{2r+1},$$

where

$$d_j = \frac{\exp\left(-\frac{1}{2}\sum_{i=1}^{n} \mu_i^2\right) t^{\frac{1}{2}n+j}(-1)^j}{D^{\frac{1}{2}} j!} \sum_{k=0}^{\infty} \frac{2^{2k} t^k E(Q^{*j} L^{*2k})}{(2k)!\Gamma(\frac{1}{2}n + j + k + 1)}.$$

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Proof. Let R represent the region where $\frac{1}{2}\sum_{i=1}^{n} a_i x_i^2 \leqq t$. Then, with $d\mathbf{x} = dx_1\, dx_2\, dx_3 \cdots dx_n$,

$$(1)\quad \begin{aligned} F(t) &= (2\pi)^{-\frac{1}{2}n} \int_R \cdots \int \exp\left[-\tfrac{1}{2}\sum_{i=1}^{n}(x_i - \mu_i)^2\right] d\mathbf{x} \\ &= (2\pi)^{-\frac{1}{2}n} \exp\left(-\tfrac{1}{2}\sum_{i=1}^{n}\mu_i^2\right)\int_R \cdots \int \exp\left(-\tfrac{1}{2}\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} x_i\,\mu_i\right) d\mathbf{x}. \end{aligned}$$

Expanding the exponential in the integrand, we get

$$F(t) = C \int_R \cdots \int \left\{\sum_{j=0}^{\infty}(1/j!)\left(-\tfrac{1}{2}\sum_{i=1}^{n} x_i^2\right)^j\right\}\left\{\sum_{k=0}^{\infty}(1/k!)\left(\sum_{i=1}^{n} x_i\,\mu_i\right)^k\right\} d\mathbf{x},$$

where $C = (2\pi)^{-\frac{1}{2}n}\exp\left(-\frac{1}{2}\sum_{i=1}^{n}\mu_i^2\right)$, i.e.,

$$(2)\quad F(t) = C\sum_{j,k=0}^{\infty}\left[(-\tfrac{1}{2})^j/(j!k!)\right]\int_R \cdots \int \left(\sum_{i=1}^{n} x_i^2\right)^j\left(\sum_{i=1}^{n} x_i\,\mu_i\right)^k d\mathbf{x},$$

$$(3)\quad \begin{aligned} F(t) = C\sum_{j,k=0}^{\infty}\left[(-\tfrac{1}{2})^j/(j!k!)\right]\sum_{\pi\text{'s}}\sum_{\eta\text{'s}} \frac{j!k!\mu_1^{\eta_1}\cdots\mu_n^{\eta_n}}{\pi_1!\cdots\pi_n!\eta_1!\cdots\eta_n!} \\ \int_R \cdots \int \prod_{l=1}^{n} x_l^{2\pi_l+\eta_l}\, d\mathbf{x}, \end{aligned}$$

where $\sum_{\pi\text{'s}}$ and $\sum_{\eta\text{'s}}$ denote respectively the sums taken over all non-negative integral π_i's and η_i's subject to the respective conditions

$$\pi_1 + \pi_2 + \cdots \pi_n = j, \qquad \eta_1 + \eta_2 + \cdots \eta_n = k.$$

Now by a well-known formula of Dirichlet it is easy to show that the right-most n-tuple integral in (3) is 0, if any η_l is odd,

$$\frac{(2t)^{j+\frac{1}{2}(k+n)}\Gamma[\frac{1}{2}(2\pi_1+\eta_1+1)]\cdots\Gamma[\frac{1}{2}(2\pi_n+\eta_n+1)]}{D^{\frac{1}{2}}\Gamma[\frac{1}{2}(k+n)+j+1]a_1^{\pi_1+\frac{1}{2}\eta_1}\cdots a_n^{\pi_n+\frac{1}{2}\eta_n}},$$

if all η_l are even, and so we need consider only even values of $\sum \eta_l = k$. Hence

$$(5)\quad \int_R \cdots \int \left(\sum_{i=1}^{n} x_i^2\right)^j\left(\sum_{i=1}^{n} x_i\,\mu_i\right)^{2k+1} d\mathbf{x} = 0,$$

$$(6)\quad \begin{aligned} \int_R \cdots \int \left(\sum_{i=1}^{n} x_i^2\right)^j\left(\sum_{i=1}^{n} x_i\,\mu_i\right)^{2k} d\mathbf{x} = \frac{j!(2k)!(2t)^{\frac{1}{2}(j+k+n)}}{D^{\frac{1}{2}}\Gamma(\frac{1}{2}n+j+k+1)} \\ \sum_{\pi\text{'s}}\sum_{\eta\text{'s}}\frac{\mu_1^{2\eta_1}\cdots\mu_n^{2\eta_n}\Gamma[\frac{1}{2}(2\pi_1+2\eta_1+1)]\cdots\Gamma[\frac{1}{2}(2\pi_n+2\eta_n+1)]}{\pi_1!\cdots\pi_n!(2\eta_1)!\cdots(2\eta_n)!a_1^{\pi_1+\eta_1}\cdots a_n^{\pi_n+\eta_n}}, \end{aligned}$$

where $\sum_{\eta\text{'s}}$ means summation over all non-negative integral $\eta_1 \cdots \eta_n$ such that $\sum_{l=1}^{n}\eta_l = k$. The problem is to evaluate this last expression. Recalling that, if x_i is $N(\mu_i, 1)$, then $E\{\frac{1}{2}(x_i - \mu_i)^2\}^r = \Gamma(r + \frac{1}{2})/\Gamma(\frac{1}{2})$, we find

$$E(Q^{*j}L^{*2k})$$

$$(7)\qquad \begin{aligned} &= E\left\{\tfrac{1}{2}\sum_{i=1}^{n} a_i^{-1}(x_i - \mu_i)^2\right\}^j \left\{2^{-\frac{1}{2}}\sum_{i=1}^{n} a_i^{-\frac{1}{2}}(x_i - \mu_i)\mu_i\right\}^{2k} \\ &= 2^{-j-k}\sum_{\pi\text{'s}}\sum_{\eta\text{'s}} \frac{j!(2k)!\,\mu_1^{2\eta_1}\cdots\mu_n^{2\eta_n}E[(x_1-\mu_1)^{2\pi_1+2\eta_1}\cdots(x_n-\mu_n)^{2\pi_n+2\eta_n}]}{\pi_1!\cdots\pi_n!\,(2\eta_1)!\cdots(2\eta_n)!\,a_1^{\pi_1+\eta_1}\cdots a_n^{\pi_n+\eta_n}}. \end{aligned}$$

Substituting the value of $E(x_i - \mu_i)^{2\pi_i+2\eta_i}$, we find that equation (6) is equivalent to

$$(8)\qquad \int \underset{R}{\cdots} \int \left(\sum_{i=1}^{n} x_i^2\right)^j \left(\sum_{i=1}^{n} x_i\mu_i\right)^{2k} d\mathbf{x} = \frac{(2t)^{\frac{1}{2}(j+k+n)}\{\Gamma(\frac{1}{2})\}^n}{D^{\frac{1}{2}}\Gamma(\frac{1}{2}n + j + k + l)} E(Q^{*j}L^{*2k}).$$

Hence equation (2) gives

$$(9)\qquad F(t) = \exp\left(-\tfrac{1}{2}\sum_{i=1}^{n}\mu_i^2\right)t^{\frac{1}{2}n}\sum_{j,k=0}^{\infty}\frac{(-1)^j 2^{2k}t^{j+k}E(Q^{*j}L^{*2k})}{j!(2k)!D^{\frac{1}{2}}\Gamma(\frac{1}{2}n + j + k + l)}.$$

This proves part (a) of the theorem.

To show absolute convergence for part (b) we note that, if $a = \min a_i$ and $\nu = \sum_{i=1}^{n}\mu_i^2/a_i$,

$$(10)\qquad \begin{aligned} Q^* &\leqq \sum_{i=1}^{n}(x_i - \mu_i)^2/(2a), \\ L^{*2} &\leqq \tfrac{1}{2}\nu\sum_{i=1}^{n}(x_i - \mu_i)^2, \\ E(Q^{*j}L^{*2k}) &< \nu^k\frac{\Gamma[\frac{1}{2}(j + k + l + n)]}{a^j\Gamma(\frac{1}{2}n + 1)}. \end{aligned}$$

Hence, if $F^+(t)$ is the sum of the absolute values of the terms of the series for $F(t)$, then

$$(11)\qquad F^+(t) < \frac{\exp\left(-\frac{1}{2}\sum_{i=1}^{n}\mu_i^2\right)t^{\frac{1}{2}n}\exp(t/a)\cosh(2\nu^{\frac{1}{2}}t^{\frac{1}{2}})}{D^{\frac{1}{2}}\Gamma(\frac{1}{2}n + 1)} < \infty \text{ for } t < \infty.$$

This proves part (b) of the theorem.

The bounds of part (c) are based on the fact that, if r and s are any two nonnegative integers, then, for t positive,

$$\sum_{j=0}^{2s}\frac{(-t)^j}{j!} > e^{-t} > \sum_{j=0}^{2r+1}\frac{(-t)^j}{j!}.$$

Replacing t by $\frac{1}{2}\sum_{i=1}^{n}x_i^2$ and using (1), (2) and (8), part (c) of the theorem can be established.

REMARKS.

(i) The joint moments of Q^* and L^* are easy to obtain from the joint cumulants of Q^* and L^*. If $K_{i,j}$ is the (i, j) cumulant of (Q^*, L^*) then

$$K_{r,0} = (r-1)!\,\tfrac{1}{2}\sum_{i=1}^{n} a_i^{-r} \qquad \text{for } r = 1, 2, \cdots,$$

$$K_{r,2} = r!\sum_{i=1}^{n} \mu_i^2/(2a_i^{r+1}) \qquad \text{for } r = 0, 1, 2, \cdots,$$

$$K_{r,j} = 0 \quad \text{for} \quad j = 1, 3, 4, 5, \cdots \qquad \text{and } r = 0, 1, 2, \cdots.$$

(ii) Clearly S_0, S_2, S_4, $\cdots$ is a sequence of upper bounds and S_1, S_3, S_5, $\cdots$ is a sequence of lower bounds for $F(t)$. In practice we would compute a finite number of terms and then state that $\min S_{2s} > F(t) > \max S_{2r+1}$. The absolute value of the error thus committed is not greater than

$$\min S_{2s} - \max S_{2r+1}.$$

4. Applications to the distribution of a sum of squares in dependent variates. (a) Let $y_1, y_2 \cdots y_n$ have a joint multivariate normal distribution with means $\nu_1, \cdots, \nu_n$ and variance-covariance matrix A. We wish to find the distribution of $R = \frac{1}{2}\mathbf{y}'\mathbf{y}$. Now

$$\Pr(R \leqq t) = |\mathbf{A}|^{-\frac{1}{2}}(2\pi)^{-\frac{1}{2}n} \int \cdots \int_{\frac{1}{2}\mathbf{y}'\mathbf{y} \leqq t} \exp\left[-\tfrac{1}{2}(\mathbf{y}-\boldsymbol{\nu})'\mathbf{A}^{-1}(\mathbf{y}-\boldsymbol{\nu})\right] d\mathbf{y}$$

$$= (2\pi)^{-\frac{1}{2}n} \int \cdots \int_{\frac{1}{2}\mathbf{x}'\mathbf{A}\mathbf{x} \leqq t} \exp\left[-\tfrac{1}{2}(\mathbf{x}-\boldsymbol{\mu})'(\mathbf{x}-\boldsymbol{\mu})\right] d\mathbf{x},$$

where $\mathbf{P}\mathbf{x} = \mathbf{y}$ and $\mathbf{P}\boldsymbol{\mu} = \boldsymbol{\nu}$, $\mathbf{P}^2 = \mathbf{A}$ ($\mathbf{P}$ is a symmetric matrix). We can now directly apply the theorem.

REMARKS. Combining the results of the theorem and the above application, it is easy to show that we could find the distribution of a definite (or semi-definite) quadratic form $\mathbf{y}'\mathbf{B}\mathbf{y}$, which involves as parameters the latent roots of **AB**, and with non-central parameters depending on the given means, the variance-covariance matrix **A** and the latent roots of **AB**.

(b) The complex normal distribution defined by Wooding [3] and Turin [2] has density function given by

$$\pi^{-n}|\mathbf{L}|^{-1}\exp\left[-(\mathbf{v}-\boldsymbol{\nu})^{*\prime}\mathbf{L}^{-1}(\mathbf{v}-\boldsymbol{\nu})\right]$$

where $\mathbf{v} = \mathbf{z} + i\mathbf{w}$ and $E(\mathbf{v}) = \boldsymbol{\nu}$, $\mathbf{x}^*$ is the complex conjugate of $\mathbf{x}$,

$$E[\mathbf{z} - E(\mathbf{z})][\mathbf{z} - E(\mathbf{z})]' = \mathbf{L}_1,$$

$$E[\mathbf{w} - E(\mathbf{w})][\mathbf{z} - E(\mathbf{z})]' = -E[\mathbf{z} - E(\mathbf{z})][\mathbf{w} - E(\mathbf{w})]' = \mathbf{L}_2$$

and $\mathbf{L} = \mathbf{L}_1 + i\mathbf{L}_2$ is a hermitian positive definite matrix.

Then for the distribution of $v^{*\prime}v = R$, we have

$$\Pr(R \leqq t) = \pi^{-n}\,|\mathbf{L}|^{-1} \int \cdots \int_{R \leqq t} \exp\left[-\,(\mathbf{v}-\boldsymbol{\nu})^{*\prime}\mathbf{L}^{-1}(\mathbf{v}-\boldsymbol{\nu})\right] d\mathbf{v}$$

where $d\mathbf{v} = dz_1\,dz_2\,dz_3 \cdots dz_n\,dw_1\,dw_2 \cdots dw_n = d\mathbf{z}\,d\mathbf{w}$. This is equivalent to

$$\Pr(R \leqq t) = \pi^{-n} |\mathbf{L}|^{-1} \int \cdots \int_{\mathbf{x}^{*\prime}\mathbf{x} \leqq t} \exp\left[- \sum_{j=1}^{n} a_j^{-1} \{ (x_{1j} - \mu_{1j})^2 + (x_{2j} - \mu_{2j})^2 \} \right] d\mathbf{x},$$

where $d\mathbf{x} = d\mathbf{x}_1\, d\mathbf{x}_2$, the a_j's ($a_j > 0$) are the latent roots of $\mathbf{L}$ and

$$\mathbf{v}^{*\prime}\mathbf{v} = \textstyle\sum_{j=1}^{n} \mu_{1j}^2 + \sum_{j=1}^{n} \mu_{2j}^2,$$

or

$$\Pr(R \leqq t) = \pi^{-n} \int \cdots \int_{\sum_{j=1}^{n} a_j x_j^2 \leqq t} \exp\left[-\sum_{j=1}^{n} \{ (x_{1j} - \mu_{1j}/\sqrt{a_j})^2 + (x_{2j} - \mu_{2j}/\sqrt{a_j})^2 \} \right] d\mathbf{x}.$$

where $x_j^2 = x_{1j}^2 + x_{2j}^2$. This can be easily solved by the direct use of the theorem.

Remarks. Combining the results of the theorem and the above application, it is easy to show that we could find the distribution of a definite (or semi-definite) Hermitian form $\mathbf{v}^{*\prime}\mathbf{A}\mathbf{v}$ ($\mathbf{A}$ positive definite or semi-definite), which involves as parameters the latent roots of $\mathbf{AL}$ and with non-central parameters depending on the given means, the variance-covariance matrix $\mathbf{L}$, and the latent roots of $\mathbf{AL}$.

REFERENCES

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