TESTS OF FIT BASED ON THE NUMBER OF OBSERVATIONS FALLING IN THE SHORTEST SAMPLE SPACINGS DETERMINED BY EARLIER OBSERVATIONS[1]

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1. Introduction. Suppose the random variables $X_1, X_2, \cdots$ are known to be independent and identically distributed, with a continuous cumulative distribution function which is otherwise unknown. The problem which this paper discusses is the familiar one of testing the hypothesis that the cumulative distribution function is equal to a given completely specified cumulative distribution function $G(x)$. By using the random variables $G(X_1), G(X_2), \cdots$ in place of $X_1, X_2, \cdots$, the problem becomes that of testing the hypothesis that the common cumulative distribution function of $G(X_1), G(X_2), \cdots$ is the uniform distribution function $U(x)$, where $U(x) = x$ for $0 \leqq x \leqq 1$. For the remainder of the paper, it will be assumed that the problem has been reduced to this form, so that there is no loss of generality in assuming that $G(x) = U(x)$, and that all distributions considered assign probability one to the closed interval [0, 1].

Let $Y_1(n), Y_2(n), \cdots, Y_n(n)$ denote the ordered values of $X_1, \cdots, X_n$, where $0 \leqq Y_1(n) \leqq Y_2(n) \leqq \cdots \leqq Y_n(n) \leqq 1$. For convenience, $Y_0(n)$ is defined as 0 and $Y_{n+1}(n)$ is defined as 1. $T_i'(n)$ denotes the closed interval $[Y_{i-1}(n), Y_i(n)]$, and $T_i(n)$ denotes the length of this interval, for $i = 1, \cdots, n + 1$. The $T_i'(n)$ are known as sample spacings.

Let p be a fixed quantity in the open interval (0, 1). The set $S_n(p)$ is defined as the union of the shortest sample spacing, the next shortest sample spacing, $\cdots$, until the total length of the sample spacings included in $S_n(p)$ is exactly equal to p. With probability one, this will require the use of a portion of the last sample spacing used, which for convenience will always be taken as the left-hand portion of the sample spacing broken up. The chance event $C_n(p)$ is defined as that event which occurs when and only when the random variable X_{n+1} falls in the set $S_n(p)$.

If the hypothesis of a uniform distribution is true, the chance events $C_1(p)$, $C_2(p), \cdots$ are independent events, each with probability exactly equal to p. If the hypothesis is not true, the chance events are not independent and their probabilities are not all the same. However, the definition of the set $S_n(p)$ clearly favors the inclusion of those sections of the unit interval at which the true density function is relatively high, and it seems reasonable to suppose that the conditional probability of $C_n(p)$ given $X_1, \cdots, X_n$ has a high probability of approaching some limit greater than p. This conjecture is proved by the theorem of

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Section 2. Some applications to testing the hypothesis of a uniform distribution are discussed in the rest of the paper. The tests discussed reject the hypothesis if the proportion of events $C_1(p), C_2(p), \cdots$ which occur is "too far" above p.

2. The basic theorem. Throughout this section it is assumed that the common cumulative distribution function $F(x)$ of $X_1, X_2, \cdots$ assigns all probability to the interval [0, 1], and has a derivative $f(x)$ which is bounded and has at most a finite number of discontinuities. For any nonnegative t, $Q(t; f)$ denotes $1 - \int_0^1 [1 + tf(x)] \exp[-tf(x)]dx$, and $M(t; f)$ denotes

$$1 - \int_0^1 f(x)[1 + tf(x)] \exp[-tf(x)]dx.$$

It is easily verified that $Q(t; f)$ is a continuous and strictly increasing function of t for nonnegative t, with $Q(0; f) = 0$, $\lim_{t\to\infty} Q(t; f) = 1$. Denote by $t(p)$ the unique solution in t of the equation $Q(t; f) = p$.

THEOREM. *The conditional probability of the event $C_n(p)$ given $X_1, \cdots, X_n$ converges to $M(t(p); f)$ with probability one as n increases. Also, $M(t(p); f) \geqq p$, with equality holding if and only if $f(x) = 1$ almost everywhere on* [0, 1].

The remainder of this section is devoted to proving this theorem. The introduction of some detailed notation is necessary. Let $Z_1(n), \cdots, Z_{n+1}(n)$ denote the ordered values of $T_1(n), \cdots, T_{n+1}(n)$, where $0 \leqq Z_1(n) \leqq \cdots \leqq Z_{n+1}(n)$ and $\sum_{i=1}^{n+1} Z_i(n) = 1$. With probability one, $Z_i(n) < Z_{i+1}(n)$ for $i = 1, \cdots, n$. $J_n(p)$ is defined as the largest integer such that $\sum_{i=1}^{J_n(p)} Z_i(n) < p$. The set $S_n(p)$ as defined above is the union of $J_n(p) + 1$ closed subintervals: the $J_n(p)$ subintervals $[Y_i(n), Y_{i+1}(n)]$ such that $Y_{i+1}(n) - Y_i(n) = Z_j(n)$ for some $j \leqq J_n(p)$, plus the subinterval $[Y_k(n), Y_k(n) + \Delta]$, where $Y_{k+1}(n) - Y_k(n) = Z_{J_n(p)+1}(n)$, and Δ is chosen so that the Lebesgue measure of $S_n(p)$ is exactly p. $N_n(t)$ denotes the number of the quantities $Z_1(n), \cdots, Z_{n+1}(n)$ which are no greater than $t/(n + 1)$, and $R_n(t)$ denotes $(n + 1)^{-1}N_n(t)$. $L_n(t)$ denotes $\sum_{i:Z_i(n)\leqq t/(n+1)} Z_i(n)$, and $K_n(t)$ denotes the total probability assigned by $F(x)$ to the union of the $N_n(t)$ intervals $[Y_j(n), Y_{j+1}(n)]$ such that $Y_{j+1}(n) - Y_j(n) \leqq t/(n + 1)$. That is,

$$K_n(t) = \sum_{j:Y_{j+1}(n)-Y_j(n)\leqq t/(n+1)} [F(Y_{j+1}(n)) - F(Y_j(n))].$$

LEMMA 1. *$L_n(t)$ converges to $Q(t; f)$ with probability one as n increases.*

PROOF OF LEMMA 1. $L_n(t) = \int_0^t u/(n + 1)\, dN_n(u)$, the integral being Riemann-Stieltjes. Then $L_n(t) = \int_0^t u dR_n(u) = tR_n(t) - \int_0^t R_n(u)\, du$. In [1] it was proved that

$$\sup_{u\geqq 0} \left| R_n(u) - \left[1 - \int_0^1 f(x) \exp(-uf(x))\, dx\right]\right|$$

converges to zero with probability one as n increases. Then, with probability one as n increases, $L_n(t)$ converges to

$$t\left[1 - \int_0^1 f(x) \exp(-tf(x))\, dx\right] - \int_0^t \left[1 - \int_0^1 f(x) \exp(-uf(x))\, dx\right] du,$$

and this last expression is easily seen to be equal to $Q(t; f)$, completing the proof of Lemma 1.

LEMMA 2. *$K_n(t)$ converges to $M(t; f)$ with probability one as n increases.*

PROOF OF LEMMA 2. By the assumption about $f(x)$, for any given positive γ it is possible to break the interval [0, 1] into a finite number of subintervals, such that in the interior of each subinterval there is a variation of $f(x)$ which is no greater than γ. Suppose there are $b(\gamma)$ such subintervals, the endpoints of the ith such subinterval being denoted by c_i, d_i, with $c_i < d_i$. m_i denotes $\inf_{c_i<x<d_i} f(x)$, and M_i denotes $\sup_{c_i<x<d_i} f(x)$, where $0 \leqq M_i - m_i \leqq \gamma$. q_i denotes $F(d_i) - F(c_i)$, and N_i denotes the number of values among $X_1, \cdots, X_n$ falling in the closed interval $[c_i, d_i]$. N_i has a binomial distribution with parameters n, q_i. It may be assumed that q_i is positive, since if $q_i = 0$ the interval $[c_i, d_i]$ can be ignored in what follows.

Denote by $Y_1''(i) \leqq Y_2''(i) \leqq \cdots \leqq Y_{N_i}''(i)$ the ordered values of the N_i observations in the interval $[c_i, d_i]$. $Y_0''(i)$, $Y_{N_i+1}''(i)$ denote c_i, d_i respectively. $T_j''(i)$ denotes $Y_j''(i) - Y_{j-1}''(i)$ for $j = 1, \cdots, N_i + 1$. ${}_iN_n(t)$ denotes the number of the quantities $T_1''(i), \cdots T_{N_i+1}''(i)$ which are no greater than $t/(N_i + 1)$, and ${}_iR_n(t)$ denotes $(N_i + 1)^{-1}{}_iN_n(t)$. Since $(N_i + 1)/(n + 1)$ converges to q_i with probability one as n increases, it follows from [1] that

$$\sup_{t \geqq 0} \left| {}_iR_n(t) - \left[1 - \int_{c_i}^{d_i} \frac{f(x)}{q_i} \exp\left[-\frac{tf(x)}{q_i}\right] dx\right] \right|$$

converges to zero with probability one as n increases. From this it follows by the same sort of calculation used in the proof of Lemma 1 that $\sum_{j: T_j''(i) \leqq t/(N_i+1)} T_j''(i)$ converges to

$$d_i - c_i - \int_{c_i}^{d_i} \left[1 + \frac{tf(x)}{q_i}\right] \exp\left[-\frac{tf(x)}{q_i}\right] dx = \rho_i(t),$$

say, with probability one as n increases. $\sum_{j: T_j''(i) \leqq t/(n+1)} T_j''(i)$ can be written as $\sum_{j: T_j''(i) \leqq t^*/(N_i+1)} T_j''(i)$ where $t^* = t(N_i + 1)/(n + 1)$, and because $(N_i + 1)/(n + 1)$ converges to q_i with probability one as n increases, and $\rho_i(t)$ is a continuous function of t, it follows that $\sum_{j: T_j''(i) \leqq t/(n+1)} T_j''(i)$ converges to $\rho_i(q_i t)$ with probability one as n increases.

Denote the total probability assigned by $F(x)$ to the union of all the subintervals $(Y_{j-1}''(i), Y_j''(i))$ with $Y_j''(i) - Y_{j-1}''(i) \leqq t/(n + 1)$ by $\theta_i(t)$.

$$m_i \sum_{j: T_j''(i) \leqq t/(n+1)} T_j''(i) \leqq \theta_i(t) \leqq M_i \sum_{j: T_j''(i) \leqq t/(n+1)} T_j''(i)$$

and

$$\sum_{j: T_j''(i) \leqq t/(n+1)} T_j''(i)$$

can be written as $\rho_i(q_i t) + \delta_i(n)$, where $\delta_i(n)$ converges to zero with probability one as n increases.

$$K_n(t) = \sum_{i=1}^{b(\gamma)} \theta_i(t) + \epsilon_n ,$$

where ϵ_n represents a term that takes account of the fact that at most $2b(\gamma)$ of the original subintervals $(Y_{j-1}(n), Y_j(n))$ were broken into two parts by the points c_i, d_i. Clearly, ϵ_n converges to zero with probability one as n increases. Then

$$\sum_{i=1}^{b(\gamma)} m_i\rho_i(q_i t) + \sum_{i=1}^{b(\gamma)} m_i\delta_i(n) + \epsilon_n \leqq K_n(t) \leqq \sum_{i=1}^{b(\gamma)} M_i\rho_i(q_i t) + \sum_{i=1}^{b(\gamma)} M_i\delta_i(n) + \epsilon_n .$$

By taking γ small enough, $\sum_{i=1}^{b(\gamma)} m_i\rho_i(q_i t)$ and $\sum_{i=1}^{b(\gamma)} M_i\rho_i(q_i t)$ can both be made arbitrarily close to $M(t; f)$. Since $\delta_1(n), \cdots, \delta_{b(\gamma)}(n)$, ϵ_n approach zero with probability one as n increases, Lemma 2 follows immediately.

Define the set $T_n^*(t)$ as the union of the intervals $(Y_{j-1}(n), Y_j(n))$ for all j for which $Y_j(n) - Y_{j-1}(n) \leqq t/(n+1)$. It follows from Lemma 1 that the sets $S_n(p)$ and $T_n^*(t(p))$ differ by a set whose measure approaches zero with probability one as n increases. It follows from Lemma 2 that the total probability assigned by $F(x)$ to the set $T_n^*(t(p))$ converges to $M(t(p); f)$ with probability one as n increases. Therefore the total probability assigned by $F(x)$ to the set $S_n(p)$ converges to $M(t(p); f)$ with probability one as n increases. Since the probability assigned to $S_n(p)$ by $F(x)$ is the conditional probability of the event $C_n(p)$ given $X_1, \cdots, X_n$, the first part of the theorem is proved. The second part of the theorem is a direct consequence of the following lemma.

LEMMA 3. *$Q(t; f) \leqq M(t; f)$ for all $f(x)$ and for each positive t, with equality if and only if $f(x) = 1$ almost everywhere on* [0, 1].

PROOF OF LEMMA 3. $M(t; f) - Q(t; f)$ can be written as

$$\int_0^1 [1 + tf(x)][1 - f(x)] \exp(-tf(x))\, dx.$$

The function $[1 + tf(x)] \exp(-tf(x))$ is equal to unity when $f(x) = 0$, and decreases strictly monotonically toward zero as $f(x)$ increases. Then $M(t; f) - Q(t; f)$ can be written as

$$\int_{x:f(x)\leqq 1} [1 - f(x)][1 + tf(x)] \exp(-tf(x))\, dx + \int_{x:f(x)\geqq 1} [1 - f(x)][1 + tf(x)] \exp(-tf(x))\, dx.$$

The first of these integrals is at least equal to $(1 + t)e^{-t} \int_{x:f(x)\leqq 1} [1 - f(x)]\, dx$, with equality if and only if the subset of [0, 1] where $f(x) < 1$ has measure zero, and the second of the integrals is at least equal to $(1 + t)e^{-t} \int_{x:f(x)\geqq 1} [1 - f(x)]\, dx$, with equality if and only if the subset of [0, 1] where $f(x) > 1$ has measure zero. Therefore

$$M(t; f) - Q(t; f) \geqq (1 + t)e^{-t} \left[\int_{x:f(x)\leqq 1} [1 - f(x)]\, dx + \int_{x:f(x)\geqq 1} [1 - f(x)]\, dx \right] = 0,$$

with equality if and only if $f(x) = 1$ almost everywhere on [0, 1]. This completes the proof of Lemma 3, and of the theorem.

3. Application of the theorem to a nonsequential test of fit. Let W_i denote the random variable which is equal to one if the event $C_i(p)$ occurs, and is equal to zero otherwise, for $i = 1, 2, \cdots$. Define V_n as $W_1 + \cdots + W_n$. If the hypothesis of a uniform distribution for $X_1, X_2, \cdots$ is true, then V_n has a binomial distribution with parameters n, p. If $X_1, \cdots, X_{n+1}$ are observed, a possible test of the hypothesis is to reject when $V_n/n \geqq d_n(\alpha)$, where $d_n(\alpha)$ is a constant chosen to give the desired level of significance α. If n is large, then $d_n(\alpha)$ is approximately $p + z_\alpha[p(1-p)/n]^{\frac{1}{2}}$, where $(2\pi)^{-\frac{1}{2}} \int_{z_\alpha}^{\infty} \exp(-\frac{1}{2}t^2)\, dt = \alpha$.

The consistency of this proposed test will be shown if it is shown that V_n/n converges stochastically to $M(t(p); f)$ as n increases, since $M(t(p); f) > p$ if the hypothesis is not true, and the critical value for V_n/n approaches p as n increases. The convergence of V_n/n to $M(t(p); f)$ is shown as follows.

For convenience, let $X(j)$ denote the sequence $(X_1, \cdots, X_j)$, and let Q_j denote $W_j - M(t(p); f)$. Define r_j as $E|E(W_j \mid X(j)) - M(t(p); f)|$. Since $E(W_j \mid X(j))$ is simply the conditional probability that $C_j(p)$ will occur, given $X(j)$, the theorem of Section 2 shows that r_j approaches zero as j increases. Clearly, $0 \leqq r_j \leqq 1$ for all j. If

$$i < j,\ E(Q_iQ_j) = E\{E(Q_iQ_j \mid X(j))\} = E\{Q_iE(Q_j \mid X(j))\}$$

since if $i < j$, Q_i is a uniquely determined function of $X(j)$. Also,

$$E\{Q_iE(Q_j \mid X(j))\} \leqq E\,|Q_iE(Q_j \mid X(j))| = E\{|Q_i|\,|E(Q_j \mid X(j))|\} \leqq E\,|E(Q_j \mid X(j))| = r_j,$$

the last inequality holding because $|Q_i| \leqq 1$ with probability one.

$$E(V_n/n - M(t(p); f))^2 = E\left(n^{-1}\sum_{i=1}^{n} Q_i\right)^2$$

$$= n^{-2}\left[\sum_{i=1}^{n} E(Q_i^2) + 2\sum_{i<j} E(Q_i\,Q_j)\right] < n^{-2}\left[n + 2\sum_{i<j} r_j\right] < n^{-1} + 2n^{-1}\sum_{j=1}^{n} r_j,$$

and because r_j approaches zero as j increases, this last expression approaches zero as n increases. Therefore $E(V_n/n - M(t(p); f))^2$ converges to zero as n increases, and the stochastic convergence of V_n/n to $M(t(p); f)$ follows from Chebyshev's inequality.

The value of the statistic V_n/n may change if the values of $X_1, \cdots, X_{n+1}$ are permuted. In most fixed sample size tests of the hypothesis, the statistic is invariant under permutation of the observations.

4. A sequential test of fit. When the hypothesis of uniform distribution is true, $W_1, W_2, \cdots$ are independent random variables, with $P(W_i = 1) = p$ for $i = 1, 2, \cdots$. When the hypothesis is not true, $W_1, W_2, \cdots$ are not independent, but it has been shown that $P(W_n = 1 \mid X_1, \cdots, X_n)$ converges with probability one to $M(t(p); f)$ as n increases, where $f(x)$ is the common probability density function of $X_1, X_2, \cdots$. Wald's sequential test [2] that a binomial mean has a given value p against the alternative that its value is $p_1 (p_1 > p)$, with error probabilities α, β can be applied to the test of uniform distribution, as follows. Let a_m denote

$$\frac{\log [\beta/(1 - \alpha)] + m \log [(1 - p)/(1 - p_1)]}{\log [p_1(1 - p)] - \log [p(1 - p_1)]}$$

and r_m denote

$$\frac{\log [(1 - \beta)/\alpha] + m \log [(1 - p)/(1 - p_1)]}{\log [p_1(1 - p)] - \log [p(1 - p_1)]}.$$

The test continues as long as $a_m < W_1 + \cdots + W_m < r_m$. The first time that these inequalities do not hold, the hypothesis of uniform distribution is accepted if $W_1 + \cdots + W_m \leqq a_m$, and is rejected if $W_1 + \cdots + W_m \geqq r_m$.

When the hypothesis of uniform distribution is true, $W_1, W_2, \cdots$ are independent random variables with $P(W_i = 1) = p$, and therefore the probability of acceptance and the expected number of observations of the sequential test are known, at least approximately, through the Wald approximations. In particular, when the hypothesis is true the probability of its rejection is approximately α.

Since it has been shown that when the common probability density function is $f(x)$, $P(W_n = 1 \mid X_1, \cdots, X_n)$ converges with probability one to $M(t(p); f)$ as n increases, it is tempting to say that when the common probability density function is $f(x)$, the sequential test has approximately the same properties as the Wald test for the binomial case when the binomial mean is equal to $M(t(p); f)$. However, there are certain obstacles, which will now be discussed, in the way of doing this.

The first obstacle is the following. Even if $P(W_n = 1 \mid X_1, \cdots, X_n)$ were close to $M(t(p); f)$ for all n, might the small differences lead to large differences between the properties of the proposed test and the properties of the Wald binomial test? A negative answer to this question can be given, since the properties of the Wald test vary continuously with the binomial mean.

A second obstacle is the following. Convergence with probability one as n increases does not rule out fairly large probabilities of large differences between $P(W_n = 1 \mid X_1, \cdots, X_n)$ and $M(t(p); f)$ for small values of n. Might this lead to large differences between the properties of the proposed sequential test of uniformity and the Wald binomial test? The exact answer to this question remains unknown, and may be in the affirmative in general. However, if α and β are small, a large number of observations will be taken, as is obvious from the form of the Wald test. For the later observations in the sequence, the convergence theorem applies. From the form of the decision boundaries that the Wald test applies to the random walk, it is easily seen that even large disturbances in

$P(W_n = 1 \mid X_1, \cdots, X_n)$ for a relatively few initial values of n will have only a small effect on the properties of the test. Thus this second obstacle diminishes as α and β become smaller.

A third obstacle is the following. What is the effect of those sample sequences for which $P(W_n = 1 \mid X_1, \cdots, X_n)$ is not close to $M(t(p); f)$ even for large values of n? Since it has been shown that such sample sequences have a probability approaching zero as n increases, it is clear that they cannot have much effect on the probability of rejecting the hypothesis, if α and β are small. However, these sample sequences may conceivably have a large effect on the expected sample size, since the sample size is an unbounded function over all possible sample sequences, and the expected value of an unbounded function can be greatly affected even by small changes in the probabilities of large values. Whether this effect on the expected sample size actually exists is an open question. However, the probability that the sample size will be less than m, for any fixed m, will not be affected much, since this probability is a bounded function.

5. Comparison of the sequential test with other tests. In trying to compare the sequential test of uniform distribution with existing tests of this hypothesis, a difficulty is that for very few of the existing tests (which are all predetermined sample size tests) is even the asymptotic power known. However, in [3] the asymptotic power of the test which rejects when $\sum_{i=1}^{n+1} Z_i^2(n)$ is "too large" (n is a predetermined number) is found. Furthermore, in [4] it is shown that this test is admissible among all fixed sample size tests. This test will be called the "Z test" for convenience, and will be compared to the sequential test.

For computational purposes, p will be set equal to $\frac{1}{2}$ and α will be set equal to β. $f(x)$ will be written as $1 + cr(x)$, where $|r(x)|$ is bounded,

$$\int_0^1 r(x)\,dx = 0, \qquad \int_0^1 r^2(x)\,dx = D > 0,$$

and small absolute values of c are of interest. After straightforward but somewhat lengthy calculations, it is found that $t(\frac{1}{2}) = 1.6784 + .5693Dc^2 + o(c^2)$, and using this, that $M(t(\frac{1}{2}); f) = \frac{1}{2} + .5259Dc^2 + o(c^2)$. Set $p_1 = \frac{1}{2} + .5259Dc^2$, so that the power of the sequential test of uniform distribution against the alternative $f(x) = 1 + cr(x)$ is approximately $1 - \alpha$. Denote by $E(\alpha, c)$ the expected sample size when the sequential test is used and the hypothesis of uniform distribution is true. Then from the known Wald approximations, it is found that asymptotically

$$E(\alpha, c) = \frac{(1-\alpha)\log(\alpha/(1-\alpha)) + \alpha\log((1-\alpha)/\alpha)}{\frac{1}{2}\log(1 + 1.0518Dc^2) + \frac{1}{2}\log(1 - 1.0518Dc^2)}$$
$$= \frac{2\log\alpha - 4\alpha\log\alpha + (4\alpha - 2)\log(1-\alpha)}{-1.1067D^2c^4 + o(c^4)}$$

the approximation becoming better as c becomes smaller in absolute value.

Next the sample size necessary to give the Z test level of significance α and power $1 - \alpha$ against the alternative $f(x) = 1 + cr(x)$ is found. Denote this sample size by $N(\alpha, c)$, and denote by $k(\alpha)$ the quantity satisfying

$$(2\pi)^{-\frac{1}{2}} \int_{k(\alpha)}^{\infty} \exp\left(-\tfrac{1}{2}t^2\right) dt = \alpha$$

It is found directly from [3] that $N(\alpha, c)$ is asymptotically the solution for n in the equation

$$\frac{n^{\frac{1}{2}}\left[1 - \int_0^1 (1 + cr(x))^{-1}\, dx\right] + k(\alpha)}{\left\{2\int_0^1 (1 + cr(x))^{-3}\, dx - \left[\int_0^1 (1 + cr(x))^{-1}\, dx\right]^2\right\}^{\frac{1}{2}}} = k(1 - \alpha),$$

the approximation becoming better as α decreases. Since $k(1 - \alpha) = -k(\alpha)$, $N(\alpha, c)$ is asymptotically equal to

$$k^2(\alpha)\left[\frac{2 + 10Dc^2 + o(c^2) + 2(1 + 10Dc^2 + o(c^2))^{\frac{1}{2}}}{D^2c^4 + o(c^4)}\right].$$

From page 166 of [5], we find that $-2 \log \alpha$ approaches $\log 2\pi + 2 \log k(\alpha) + k^2(\alpha)$ asymptotically as α approaches zero. From this, it follows that $\lim_{\alpha \to 0} N(\alpha, c)/E(\alpha, c) = 4.4268 + \delta(c)$, where $\lim_{c \to 0} \delta(c) = 0$. Thus for α and c near zero, the sequential test of uniform distribution requires on the average only 1/4.4268 times the number of observations required by the Z test of the same size and power against the alternative $f(x) = 1 + cr(x)$, when the hypothesis is true. This is a substantial saving.

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