SEQUENTIAL χ^2- AND T^2-TESTS[1]

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1. Summary. Consider a multivariate normal population with mean $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_p)$ and covariance matrix $\boldsymbol{\Sigma}$. Let $\boldsymbol{\mu}_0$ be a vector of constants, ${}_n\bar{\mathbf{x}}$ a vector of sample means based on n observations, and $\mathbf{S}_n$ the corresponding sample covariance matrix. The statistics considered are

$$\chi_n^2 = n({}_n\bar{\mathbf{x}} - \boldsymbol{\mu}_0)\boldsymbol{\Sigma}^{-1}({}_n\bar{\mathbf{x}} - \boldsymbol{\mu}_0)' \tag{1.1}$$

and

$$T_n^2 = n({}_n\bar{\mathbf{x}} - \boldsymbol{\mu}_0)\mathbf{S}_n^{-1}({}_n\bar{\mathbf{x}} - \boldsymbol{\mu}_0)'. \tag{1.2}$$

It is shown that probability-ratio tests for a sequential test of the composite hypothesis,

$$H_0 : (\boldsymbol{\mu} - \boldsymbol{\mu}_0)\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \boldsymbol{\mu}_0)' = \lambda_0^2 \tag{1.3}$$

against the alternative

$$H_1 : (\boldsymbol{\mu} - \boldsymbol{\mu}_0)\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \boldsymbol{\mu}_0)' = \lambda_1^2 \tag{1.4}$$

may be based on

$$p_{1n}/p_{0n} = [\exp - n(\lambda_1^2 - \lambda_0^2)/2]\, {}_0F_1(p/2; n\lambda_1^2\chi_n^2/4)/{}_0F_1(p/2; n\lambda_0^2\chi_n^2/4) \tag{1.5}$$

when $\boldsymbol{\Sigma}$ is known and

$$p_{1n}/p_{0n} = [\exp - n(\lambda_1^2 - \lambda_0^2)/2]\, {}_1F_1[n/2, p/2; n\lambda_1^2T_n^2/2(n - 1 + T_n^2)]/ {}_1F_1[n/2, p/2; n\lambda_0^2T_n^2/2(n - 1 + T_n^2)] \tag{1.6}$$

when $\boldsymbol{\Sigma}$ is unknown and must be estimated from the sample. The sequential χ^2-test is associated with (1.5) and the sequential T^2-test with (1.6). ${}_0F_1$ and ${}_1F_1$ are respectively forms of the generalized hypergeometric function ${}_pF_q(a_1, \cdots, a_p; c_1, \cdots, c_q; x)$, the second being the confluent hypergeometric function.

It is shown that the use of these probability ratios in sequential tests results in Type I and Type II errors of approximately α and β when these values are used to obtain bounds on the probability ratios in the traditional way. It is also shown that the sequential tests terminate with probability unity. Bounds on the prob-

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ability ratios are translated into bounds on χ_n^2 and T_n^2 themselves and tables have been prepared with more tables in preparation.

Procedures are also given to test sequentially whether or not two samples come from populations with the same means. The χ^2-test is generalized to give simultaneous sequential tests on both the means and the covariance matrix.

The average sample number functions (ASN functions) are considered and approximations to them suggested. The operating characteristic functions (OC functions) are difficult to investigate and essentially are only known approximately at λ_0^2 and λ_1^2.

2. Introduction. Modern techniques of sequential analysis were largely inspired by the work of Wald, summarized in his book [27]. This work was motivated by the need to cut down on the amount of work necessary in the acceptance sampling of military supplies.

Wald's procedures are based on a probability-ratio test and were largely developed for the test of a simple hypothesis against a simple alternative. For composite hypotheses, Wald proposed a *method of weight functions* but the method is cumbersome and no method of insuring optimum weights is available. Goldberg, as reported by Wallis [28], and Nandi [19] proposed a *method of frequency functions*. This method is now generally used in considering composite hypotheses and will be used here.

The method of frequency functions was used to develop the sequential t-test in the univariate case. This work was done independently by Rushton [24] [25] and Arnold [21]. We extend these methods to the multivariate problem to obtain the sequential T^2-test and also consider a sequential χ^2-test. Applications are of consequence for, in the inspection of complex items, a number of characteristics are measured. Observations on these characteristics are often correlated and univariate sequential methods applied to each characteristic lead to confusion.

For the method of frequency functions, *observations* are successive values of the test-statistic and hence observations are no longer independent. Wald showed that the probability-ratio test could be used to obtain bounds on the test-statistic even though observations are not independent, but his work on termination, OC functions, and ASN functions no longer applies. Barnard [2] [3] and Cox [5] have independently established conditions under which the frequency function of a test statistic might be used in a sequential probability-ratio test and still guarantee approximately the risks α and β. We quote Cox's theorem for later use.

THEOREM. "*Let* $\mathbf{x} = [x_1, \cdots, x_n]$ *be random variables whose probability density function* (p.d.f.) *depends on unknown parameters* $\theta_1, \cdots, \theta_p$. *The* x_i *themselves may be vectors. Suppose that*

(i) $t_1, \cdots, t_p$ *are a functionally independent jointly sufficient set of estimators for* $\theta_1, \cdots, \theta_p$;

(ii) *the distribution of* t_1 *involves* θ_1 *but not* $\theta_2, \cdots, \theta_p$;

(iii) $u_1, \cdots, u_m$ *are functions of* $\mathbf{x}$ *functionally independent of each other and of* $t_1, \cdots, t_p$;

(iv) *there exists a set* $\mathcal{S}$ *of transformations of* $\mathbf{x} = [x_1, \cdots, x_n]$ *into* $\mathbf{x}^* = [x_1^*, \cdots, x_n^*]$ *such that*

(a) $t_1, u_1, \cdots, u_m$ *are unchanged by all transformations in* $\mathcal{S}$;

(b) *the transformation of* $t_2, \cdots, t_p$ *into* $t_2^*, \cdots, t_p^*$ *defined by each transformation in* $\mathcal{S}$ *is one-to-one*;

(c) *if* $T_2, \cdots, T_p$ *and* $T_2^*, \cdots, T_p^*$ *are two sets of values of* $t_2, \cdots, t_p$ *each having non-zero probability density under at least one of the distributions of* $\mathbf{x}$, *then there exists a transformation in* $\mathcal{S}$ *such that if* $t_2 = T_2, \cdots, t_p = T_p$, *then* $t_2^* = T_2^*, \cdots, t_p^* = T_p^*$.

Then the joint p.d.f. *of* $t_1, u_1, \cdots, u_m$ *factorizes into*

$$g(t_1 \mid \theta_1)\ell(u_1, \cdots, u_m, t_1),$$

where g *is the* p.d.f. *of* t_1 *and* ℓ *does not involve* θ_1." The proof of this theorem is given in Cox's paper.

The application of the theorem is straight-forward. The theorem permits factorization of the sample p.d.f. in such a way that the probability ratio attempted from the sample p.d.f.'s under null and alternative hypotheses reduces to the probability ratio for the test statistic under null and alternative hypotheses. The composite hypotheses have been reduced to simple hypotheses on a single parameter involved in the distributions of the test statistics. In repetition,

$$p_{1n}/p_{0n} = g(t_{1n} \mid \theta_1)/\, g(t_{1n} \mid \theta_0) \tag{2.1}$$

in the notation of the theorem and where t_{1n} is the statistic t_1 based on n observations. The sequential test is as follows:

(i) Accept H_0 if $p_{1n}/p_{0n} \leqq \beta/(1 - \alpha)$.

(ii) Accept H_1 if $p_{1n}/p_{0n} \geqq (1 - \beta)/\alpha$.

(iii) Continue sampling if $\beta/(1 - \alpha) < p_{1n}/p_{0n} < (1 - \beta)/\alpha$.

Provided that the probability is one that the test terminates, the probabilities of error under the null and alternative hypotheses are approximately α and β respectively (Wald [27], p. 43).

We test H_0 of (1.3) against H_1 of (1.4) using χ_n^2 in (1.1) or T_n^2 in (1.2) depending on whether $\mathbf{\Sigma}$ is known or not. For the sequential χ^2-test, the probability ratio is the ratio of two non-central χ^2-densities as shown in (1.5). For the sequential T^2-test, the probability is the ratio of two non-central T^2-densities as shown in (1.6). In both (1.5) and (1.6) some simple combinations of terms have been made to obtain the forms shown.

3. Fulfillment of the conditions of Cox's Theorem. Verification of the conditions of Cox's Theorem has not been included by other authors using the theorem. Since we have not found the necessary verifications trivial either for this paper or others already published, we include a sketch of the required demonstrations.

Let $\mathbf{x}$ be the vector $(\mathbf{x}_1, \cdots, \mathbf{x}_n)$ where $\mathbf{x}_i$ is the ith observation vector on

p multivariate normal variates. Then $\mathbf{x}$ consists of n independent, equally distributed, multivariate normal observation vectors. The vector of variate means is $\boldsymbol{\mu}$ and the dispersion matrix is $\boldsymbol{\Sigma}$ assumed known for the χ^2-test and unknown for the T^2-test. The sample covariance matrix corresponding to $\boldsymbol{\Sigma}$ is $\mathbf{S}_n$. We may let $\dot{\mathbf{x}}_i = \mathbf{x}_i - \boldsymbol{\mu}_0$ and $\dot{\mathbf{x}} = (\dot{\mathbf{x}}_1, \cdots, \dot{\mathbf{x}}_n)$ and regard $\dot{\mathbf{x}}$ as the original observation matrix of the Cox Theorem. Note that $\mathbf{S}_n$ is invariant under such changes in location.

Condition (i). A vector of sufficient statistics for the elements of $\boldsymbol{\mu}$ is ${}_n\bar{\mathbf{x}}$ and for $\dot{\boldsymbol{\mu}} = \boldsymbol{\mu} - \boldsymbol{\mu}_0$ is ${}_n\bar{\dot{\mathbf{x}}}$; the elements of $\mathbf{S}_n$ and ${}_n\bar{\mathbf{x}}$ are sufficient for $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. (Cf. Anderson [1], Sec. 3.3.3.) Define $\mathbf{GG}' = \boldsymbol{\Sigma}^{-1}$ and $\mathbf{EE}' = \mathbf{S}_n^{-1}$. We transform so that

$$ {}_n\mathbf{y}' = n^{\frac{1}{2}}\boldsymbol{\Delta}'({}_n\bar{\mathbf{x}} - \boldsymbol{\mu}_0)' = n^{\frac{1}{2}}\boldsymbol{\Delta}'\,{}_n\bar{\dot{\mathbf{x}}}' \tag{3.1} $$

and

$$ \boldsymbol{\eta}' = \mathbf{G}'(\boldsymbol{\mu} - \boldsymbol{\mu}_0)' = \mathbf{G}'\dot{\boldsymbol{\mu}}' \tag{3.2} $$

with $\boldsymbol{\Delta} = \mathbf{G}$ for the χ^2-test and $\boldsymbol{\Delta} = \mathbf{E}$ for the T^2-test. A spherical transformation is used to transform ${}_n\mathbf{y}$ to χ_n^2 os T_n^2, $a_{1n}, \cdots, a_{p-1,n}$, χ_n^2 os $T_n^2 \geqq 0, 0 \leqq a_{in} \leqq \pi$, $i = 1, \cdots, (p-2)$, $0 \leqq a_{p-1,n} \leqq 2\pi$. A similar transformation transforms $\boldsymbol{\eta}$ to $\lambda^2 = (\boldsymbol{\mu} - \boldsymbol{\mu}_0)\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \boldsymbol{\mu}_0)'$, $\alpha_1, \cdots, \alpha_{p-1}$, $\lambda^2 \geqq 0, 0 \leqq \alpha_i \leqq \pi$, $i = 1, \cdots, (p-2)$, $0 \leqq \alpha_{p-1} \leqq 2\pi$. The transformations on ${}_n\bar{\mathbf{x}}$ and $\boldsymbol{\mu}$ and on ${}_n\bar{\dot{\mathbf{x}}}$ and $\dot{\boldsymbol{\mu}}$ are one-to-one and it follows that $\chi_n^2, a_{1n}, \cdots, a_{p-1,n}$ are a set of sufficient statistics for $\lambda^2, \alpha_1, \cdots, \alpha_{p-1}$ and $T_n^2, a_{1n}, \cdots, a_{p-1,n}, \mathbf{S}_n$ for $\lambda^2, \alpha_1, \cdots, \alpha_{p-1}, \boldsymbol{\Sigma}$. We associate the sets of statistics with $t_1, \cdots, t_p$ in the theorem and the sets of parameters with $\theta_1, \cdots, \theta_p$ of the theorem; condition (i) follows.

Condition (ii). It was shown by Fisher [8] that the marginal distribution of χ_n^2 involves only λ^2 and Hsu [14] and Bose and Roy [4] have the result for T_n^2. Condition (ii) is met when λ^2 is associated with θ_1 and χ_n^2 or T_n^2 with t_1.

Condition (iii). To consider the third condition of the theorem, we associate $u_1, \cdots, u_m$ with $\chi_1^2, \cdots, \chi_{n-1}^2$ or $T_{p+1}^2, \cdots, T_{n-1}^2$. It is necessary to show that the statistics in the sets are functionally independent of each other and of χ_n^2, $a_{1n}, \cdots, a_{p-1,n}$ or of $T_n^2, a_{1n}, \cdots, a_{p-1,n}, \mathbf{S}_n$.

Condition (iii) seems intuitively obvious. Formal proof depends on a series of transformations. We rearrange the elements of $\dot{\mathbf{x}}$ to yield a p by n matrix $\dot{\mathbf{X}}'$ with (i, j) element $\dot{x}_{ij} = x_{ij} - \mu_{i0}$, $i = 1, \cdots, p, j = 1, \cdots, n$. Let

$$ \mathbf{Y} = \mathbf{M}\dot{\mathbf{X}} \tag{3.3} $$

where $\mathbf{M}$ is the non-singular n-square matrix with jth row given by

$$ [j^{-\frac{1}{2}}, \cdots, j^{-\frac{1}{2}}, 0, \cdots, 0], $$

the number of non-zero elements being j. The rows of $\mathbf{Y}$ now depend on the sample means of the p variates, the jth row on the means of the first j observations, $j = 1, \cdots, n$. At the next stage we transform $\mathbf{Y}_j$, the jth row of $\mathbf{Y}$, so that

$$ \mathbf{Z}_j = \mathbf{Y}_j\boldsymbol{\Delta}_j \tag{3.4} $$

where $\mathbf{\Delta}_j = \mathbf{G}$ for the χ^2-test, $j = 1, \cdots, n$, and $\mathbf{\Delta}_j = \mathbf{E}_j$, $\mathbf{E}_j\mathbf{E}_j' = \mathbf{S}_j$, $j = p + 1, \cdots, n$, for the T^2-test; $\mathbf{S}_j$ is the sample dispersion matrix similar to $\mathbf{S}_n$ but based on the first j observation vectors. A spherical transformation is now applied to each $\mathbf{Z}_j$. For $\mathbf{Z}_j$, we obtain new variables χ_j^2, $a_{1j}, \cdots, a_{p-1,j}$ with $\chi_j^2 = \sum_{i=1}^p z_{ij}^2 = j\, {}_j\bar{\mathbf{x}}\mathbf{\Sigma}^{-1}\, {}_j\bar{\mathbf{x}}'$ or T_j^2, $a_{1j}, \cdots, a_{p-1,j}$ with $T_j^2 = \sum_{i=1}^p z_{ij}^2 = j\, {}_j\bar{\mathbf{x}}\mathbf{S}_j^{-1}\, {}_j\bar{\mathbf{x}}'$. For the χ^2-test, through the series of non-singular, one-to-one transformations sketched here, $\mathbf{x}$ or $\dot{\mathbf{x}}$ has been transformed to a set of new variables such that $\chi_1^2, \cdots, \chi_{n-1}^2$ are functionally independent of each other and of χ_n^2, $a_{1n}, \cdots, a_{p-1,n}$. Similarly, for the T^2-test, each successive T_j^2 is a function of one more row of $\dot{\mathbf{X}}$ and T_n^2, $a_{1n}, \cdots, a_{p-1,n}$, $\mathbf{S}_n$ all depend on all rows of $\dot{\mathbf{X}}$. Condition (iii) follows from these considerations.

Condition (iv). We rewrite Condition (iv) in terms of the present problems: There exists a set of transformations $\mathcal{S}$ of $\mathbf{X}$ into $\mathbf{X}^*$ such that:

(a) $\chi_1^2, \cdots, \chi_n^2$ (or $T_{p+1}^2, \cdots, T_n^2$) are unchanged by all transformations in $\mathcal{S}$.

(b) The transformation of $a_{1n}, \cdots, a_{p-1,n}$ (and $\mathbf{S}_n$ for the T^2-test) into $a_{1n}^*, \cdots, a_{p-1,n}^*$ (and $\mathbf{S}_n^*$) defined by each transformation in $\mathcal{S}$ is one-to-one.

(c) If $A_1, \cdots, A_{p-1}$, (and $\mathfrak{S}_n$) and $A_1^*, \cdots, A_{p-1}^*$, (and $\mathfrak{S}_n^*$) are two sets of values of $a_{1n}, \cdots, a_{p-1,n}$ (and $\mathbf{S}_n$), each having non-zero probability density under at least one of the distributions of $\mathbf{X}$, there exists a transformation in $\mathcal{S}$ such that, if $a_{1n} = A_1, \cdots, a_{p-1,n} = A_{p-1}$, $(\mathbf{S}_n = \mathfrak{S}_n)$, then $a_{1n}^* = A_1^*, \cdots, a_{p-1,n}^* = A_{p-1}^*$, $(\mathbf{S}_n = \mathfrak{S}_n^*)$.

The necessary classes of transformations are

$$\dot{\mathbf{X}}^* = \dot{\mathbf{X}}\mathbf{G}\mathbf{B}\mathbf{G}^{-1} \quad \text{or} \quad \dot{\mathbf{X}}^* = \dot{\mathbf{X}}\mathbf{E}\mathbf{B}\mathbf{C}' \tag{3.5}$$

respectively for the χ^2- and T^2-tests where $\mathbf{B}$ is any p by p orthogonal matrix and $\mathbf{C}$ is any non-singular, triangular, p by p matrix.

We first consider the χ^2-test. From (3.3) and (3.4), $\mathbf{Z} = \mathbf{M}\dot{\mathbf{X}}\mathbf{G}$ and $\mathbf{ZB} = \mathbf{M}\dot{\mathbf{X}}\mathbf{GB} = \mathbf{M}\dot{\mathbf{X}}^*\mathbf{G} = \mathbf{Z}^*$. The transformation from $\mathbf{Z}$ to $\mathbf{Z}^*$ is orthogonal. Parts (a) and (b) of Condition (iv) follow at once. The sums of squares of elements in rows of $\mathbf{Z}$ equal the corresponding sums of squares for $\mathbf{Z}^*$ and are $\chi_1^2, \cdots, \chi_n^2$. The transformation of $\mathbf{Z}_n$ into $\mathbf{Z}_n^*$ is one-to-one for each $\mathbf{B}$ and, since χ_n^2, $a_{1n}, \cdots, a_{p-1,n}$ follow from a spherical transformation on the elements of $\mathbf{Z}_n$ and the corresponding transformation applies to $\mathbf{Z}_n^*$, we have a one-to-one transformation of $a_{1n}, \cdots, a_{p-1,n}$ to $a_{1n}^*, \cdots, a_{p-1,n}^*$.

If $A_1, \cdots, A_{p-1}$ and $A_1^*, \cdots, A_{p-1}^*$ are two sets of values of $a_{1n}, \cdots, a_{p-1,n}$ suitably restricted between 0 and π or 0 and 2π as the case may be, then $\mathbf{Z}_n$ and $\mathbf{Z}_n^*$ may be evaluated except for the scalar χ_n which is the same in both cases. If these specified values yield $\mathbf{Z}_n$ and $\mathbf{Z}_n^*$, they are related by $\mathbf{Z}_n^* = \mathbf{Z}_n\mathbf{B}$ and this equation defines $(p - 1)$ independent equations on the elements of $\mathbf{B}$. There are also $p(p + 1)/2$ additional equations on the elements of $\mathbf{B}$ imposed through the requirement that $\mathbf{B}$ be orthogonal. The solution for the p^2 elements of $\mathbf{B}$ is not unique (except for $p = 2$) but matrices $\mathbf{B}$ satisfying the requirements may be found and this is sufficient for (c) of Condition (iv).

For the T^2-test, $\mathbf{Z}_j^*/j^{\frac{1}{2}} = {}_j\bar{\mathbf{x}}^* = {}_j\bar{\mathbf{x}}\mathbf{EBC}'$ and $\mathbf{S}_j^* = \mathbf{CB}'\mathbf{E}'\mathbf{S}_j\mathbf{EBC}'$,

$j = p + 1, \cdots, n$. Note that $\mathbf{S}_n^* = \mathbf{CC}'$ because $\mathbf{E}'\mathbf{S}_n\mathbf{E} = \mathbf{I}$. Part (a) of Condition (iv) follows since

$$T_j^{*2} = j\ {}_j\bar{\mathbf{x}}^*\mathbf{S}_j^{*-1}\ {}_j\bar{\mathbf{x}}^{*\prime} = j\ {}_j\bar{\mathbf{x}}\mathbf{EBC}'\mathbf{C}'^{-1}\mathbf{B}'\mathbf{E}^{-1}\mathbf{S}_j^{-1}\mathbf{E}'^{-1}\mathbf{BC}^{-1}\mathbf{CB}'\mathbf{E}_j'\ {}_j\bar{\mathbf{x}}'$$
$$= j\ {}_j\bar{\mathbf{x}}\mathbf{S}_j^{-1}\ {}_j\bar{\mathbf{x}}' = T_j^2, \qquad j = p + 1, \cdots, n.$$

From (3.3), (3.4) and (3.5) it follows that $\mathbf{Z}_n^* = \mathbf{Z}_n\mathbf{B}$. Given $\mathbf{B}$ and $\mathbf{C}$, the transformations of $\mathbf{S}_n$ into $\mathbf{S}_n^*$ and $\mathbf{Z}_n$ into $\mathbf{Z}_n^*$ are one-to-one and the transformation of $\mathbf{Z}_n$ into $\mathbf{Z}_n^*$ actually transforms $T_n^2, a_{1n}, \cdots, a_{p-1,n}$ into $T_n^{*2}, a_{1n}^*, \cdots, a_{p-1,n}^*$ with $T_n^2 = T_n^{*2}$. Part (b) is thus verified. In regard to Part (c), the existence of an appropriate $\mathbf{B}$ is demonstrated exactly as for the χ^2-test and $\mathbf{C}$ is defined by $\mathfrak{S}_n^*$ and $\mathbf{E}$ by $\mathfrak{S}_n$. Hence the required transformation in $\mathcal{S}$ exists and Part (c) also follows for the T^2-test.

All of the conditions of the theorem have been fulfilled for both the χ^2- and the T^2-tests. Hence the joint p.d.f.'s of $\chi_1^2, \cdots, \chi_n^2$ or $T_{p+1}^2, \cdots, T_n^2$ factor into $g(\chi_n^2 \mid n\lambda^2)\ell(\chi_1^2, \cdots, \chi_n^2)$ or $g(T_n^2 \mid n\lambda^2)\ell(T_{p+1}^2, \cdots, T_n^2)$ and p_{1n}/p_{0n} can be written as $g(\chi_n^2 \mid n\lambda_1^2)/g(\chi_n^2 \mid n\lambda_0^2)$ or $g(T_n^2 \mid n\lambda_1^2)/g(T_n^2 \mid n\lambda_0^2)$.

The first of these is the ratio of two non-central χ^2- densities with p degrees of freedom and non-centrality parameters $n\lambda_1^2$ and $n\lambda_0^2$ reducing to (1.5); the second is the ratio of two non-central T^2-densities with degrees of freedom p and non-centrality parameters $n\lambda_1^2$ and $n\lambda_0^2$ reducing to (1.6).

In many situations $\lambda_0^2 = 0$ and then (1.5) reduces to

$$(3.6) \qquad p_{1n}/p_{0n} = e^{-\frac{1}{2}n\lambda_1^2}\ {}_0F_1(p/2; n\lambda_1^2\chi_n^2/4)$$

while (1.6) becomes

$$(3.7) \qquad p_{1n}/p_{0n} = e^{-\frac{1}{2}n\lambda_1^2}\ {}_1F_1[n/2, p/2; n\lambda_1^2T_n^2/2(n - 1 + T_n^2)].$$

Furthermore, if $p = 1$, (3.6) reduces to

$$(3.8) \qquad p_{1n}/p_{0n} = e^{-\frac{1}{2}n\lambda_1^2} \cosh\left[\lambda_1 \sum_{j=1}^{n} (x_j - \mu_0)/\sigma\right]$$

and (3.7) to

$$(3.9) \qquad p_{1n}/p_{0n} = e^{-\frac{1}{2}n\lambda_1^2}\ {}_1F_1[n/2, \tfrac{1}{2}; n\lambda_1^2t^2/2(n - 1 + t^2)].$$

Equation (3.8) is equation 9.5, page 135, in Wald's book; equation (3.9) is equation 5 given by Rushton [25] for the univariate sequential t-test.

A referee and W. J. Hall have pointed out that Cox's Theorem may be by-passed through use of an unpublished theorem of Stein and the fact that χ_n^2 and T_n^2 are maximal invariants of sufficient statistics under certain groups of linear transformations implying that they are sufficient for any invariant statistics. This means that χ_n^2 is sufficient for $(\chi_1^2, \cdots, \chi_n^2)$ and T_n^2 is sufficient for $(T_{p+1}^2, \cdots, T_n^2)$ and the factorizations resulting from Cox's Theorem are immediate. The reader is referred to Hall [12]. We have not relied on this approach for reasons set forth at the beginning of this section and because the transformations involved are of interest in themselves.

4. Termination proofs. Let $\underline{\chi}_n^2$ and $\bar{\chi}_n^2$ be the boundary values for χ_n^2 corresponding to $p_{1n}/p_{0n} = \beta/(1 - \alpha)$ and $p_{1n}/p_{0n} = (1 - \beta)/\alpha$ where p_{1n}/p_{0n} is defined in (1.5).

We assert that

$$P(\text{Sample Size} > n) \leqq P(\underline{\chi}_n^2 < \chi_n^2 < \bar{\chi}_n^2) = P_n$$

and proof of termination follows if we show that $\lim_{n\to\infty} P_n = 0$. This approach is similar to that given by Ray [23] who considered sequential analysis of variance.

Set $U_n^2 = \chi_n^2/n$ and consider the corresponding limits $\underline{U}_n^2$ and $\bar{U}_n^2$. Erdelyi et al [7] shows that

$${}_0F_1(c; x) = e^{-2\sqrt{x}}\, {}_1F_1[(2c - 1)/2, 2c - 1; 4\sqrt{x}]$$

and, when this is applied in (1.5) and χ_n^2 replaced by nU_n^2, we have

$$p_{1n}/p_{0n} = f_n(U_n^2) = [\exp [\{-n(\lambda_1^2 - \lambda_0^2)/2\} - (n^2\lambda_1^2U_n^2)^{\frac{1}{2}} + (n^2\lambda_0^2U_n^2)^{\frac{1}{2}}]] \frac{{}_1F_1[(p - 1)/2, p - 1; 2(n^2\lambda_1^2U_n^2)^{\frac{1}{2}}]}{{}_1F_1[(p - 1)/2, p - 1; 2(n^2\lambda_0^2U_n^2)^{\frac{1}{2}}]}.$$

Intersections of the family of curves $y = g_n(U^2) = \ln f_n(U^2)$ and of $y = \ln [\beta/(1 - \alpha)]$ and $y = \ln [(1 - \beta)/\alpha]$ determine $\underline{U}_n^2$ and $\bar{U}_n^2$ respectively.

It can be shown that

$$g_n'(U^2) > 0 \qquad \text{for } U^2 > 0$$

and

$$\lim_{n\to\infty} g_n'(U^2) = \infty.$$

It may be demonstrated also that

$$g_n(U^2) = n[\{-(\lambda_1^2 - \lambda_0^2)/2\} + (U^2)^{\frac{1}{2}}[(\lambda_1^2)^{\frac{1}{2}} - (\lambda_0^2)^{\frac{1}{2}}] + O(1/n)].$$

These results are straightforward, being based on the proposition that

$${}_1F_1(a, c; x) = [\Gamma(c)/\Gamma(a)]e^x x^{a-c}[1 + O(|x|^{-1})]$$

given by Erdelyi. It follows that $y = g_n(U^2)$ defines a family of curves starting at $y = -n(\lambda_1^2 - \lambda_0^2)/2$ for $U^2 = 0$ and increasing strictly to $+\infty$ as $U^2 \to \infty$. Hence $g_n(U^2) = 0$ has one root and it is

$$U_0^2 = [\lambda_1^2 + 2(\lambda_1^2\lambda_0^2)^{\frac{1}{2}} + \lambda_0^2]/4 + O(1/n).$$

The intersection of the horizontal line $y = \ln [\beta/(1 - \alpha)]$ with $y = g_n(U^2)$ occurs where

$$-[(\lambda_1^2 - \lambda_0^2)/2] + (U^2)^{\frac{1}{2}}[(\lambda_1^2)^{\frac{1}{2}} - (\lambda_0^2)^{\frac{1}{2}}] + O(1/n) = n^{-1}\ln [\beta/(1 - \alpha)]$$

and hence

$$\underline{U}_n^2 = [\lambda_1^2 + 2(\lambda_1^2\lambda_0^2)^{\frac{1}{2}} + \lambda_0^2]/4 + O(1/n) = U_0^2 + O(1/n).$$

Similarly,

$$\bar{U}_n^2 = U_0^2 + O(1/n).$$

Consider U_n^2. This is a random variable that converges stochastically to $\lambda^2 = (\mathbf{\mu} - \mathbf{\mu}_0)\mathbf{\Sigma}^{-1}(\mathbf{\mu} - \mathbf{\mu}_0)'$ as $n \to \infty$. If $\lambda^2 \neq U_0^2$, the sequential process terminates with probability 1. If $\lambda^2 = U_0^2$, more powerful methods are required.

The work of David and Kruskal [6] suggested the following argument. Let P_T be the probability of termination of the sequential test.

$$P_T \geqq 1 - P(\underline{\chi}_n^2 \leqq \chi^2 \leqq \bar{\chi}_n^2) \geqq 1 - (\bar{\chi}_n^2 - \underline{\chi}_n^2) \sup_{(\underline{\chi}_n^2, \bar{\chi}_n^2)} h(\chi^2; p, n\lambda^2)$$

where $h(\chi^2; p, n\lambda^2)$ is the noncentral chi-square density with p degrees of freedom and parameter of noncentrality $n\lambda^2$. We show below that $\lim_{n\to\infty} (\bar{\chi}_n^2 - \underline{\chi}_n^2)$ is finite and $\lim_{n\to\infty} \sup_{(\underline{\chi}_n^2, \bar{\chi}_n^2)} h(\chi^2; p, n\lambda^2) = 0$ as may be shown by examination of that density. Hence $P_T \geqq \lim_{n\to\infty} [1 - P(\underline{\chi}_n^2 \leqq \chi^2 \leqq \bar{\chi}_n^2)] = 1$ and $P_T = 1$. It remains to show that $\lim_{n\to\infty} (\bar{\chi}_n^2 - \underline{\chi}_n^2)$ is finite.

We use U_n^2, $\bar{U}_n^2$ and $\underline{U}_n^2$ and recall that $\bar{U}_n^2$ and $\underline{U}_n^2$ are such that $g_n(\underline{U}_n^2) = \ln [\beta/(1 - \alpha)]$ and $g_n(\bar{U}_n^2) = \ln [\alpha/(1 - \beta)]$. Consider $g_n(U_n^2)$ in the interval $(\underline{U}_n^2, \bar{U}_n^2)$. We apply the law of the mean in this interval and divide both sides by n to obtain

$$\frac{\ln\left(\frac{1-\beta}{\beta} \cdot \frac{1-\alpha}{\alpha}\right)}{n(\bar{U}_n^2 - \underline{U}_n^2)} = \frac{g_n'[\underline{U}_n^2 + \theta(\bar{U}_n^2 - \underline{U}_n^2)]}{n} \tag{4.1}$$

where $0 \leqq \theta \leqq 1$.

$$g_n'(U^2) = \tfrac{1}{2}\left(\frac{n^2\lambda_1^2}{U^2}\right)^{\frac{1}{2}}\left\{\frac{{}_1F_1[(p+1)/2, p; 2(n^2\lambda_1^2U^2)^{\frac{1}{2}}]}{{}_1F_1[(p-1)/2, p-1; 2(n^2\lambda_1^2U^2)^{\frac{1}{2}}]} - 1\right\}$$
$$- \tfrac{1}{2}\left(\frac{n^2\lambda_0^2}{U^2}\right)^{\frac{1}{2}}\left\{\frac{{}_1F_1[(p+1)/2, p; 2(n^2\lambda_0^2U^2)^{\frac{1}{2}}]}{{}_1F_1[(p-1)/2, p-1; 2(n^2\lambda_0^2U^2)^{\frac{1}{2}}]} - 1\right\}.$$

But

$${}_1F_1(a, c; x) = [\Gamma(c)/\Gamma(a)]e^x x^{a-c}[1 + O(|x|^{-1})] \quad \text{and}$$
$$\frac{g_n'(U^2)}{n} = \tfrac{1}{2}\left[\left(\frac{\lambda_1^2}{U^2}\right)^{\frac{1}{2}} - \left(\frac{\lambda_0^2}{U^2}\right)^{\frac{1}{2}}\right] + O\left(\frac{1}{n}\right).$$

Now

$$\underline{U}_n^2, \bar{U}_n^2 \to U_0^2 = \left(\frac{(\lambda_1^2)^{\frac{1}{2}} + (\lambda_0^2)^{\frac{1}{2}}}{2}\right)^2$$

and in the neighborhood of U_0^2 [in the interval $(\underline{U}_n^2, \bar{U}_n^2)$],

$$\frac{g'_n(U^2)}{n} = \frac{(\lambda_1^2)^{\frac{1}{2}} - (\lambda_0^2)^{\frac{1}{2}}}{(\lambda_1^2)^{\frac{1}{2}} + (\lambda_0^2)^{\frac{1}{2}}} + O(1/n)$$

and $\lim_{n\to\infty} [g'_n(U^2)/n] > 0$ in the neighborhood when $\lambda_1^2 > \lambda_0^2$ as required. Returning to (4.1), we find that the right-hand side has a finite limit and consequently $n(\bar{U}_n^2 - \underline{U}_n^2) = \bar{\chi}_n^2 - \chi_n^2$ has a finite limit.

The termination proof for the sequential χ^2-test is now complete.

It is well known that the non-central T^2-distribution approaches the non-central χ^2-distribution asymptotically with n. Then the argument above applies. $\underline{T}_n^2$ and $\bar{T}_n^2$, the boundary values for the Sequential T^2-test, approach $n[\lambda_1^2 + 2(\lambda_1^2\lambda_0^2)^{\frac{1}{2}} + \lambda_0^2]/4$ also.

5. Two-sample cases. The sequential techniques discussed can also be used for two-sample tests with paired observation vectors. Let the first population have mean vector $\mathbf{\mu}^{(1)}$ and dispersion matrix $\mathbf{\Sigma}_{11}$, the second $\mathbf{\mu}^{(2)}$ and $\mathbf{\Sigma}_{22}$. Suppose further that the cross-covariance matrix is $\mathbf{\Sigma}_{12}$. Let $\mathbf{\mu} = \mathbf{\mu}^{(1)} - \mathbf{\mu}^{(2)}$ and $\mathbf{x}_i = \mathbf{x}_i^{(1)} - \mathbf{x}_i^{(2)}$, $i = 1, \cdots, n$ where $\mathbf{x}_i^{(1)}$ and $\mathbf{x}_i^{(2)}$ are respectively the ith observation vectors for populations 1 and 2. The dispersion matrix of $\mathbf{x}$ is $\mathbf{\Sigma}_{11} + \mathbf{\Sigma}_{22} - \mathbf{\Sigma}_{12} - \mathbf{\Sigma}'_{12}$. Now when $\mathbf{\Sigma}_{11}$, $\mathbf{\Sigma}_{22}$ and $\mathbf{\Sigma}_{12}$ are known, the two sample problem is reduced to an application of the Sequential χ^2-test.

When the variance-covariance matrices are not known and must be estimated, the situation is even simpler. Again we define $\mathbf{\mu} = \mathbf{\mu}^{(1)} - \mathbf{\mu}^{(2)}$ and use variates $\mathbf{x} = \mathbf{x}^{(1)} - \mathbf{x}^{(2)}$. $\mathbf{\Sigma}_{11} + \mathbf{\Sigma}_{22} - \mathbf{\Sigma}_{12} - \mathbf{\Sigma}'_{12}$ is estimated directly from the observation vectors $\mathbf{x}_i = \mathbf{x}_i^{(1)} - \mathbf{x}_i^{(2)}$, $i = 1, \cdots, n$ and this problem is reduced to that handled by the Sequential T^2-test.

6. The ASN functions. In the planning of sequential experiments information on the expected sample sizes (S) is desirable. Wald [27] established approximate procedures for determining the ASN function when sequential observations are independent. These procedures have not been demonstrated to be valid when successive values, not independent, of a test statistic are considered.

Johnson [18] circumvented the problem by sampling additional groups rather than additional items within a group and hence considered successive independent values of a test statistic. This could be done for the Sequential χ^2- and T^2-tests but usually would require too many observations.

Rushton [24, 25], in discussing the univariate sequential t-test, reduced the problem to a one-parameter problem by assuming that, after a number of observations, the variance was known. This led to lower bounds on the ASN function. This procedure does not seem applicable in the multivariate case because, even if $\mathbf{\Sigma}$ is assumed known, we are still dealing with composite hypotheses.

A third approach is the Monte Carlo technique. This has been used for the univariate sequential t-test and Freund and Appleby [9] have studied our tests.

A fourth method may be attempted. The method is based on the fact that, if one ignores excesses over the boundaries at the termination of a sequential test,

$$\mathcal{E}[\ln (p_{1n}/p_{0n})] = (1 - \alpha)\ln [\beta/(1 - \alpha)] + \alpha \ln [(1 - \beta)/\alpha] \tag{6.1}$$

where H_0 is true and

$$\mathcal{E}[\ln (p_{1n}/p_{0n})] = \beta \ln [\beta/(1 - \alpha)] + (1 - \beta) \ln [(1 - \beta)/\alpha] \tag{6.2}$$

where H_1 is true. Now, in general, $\ln (p_{1n}/p_{0n})$ will depend on n and a test statistic T_n based on the first n observations. What is needed is to express $\mathcal{E}[\ln (p_{1n}/p_{0n})]$ as a function of $\mathcal{E}(n)$, the ASN number, (and of the parameters involved) and to then solve (6.1) and (6.2) for $\mathcal{E}_0(n)$ and $\mathcal{E}_1(n)$, the required ASN numbers under H_0 and H_1 respectively. But a way of doing this has not been found for any sequential tests of composite hypotheses so far as can be ascertained by the authors. Bhate, in unpublished work, proposed approximating to $\mathcal{E}[\ln (p_{1n}/p_{0n})]$ by replacing T_n and n in $\ln (p_{1n}/p_{0n})$ by $\mathcal{E}[T_n \mid n = \mathcal{E}(n)]$, the fixed sample-size expectation of T_n given that $n = \mathcal{E}(n)$, and $\mathcal{E}(n)$ respectively. The expectation $\mathcal{E}[T_n \mid n = \mathcal{E}(n)]$ is obtained under H_0 for (6.1) and under H_1 for (6.2). This procedure is seen intuitively to give a "central value" for the distribution of $\ln (p_{1n}/p_{0n})$ and, upon appropriate substitutions in (6.1) and (6.2), to give equations in $\mathcal{E}_0(n)$ and $\mathcal{E}_1(n)$, values of $\mathcal{E}(n)$ under H_0 and H_1 respectively, for solution. The method, crude though it may appear, has been used in a number of situations, for example, by Ray [22] for sequential analysis of variance and most recently by Hajnal [11] for a two-sample sequential t-test.

We have tried the method for the sequential χ^2- and T^2-tests of this paper. Since $\mathcal{E}[\chi_n^2 \mid n = \mathcal{E}(n)] = (p + n\lambda^2) \mid_{n=\mathcal{E}(n)}$ and, using p_{1n}/p_{0n} in (1.5), we obtained the equations

$$\begin{aligned} -\tfrac{1}{2}n(\lambda_1^2 - \lambda_0^2) &+ \ln {}_0F_1[p/2; n\lambda_1^2(p + n\lambda_0^2)/4] \\ &- \ln {}_0F_1[p/2; n\lambda_0^2(p + n\lambda_0^2)/4] \\ &= (1 - \alpha) \ln [\beta/(1 - \alpha)] + \alpha \ln [(1 - \beta)/\alpha] \end{aligned} \tag{6.3}$$

and

$$\begin{aligned} -\tfrac{1}{2}n(\lambda_1^2 - \lambda_0^2) &+ \ln {}_0F_1[p/2; n\lambda_1^2(p + n\lambda_1^2)/4] \\ &- \ln {}_0F_1[p/2; n\lambda_0^2(p + n\lambda_1^2)/4] \\ &= \beta \ln [\beta/(1 - \alpha)] + (1 - \beta) \ln [(1 - \beta)/\alpha]. \end{aligned} \tag{6.4}$$

For brevity, these equations have been written in terms of n but solution of (6.3) for n yields $\mathcal{E}_0(n)$ and of (6.4), $\mathcal{E}_1(n)$, both for the χ^2-test. For the T^2-test, let $x = T_n^2/2(n - 1 + T_n^2)$ and note that

$$\begin{aligned} \mathcal{E}[x \mid n = \mathcal{E}(n)] = (n\lambda^2/2)\{1 &- [\exp (-n\lambda^2/2)][(n - p)/n] \\ &\cdot {}_1F_1[n/2, (n + 2)/2; n\lambda^2/2]\}\mid_{n=\mathcal{E}(n)} \end{aligned}$$

following Wishart [29]. With p_{1n}/p_{0n} in (1.6), the equations for the T^2-test corresponding with (6.3) and (6.4) are

$$\begin{aligned} -\tfrac{1}{2}n(\lambda_1^2 - \lambda_0^2) + \ln {}_1F_1[n/2, p/2; n\lambda_1^2\mathcal{E}_0(x)] - \ln {}_1F_1[n/2, p/2; n\lambda_0^2\mathcal{E}_0(x)] \\ = (1 - \alpha) \ln [\beta/(1 - \alpha)] + \alpha \ln [(1 - \beta)/\alpha] \end{aligned} \tag{6.5}$$

and

$$(6.6)\quad -\tfrac{1}{2}n(\lambda_1^2 - \lambda_0^2) + \ln {}_1F_1[n/2, p/2; n\lambda_1^2\mathcal{E}_1(x)] - \ln {}_1F_1[n/2, p/2; n\lambda_0^2\mathcal{E}_1(x)] = \beta\ln[\beta/(1-\alpha)] + (1-\beta)\ln[(1-\beta)/\alpha]$$

where $\mathcal{E}_0(x) = \mathcal{E}[x \mid n = \mathcal{E}(n)]|_{\lambda^2=\lambda_0^2}$ and $\mathcal{E}_1(x) = \mathcal{E}[x \mid n = \mathcal{E}(n)]|_{\lambda^2=\lambda_1^2}$.

Again we have left (6.5) and (6.6) in terms of n for simplicity but note that the solution of (6.5) for n yields $\mathcal{E}_0(n)$ and of (6.6), $\mathcal{E}_1(n)$.

Solutions of (6.3) and (6.4) or of (6.5) and (6.6) do seem to provide the desired guides on the numbers of experimental units required for the planning of sequential experimentation. Solution of these equations can be accomplished iteratively with a high-speed computer and, since applications are likely to be repetitive, this need only be done initially in setting up an experimental or control program.

The principal justification for the Bhate method of approximating ASN numbers is that results agree sufficiently well with Monte Carlo studies for practical purposes. For verification of this, the reader is referred to the paper by Freund and Appleby. One other study, conducted by K. J. Arnold (Natl. Bur. Stds. [21]), is available for the sequential t-test with $p = 1$, $\lambda_0^2 = 0$, $\lambda_1^2 = 1.0$ and $\alpha = \beta = .05$. In that study 500 sets of observations were sampled for the two values of λ^2; the average sample size to reach a decision under H_0 was 10.0 while the conjectural value was 10.7 and under H_1 was 11.2 compared to 9.7. It is interesting to note that the actual α- and β-values from this study were .044 and .034 respectively, somewhat different from the intended values. Ray [22] used this second example also but a rounding error occurs which makes his conjectured values appear to be closer to Arnold's results than they really are.

7. Generalized χ^2- and T^2-statistics. In addition to the χ^2- and T^2-statistics already discussed, there are two others in each case which deserve mention and complete the families of χ^2- and T^2-statistics (Hotelling [13]). We adopt Hotelling's notation and also drop the subscript n used on χ^2, T^2, $\mathbf{S}$ and $\bar{\mathbf{x}}$. The χ^2-test so far considered is χ_M^2 in Hotelling's notation and $\mathbf{\Sigma}$ becomes $\mathbf{\Sigma}_0$. Now $\lambda^2 = (\boldsymbol{\mu} - \boldsymbol{\mu}_0)\mathbf{\Sigma}_0^{-1}(\boldsymbol{\mu} - \boldsymbol{\mu}_0)'$ becomes λ_M^2 and the sequential χ^2-test is for the hypotheses,

$$(7.1)\quad \begin{aligned} H_0 &: \lambda_M^2 = \lambda_{M_0}^2, \\ H_1 &: \lambda_M^2 = \lambda_{M_1}^2. \end{aligned}$$

The second statistic to be considered is

$$(7.2)\quad \chi_D^2 = (n-1)\operatorname{tr}\mathbf{S}\mathbf{\Sigma}_0^{-1}$$

for hypotheses

$$(7.3)\quad \begin{aligned} H_0 &: \mathbf{\Sigma} = \mathbf{\Sigma}_0 \ (\text{or } \lambda_D^2 = \operatorname{tr}\mathbf{\Sigma}\mathbf{\Sigma}_0^{-1} = \lambda_{D_0}^2), \\ H_1 &: \mathbf{\Sigma} > \mathbf{\Sigma}_0 \ (\text{or } \lambda_D^2 = \lambda_{D_1}^2 > \lambda_{D_0}^2). \end{aligned}$$

Usually we shall want $\lambda_{D_0}^2 = p$ but this is not essential. χ_D^2 is distributed like χ^2 with $(n-1)p$ degrees of freedom and noncentrality parameter $(n-1)\lambda_D^2$. χ_M^2 and χ_D^2 are multivariate extensions of univariate tests based on $\bar{\mathbf{x}}$ and s.

The sum $\chi_O^2 = \chi_M^2 + \chi_D^2$ is a measure of the overall variation of the sample from standard. χ_O^2 is distributed like χ^2 with np degrees of freedom and parameter of noncentrality $\lambda_O^2 = n\lambda_M^2 + (n-1)\lambda_D^2$. An alternative form for χ_O^2 is

$$\chi_O^2 = \sum_{i=1}^{n} \chi_i^2 \tag{7.4}$$

where $\chi_i^2 = (\mathbf{x}_i - \boldsymbol{\mu}_0)\boldsymbol{\Sigma}_0^{-1}(\mathbf{x}_i - \boldsymbol{\mu}_0)'$, $i = 1, \cdots, n$. χ_D^2 could be obtained by subtraction, $\chi_O^2 - \chi_M^2$, and $\mathbf{S}$ need not be computed. Logical hypotheses for use with χ_O^2 appear to be

$$\begin{aligned} H_0 &: n\lambda_O^2 = n\lambda_{M_0}^2 + (n-1)\lambda_{D_0}^2 = n\lambda_{O_0}^2 \\ H_1 &: n\lambda_O^2 = n\lambda_{M_1}^2 + (n-1)\lambda_{D_1}^2 = n\lambda_{O_1}^2 . \end{aligned} \tag{7.5}$$

Sequential tests may be developed for (7.3) and (7.5) based on χ_D^2 and χ_O^2. The probability-ratio statistics are respectively

$$p_{1n}/p_{0n} = e^{-(n-1)(\lambda_{D_1}^2 - \lambda_{D_0}^2)/2} \frac{{}_0F_1[(n-1)p/2; (n-1)\lambda_{D_1}^2\chi_D^2/4]}{{}_0F_1[(n-1)p/2; (n-1)\lambda_{D_0}^2\chi_D^2/4]}$$

and

$$p_{1n}/p_{0n} = e^{-n(\lambda_{O_1}^2 - \lambda_{O_0}^2)/2} \frac{{}_0F_1[np/2; n\lambda_{O_1}^2\chi_O^2/4]}{{}_0F_1[np/2; n\lambda_{O_0}^2\chi_O^2/4]}.$$

These sequential χ^2-tests are developed just as the one based on χ_M^2 and would depend on the same set of tables for values of $\underline{\chi}^2$ and $\bar{\chi}^2$ except that often for χ_M^2 we would have $\lambda_{M_0}^2 = 0$ and here tables are required for cases where neither null nor non-null values of λ^2 are zero.

If the family of sequential χ^2-test were used, say, in sampling inspection, the inspector could ascertain after each item inspected

(i) whether or not the sample means differed significantly from standard,

(ii) whether or not the variation about sample means was greater than the preassigned $\boldsymbol{\Sigma}_0$, and

(iii) whether or not the overall variability of the sample is larger than should have been expected.

Generalizations of the sequential T^2-test are not directly available; generalizations of the non-sequential T^2-test were developed. T^2 in this paper corresponds to T_M^2 in Hotelling's notation. T_D^2 of Hotelling generally represents the variability in a subgroup of an experiment compared to, say, the average subgroup variability of an experiment. Rarely would such situations occur in sequential experimentation. Somewhat more conceivable is the situation where a sequential T_M^2 test is run in parallel with a χ_D^2-test, the test on variances is based on previous experience but the test on means depends only on the variability of the sample itself. T_M^2 and T_O^2 are useful statistics in the multivariate analysis of variance and could perhaps be used in sequential multivariate schemes when more is known about the forms of their distributions. Sequential tests for the roots of determinantal equations might also prove useful and feasible but computational procedures would be difficult.

8. Discussion. We now discuss some problems that arise in using the sequential methods developed in this paper.

(i) *Tables.* Direct applications of our sequential procedures involve comparison of the probability ratio at each stage with $\beta/(1 - \alpha)$ and $(1 - \beta)/\alpha$. This is laborious and requires evaluation of either ${}_0F_1(c; x)$ or ${}_1F_1(a, c; x)$ after each observation. Tables of both functions are available (Jackson [15], Nath [20], Rushton and Lang [26]) but Lagrangian interpolation of the logarithms of these functions is still necessary in most cases. It is better to prepare tables of the boundary values $\underline{\chi}_n^2$ and $\bar{\chi}_n^2$ and $\underline{T}_n^2$ and $\overline{T}_n^2$ so that only the test statistic is computed in applications. Tables now completed for $\alpha = \beta = .05$ are given by Jackson and Bradley [16] and show $\underline{\chi}_n^2$, $\bar{\chi}_n^2$, $\underline{T}_n^2$ and $\overline{T}_n^2$ for $p = 2\ (1)\ 9$; $\lambda_0^2 = 0$; $\lambda_1^2 = .5, 1.0, 2.0$; maximum n: 60 for $\lambda_1^2 = .5$, 45 for $\lambda_1^2 = 1.0$, 30 for $\lambda_1^2 = 2.0$. R. J. Freund with Jackson at the Virginia Polytechnic Institute has completed some additional tables and a report [10] has been prepared. Publication of a separate volume of tables is contemplated when this work is complete.

(ii) *Determination of H_0 and H_1.* Specifications of values of the noncentrality parameter λ^2 lead to difficult administrative decisions. For sequential tests for means we would often take $\lambda_0^2 = 0$ corresponding to $\boldsymbol{\mu} = \boldsymbol{\mu}_0$. Determination of λ_1^2 is much more difficult in the multivariate case than for the univariate case since a p-dimensional ellipsoid related to problem specifications must be visualized. No single rule on specifying λ_1^2 can be given and each problem has to be handled individually. Jackson and Bradley give some examples in connection with the sampling inspection of ballistic missiles and a paper showing these applications has been accepted for publication [17].

Sequential procedures should also be extended to cover the use of one-sided tolerances and essentially generalize the work of Goldberg (Wallis [28]).

(iii) OC *and* ASN *functions and truncation.* No explicit or even approximate expressions yet exist for the OC and ASN functions when the hypotheses under consideration are composite. Until such time as these expressions can be found, we must rely on Monte Carlo evaluations for a description of these properties. Little or no work has yet been done regarding truncation of sequential tests of composite hypotheses. Again, until such expressions are available, we must rely on Monte Carlo studies to show us the effect of truncation on the OC and ASN functions.

(iv) *Grouping.* These techniques were originally designed for the sampling of ballistic missiles, items which involve considerable expense. However, for a low-cost, high-volume process, sequential sampling by groups might be preferable to item-by-item sampling. Except for a few isolated cases like the binomial, no optimum procedures have been worked out for sequential sampling by groups. The general procedure recommended by Wald, in our case, would be to take groups of say m observations per group and compare the resultant χ_n^2 or T_n^2, as the case may be, with the corresponding $\underline{\chi_n^2}$ and $\overline{\chi_n^2}$ or $\underline{T_n^2}$ and $\overline{T_n^2}$ where n is now equal to m, $2m$, $3m$, $\cdots$, etc. The effect of this procedure is to increase the average sample number and to decrease the size of α and β. Except for empirical studies, the magnitudes of these changes are unknown but the directions of the changes are such that they compensate for each other to some extent.

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