CONDITIONAL PROBABILITY OPERATORS[1]

By Robert Cogburn

University of California, Berkeley

0. Introduction and summary. The study of laws of random variables is facilitated by various methods of representing these laws. The distribution function and characteristic function have played an important role, and the functional representation of the law of a random variable X as a mapping T of the bounded Borel functions into the real line given by $Tg = Eg(X)$ connects many problems in probability theory with functional analysis.

Similarly, in the study of conditional probability laws various representations of these conditional laws are desirable. In particular, in the study of Markov processes an operator representation has proven most useful. In this paper we develop the representation of the conditional law of a random variable in a way analogous to the functional representation mentioned above. Thus, if X is a random variable on a probability space $(\Omega, \mathcal{A}, P)$ and $\mathcal{A}^*$ is a sub-σ-field of $\mathcal{A}$, then we introduce the $\mathcal{A}^*$ conditional operator T of X mapping the bounded Borel functions into $L_\infty(\Omega, \mathcal{A}^*, P)$ and given by $Tg = E^{\mathcal{A}^*}g(X)$.

The first four sections of the paper develop a rudimentary theory of such operators. The use of these operators in probability theory leads one to consider various operator topologies, etc., weaker than those usually studied in functional analysis. Most of the material presented in this development is quite elementary, and many properties of these operators having potential interest have not even been mentioned. Nevertheless, it is hoped that this exposition may suggest further use of the operator representation of conditional distributions in probability theory, and that the properties of these operators relevant to probability theory will be investigated more systematically.

In the final section conditional probability operators are applied to a mixing problem. A stationary process is said to have central structure if, under conditions similar to those of the central limit problem for independent random variables, conditional limit laws given the invariant σ-field are infinitely divisible. It is shown that central structure is closely related to a type of uniform ergodicity.

1. Order continuous operators on G to L_∞. In this section we will be concerned with the space G of all bounded real-valued Borel functions on the Borel line $(R, \mathcal{B})$, and the space $L_\infty(S, \mathcal{S}, \mu)$ of an arbitrary finite positive measure space $(S, \mathcal{S}, \mu)$. The designation $(S, \mathcal{S}, \mu)$ will be omitted from the notation except when it is needed for clarity. The space G under the uniform norm, $\|g\| = \sup_{x \varepsilon R} |g(x)|$, is a Banach space, and under the order relation, $g \leqq h$ if $g(x) \leqq h(x)$, $x \varepsilon R$, G is a conditionally σ-complete lattice in McShane's terminology [8, p. 9],

Received October 3, 1961.

[1] Prepared with the partial support of National Science Foundation Grant G-14648.

that is, every bounded countable set of elements of G has a supremum and an infimum in G. Moreover, G is a Banach lattice, that is, the norm and order relation are connected by the property.

$$|f| \leqq |g| \Rightarrow \|f\| \leqq \|g\|.$$

The space L_∞ is, of course, a Banach space with the norm, $\|X\|_\infty = \mu -$ ess sup $|X|$, and there is a natural order relation, $X \leqq Y$ if $\mu[s: X(s) > Y(s)] = 0$. With this relation, L_∞ is a Banach lattice and is conditionally complete (see [2, p. 302]), that is, any bounded set of elements in L_∞ has a supremum and infimum in L_∞.

The symbols $\vee$ and $\wedge$ will denote the usual lattice operations, supremum and infimum, respectively. We say that a sequence of elements $l_1, l_2, l_3, \cdots$ of a vector lattice $\mathfrak{L}$ converge in order to $l \varepsilon \mathfrak{L}$, denoted $l_n \xrightarrow{o} l$ or $l = o-\lim l_n$, if $\wedge l_n$ and $\vee l_n$ exist and

$$\bigwedge_n \bigvee_{m \geqq n} l_m = \bigvee_n \bigwedge_{m \geqq n} l_m = l.$$

Clearly, order convergence of sequences in G is simply bounded (in sense of norm) pointwise convergence, and order convergence in L_∞ is bounded (in sense of norm) $\mu -$ a.e. convergence.

Convergences in terms of sequences will frequently be introduced in this paper, but such convergences extend immediately to countable generalized sequences (for example, double sequences) and such extensions will be used without further comment.

It is to be understood that when we write $X \leqq Y$, $X \geqq Y$ or $X = Y$ for $X, Y \varepsilon L_\infty$, these symbols denote the corresponding order relations and not the pointwise relations $X(s) \leqq Y(s)$, $s \varepsilon S$, etc.

We introduce several subspaces of G. Namely, the space C of bounded continuous functions; C_∞ of continuous functions having a limit at infinity ($g \varepsilon C_\infty$ if $g \varepsilon C$ and $g(\infty) = \lim_{x \to \pm\infty} g(x)$ exists); C_0 of continuous functions vanishing at infinity ($g \varepsilon C_0$ if $g \varepsilon C_\infty$ and $g(\infty) = 0$); C_{00} of continuous functions with compact support; and finally the space AP_∞ of continuous functions almost periodic at infinity defined as follows: $g \varepsilon AP_\infty$ if $g \varepsilon C$ and, for every $\epsilon > 0$, there exists a periodic function g_ϵ and a compact set K_ϵ such that $|g - g_\epsilon| < \epsilon$ for $x \not\varepsilon K_\epsilon$. Note that, since g is continuous, g_ϵ can be chosen to be continuous. Clearly,

$$G \supset C \supset AP_\infty \supset C_\infty \supset C_0 \supset C_{00}.$$

The subspaces C, AP_∞, C_∞ and C_0 are closed in the normed typology of G, and the closure of C_{00} in this topology is C_0. On the other hand, the closure of C_{00} under passages to the limit in order by sequences is G.

Since it is closed in the normed topology, the subspace C_0 with its relative topology is a Banach space, and, according to the Riesz representation theorem and its classical extensions, the adjoint C_0^* of C_0 is isometrically isomorphic to the space Φ of all finite real-valued countably additive set functions on the

Borel line. Moreover, every $\varphi \varepsilon \Phi$ determines a continuous linear functional on all of G through the relation $\varphi g = \int g(x)\varphi(dx)$, hence these functionals are in G^*, and we use the same notation for elements of Φ and the corresponding functionals. The Φ topology of G is defined to be the coarsest topology of G for which all the functionals in Φ are continuous. Since Φ contains measures concentrated at a single point, it follows from the dominated convergence and uniform boundedness theorems that a sequence $g_n \varepsilon G$ converges to g in the Φ topology of G if and only if the g_n are uniformly bounded and converge pointwise to g. Thus order convergence and convergence in the Φ topology of G coincide for sequences.

The space L_∞ is isometrically isomorphic to the adjoint of L_1, and the canonical mapping of elements X of L_1 into elements $\hat{X}$ of the adjoint, L_∞^*, of L_∞ is given by

$$\hat{X}Y = \int XY \, d\mu, \qquad Y \varepsilon L_\infty$$

(see [2, p. 289]). We let $\hat{L}_1$ denote the canonical image of L_1 in L_∞^*. Note that L_1 is norm determining for L_∞, that is, for $X \varepsilon L_\infty$,

$$\|X\|_\infty = \sup_{Y \varepsilon L_1, \|Y\|_1 \leqq 1} \hat{Y}X,$$

where $\|Y\|_1 = \int |Y| \, d\mu$ is the L_1 norm.

Convergence in norm and weak convergence in L_∞ are usually too strong for our purposes. Besides order convergence, we introduce bounded convergence in measure and convergence in the L_1 topology of L_∞ (to be called simply the L_1 topology): a sequence X_n *converges boundedly in measure* to X, denoted $X_n \to X \quad b - \mu$, if the X_n are uniformly bounded in norm and X_n converges in μ-measure to X. And X_n *converges in the* L_1 *topology to* X, denoted $X_n \to X \quad L_1$, if for every $Y \varepsilon L_1$, $\hat{Y}X_n \to \hat{Y}X$.

We have introduced five notions of convergence in L_∞. For sequences these convergences are simply related, namely,

norm $\Rightarrow$ weak $\Rightarrow$ order $\Rightarrow$ bounded in measure $\Rightarrow$ L_1 convergence.

Except for the second implication these relations are classical and will be found discussed, for example, in Dunford and Schwartz [2]. To prove the second relation we begin by putting a standard relation of measure theory into a convenient form.

LEMMA 1. *A sequence $X_n \varepsilon L_\infty$ converges in order to some $X \varepsilon L_\infty$, if and only if*

$$Y_{m,n} = \bigvee_{m \leqq k \leqq n} X_k - \bigwedge_{m \leqq k \leqq n} X_k$$

converges to 0 *in the L_1 topology as $m, n \to \infty$.*

PROOF. The only if assertion follows from the dominated convergence theorem. To prove the converse, observe that if the $Y_{m,n}$ converge to 0 in the L_1 topology, then by the uniform boundedness theorem, the $Y_{m,n}$, hence the X_n, are uniformly bounded in norm. Now assume that the X_n do not converge in order.

Then

$$\bigwedge_m \bigvee_{k \geqq m} X_k - \bigvee_m \bigwedge_{k \geqq m} X_k = Y > 0.$$

But as $n \to \infty$,

$$Y \leqq \bigvee_{k \geqq m} X_k - \bigwedge_{k \geqq m} X_k \leftarrow Y_{m,n} \qquad \mu - \text{a.e.},$$

and since the $Y_{m,n}$ are uniformly bounded, the dominated convergence theorem yields

$$\lim_n \int Y_{m,n}\, d\mu \geqq \int Y\, d\mu > 0.$$

Upon taking the function identically 1 in L_1, the lemma follows *ab contrario*.

THEOREM 1. *If a sequence in L_∞ converges weakly, then it also converges in order.*

PROOF. According to a basic theorem of Krein and Kakutani [4] the space L_∞ is isometric and lattice isomorphic to the space $C(S_1)$ of all bounded continuous real-valued functions on a compact Hausdorff space S_1. Let Υ denote an order preserving isomorphism from L_∞ to $C(S_1)$. Then if X_n converges weakly to X in L_∞, is follows that ΥX_n converges weakly to ΥX in $C(S_1)$. But weak convergence in $C(S_1)$ is equivalent to bounded pointwise convergence (see [2, p. 265]), hence $\gamma_{m,n} = \bigvee_{m \leqq k \leqq n} \Upsilon X_k - \bigwedge_{m \leqq k \leqq n} \Upsilon X_k \to 0$ as $m, n \to \infty$ in the weak topology of $C(S_1)$. Now let $Y_{m,n}$ be defined as in Lemma 1. Then since Υ is order preserving, $\Upsilon^{-1}\gamma_{m,n} \geqq Y_{m,n} \geqq 0$. But $\Upsilon^{-1}\gamma_{m,n} \to 0$ as $m, n \to \infty$ in the weak topology of L_∞, hence in the L_1 topology, and it follows that $Y_{m,n} \to 0$ as $m, n \to \infty$ in the L_1 topology. The theorem follows from Lemma 1.

COROLLARY. *For monotone sequences in L_∞*

1. *Convergence in norm and weak convergence coincide.*
2. *Convergence in order, bounded convergence in measure and convergence in the L_1 topology coincide.*

PROOF. Let $X_n \downarrow 0$ weakly. Then $\Upsilon X_n \downarrow 0$ weakly, hence in order in $C(S_1)$. By Dini's lemma, $\|\Upsilon X_n\| \to 0$, hence $\|X_n\| \to 0$. It follows that if $X_n \downarrow X$ or $X_n \uparrow X$ weakly, then $\|X_n - X\| \to 0$. The second assertion is an immediate consequence of Lemma 1.

Given two Banach spaces B_1 and B_2, we let $B(B_1, B_2)$ denote the Banach space of all bounded linear operators on B_1 to B_2, and given two vector lattices $\mathfrak{L}_1$ and $\mathfrak{L}_2$, we let $O(\mathfrak{L}_1, \mathfrak{L}_2)$ denote the space of all order continuous linear operators on $\mathfrak{L}_1$ to $\mathfrak{L}_2$, that is, linear operators T such that $l = o - \lim l_n$ implies $Tl = o - \lim Tl_n$ for all sequences of $l_n \,\varepsilon\, \mathfrak{L}_1$. An operator T in $O(\mathfrak{L}_1, \mathfrak{L}_2)$ is *positive*, denoted $T \geqq 0$, if $l \geqq 0$ in $\mathfrak{L}_1$ implies $Tl \geqq 0$ in $\mathfrak{L}_2$.

LEMMA 2. $O(G, L_\infty) \subset B(G, L_\infty)$ *and* $O(C_0, L_\infty) = B(C_0, L_\infty)$. *Moreover, each operator $T \,\varepsilon\, O(C_0, L_\infty)$ determines a unique extension to an operator $T_1 \,\varepsilon\, O(G, L_\infty)$, and $\|T_1\| = \|T\|$.*

PROOF. Let $T \,\varepsilon\, O(G, L_\infty)$. The mapping $(X, g) \to \hat{X}Tg$ defines for each fixed $g \,\varepsilon\, G$ a bounded linear functional on L_1 and for each fixed $X \,\varepsilon\, L_1$ a bounded linear functional on G, since $\hat{X}T$ is continuous under order convergence, hence under convergence in norm in G. It follows from the uniform boundedness theorem that $\|T\| = \sup \hat{X}Tg < \infty$, where the supremum is over all $g \varepsilon G$ with $\|g\| \leqq 1$ and all $X \varepsilon L_1$ with $\|X\|_1 \leqq 1$. Thus $O(G, L_\infty) \subset B(G, L_\infty)$, and in the same way $O(C_0, L_\infty) \subset B(C_0, L_\infty)$ But an operator continuous in the normed topologies of C_0 and L_∞ is also continuous in the weak topologies, and for sequences weak and order convergence coincide on C_0, while weak convergence implies order convergence on L_∞. It follows that a bounded linear operator on C_0 to L_∞ is order continuous, hence $O(C_0, L_\infty) = B(C_0, L_\infty)$. The last assertions follow since G is the closure of its subspace C_0 under order passages to the limit by sequences, and the lemma is proved.

Operators T in $O(G, L_\infty)$ are in $B(G, L_\infty)$ and this subset of $B(G, L_\infty)$ is partially characterized by the adjoint operators T^* in $B(L_\infty^*, G^*)$.

LEMMA. 3

1. $T \varepsilon O(G, L_\infty) \Rightarrow T^*\hat{L}_1 \subset \Phi$.
2. $T \varepsilon B(G, L_\infty), T^*\hat{L}_1 \subset \Phi$ *and* $T \geqq 0 \Rightarrow T \varepsilon O(G, L_\infty)$.
3. $T \varepsilon B(G, L_\infty), T^*L_\infty^* \subset \Phi \Rightarrow T \varepsilon O(G, L_\infty)$.

PROOF. Let $T \varepsilon O(G, L_\infty)$. Then for every $X \varepsilon L_1$, the functional $T^*\hat{X} = \hat{X}T$ on G is order continuous, hence in Φ. To prove the second assertion, observe that since $T \geqq 0$ it suffices to show that $g_n \downarrow 0$ in order implies $Tg_n \to 0$ in order. But $g_n \downarrow 0$ implies Tg_n is monotone nonincreasing in L_∞, and for every $X \varepsilon L_1$, $\hat{X}Tg_n = (T^*\hat{X})g_n \to 0$ since $T^*\hat{X}$ is in Φ. The Corollary to Theorem 1 implies that $Tg_n \to 0$ in order. Finally, the hypothesis in 3 implies that, if $g_n \to 0$ in order, then $Tg_n \to 0$ weakly, hence by Theorem 1, $Tg_n \to 0$ in order, and the lemma is proved.

Hereafter we will be concerned exclusively with operators in $O(G, L_\infty)$ and the restrictions of these operators to various subspaces of G. A number of notions of convergence in $O(G, L_\infty)$ are available corresponding to selection of various subspaces of G and convergences in L_∞. Thus for any subset $G_0 \subset G$ we will say that a sequence of operators $T_n \varepsilon O(G, L_\infty)$ *converges G_0 strongly* (*weakly*; *in order*; *boundedly in measure*; *in the L_1 topology*) to an operator $T \varepsilon O(G, L_\infty)$, denoted $T_n \to T \quad G_0, s(G_0, w; G_0, o; G_0, b - \mu; G_0, L_1)$, if, for every $g \varepsilon G_0$, $T_ng \to Tg$ in norm (weakly, in order, boundedly in measure, in the L_1 topology). Of course, for a given G_0 the five types of convergence are simply related:

$$T_n \to T \quad G_0, s \Rightarrow T_n \to T \quad G_0, w \Rightarrow T_n \to T \quad G_0, o$$
$$\Rightarrow T_n \to T \quad G_0, b - \mu \Rightarrow T_n \to T \quad G_0, L_1.$$

The convergence $T_n \to T \quad G_0, b - \mu$ is clearly equivalent to $\sup \|T_ng\| < \infty$ and $\|T_ng - Tg\|_1 \to 0$ for every $g \varepsilon G_0$, and is also equivalent to $\sup \|T_ng\| < \infty$, and $\int_A T_ng \, d\mu \to \int_A Tg \, d\mu$ uniformly in $A \varepsilon \mathcal{S}$ for each $g \varepsilon G_0$. On the other hand the convergence $T_n \to T \quad G_0, L_1$ is equivalent to $\sup \|T_ng\| < \infty$ and $\int_A T_ng \, d\mu \to \int_A Tg \, d\mu$ for each $g \varepsilon G_0$ and $A \varepsilon \mathcal{S}$.

Considering the operators restricted to C_0, the convergence $T_n \to T \quad C_0, L_1$

corresponds to the topology of $O(C_0, L_\infty)$ obtained from the base

$$N(T_0 : A, B, \epsilon) = \{T: |\hat{X}Tg - \hat{X}T_0g| < \epsilon, g \varepsilon A, X \varepsilon B\},$$

where $T_0 \varepsilon O(C_0, L_\infty)$, A is a finite subset of C_0, B is a finite subset of L_1 and $\epsilon > 0$. This topology will be called simply the *L_1 topology of $O(C_0, L_\infty)$*. Let Q denote the closed unit sphere in $O(C_0, L_\infty)' = B(C_0, L_\infty)$ in the normed topology.

LEMMA 4. *Q is closed in the L_1 topology of $O(C_0, L_\infty)$*. For, if $\|T\| > 1$, then there exist $g \varepsilon C_0$ with $\|g\| \leqq 1$ and $X \varepsilon L_1$ with the L_1 norm, $\|X\|_1 \leqq 1$, such that $|\hat{X}Tg| = \delta > 1$. But then $N(T: \{g\}, \{X\}, \delta - 1)$ is disjoint from Q, and the lemma follows.

The next proposition is a variation on Alaoglu's theorem.

THEOREM 2. *Q is compact in the L_1 topology of $O(C_0, L_\infty)$.*

PROOF. Furnish $\mathfrak{P} = \prod_{g \varepsilon C_0} L_\infty^{(g)}$ with the product topology obtained by giving each $L_\infty^{(g)} = L_\infty$ the L_1 topology. Then the mapping $\tau: T \to \prod_{g \varepsilon C_0} Tg$ is a homeomorphism of $O(C_0, L_\infty)$ with the L_1 topology onto a subset of $\mathfrak{P}$ with its relative topology. By Lemma 4, τQ is closed. But for every $g \varepsilon C_0$, $pr_g \tau Q$ is bounded by $\|g\|$, hence the closure of this set is compact by Alaoglu's theorem. By Tychonoff's theorem $\prod_{g \varepsilon C_0} \overline{(pr_g \tau Q)}$ is compact, hence the closed subset τQ is compact, and the theorem is proved.

COROLLARY. *A subset of $O(C_0, L_\infty)$ is compact in the L_1 topology, if and only if it is closed in the L_1 topology and bounded in the normed topology.*

We let Δ_a denote the functional on G defined by $\Delta_a g = g(a)$, $g \varepsilon G$, and we say that an operator T on G to L_∞ is a simple operator if for some finite number n $T = \sum_{k=1}^{n} X_k \Delta_{a_k}$ where $X_1, \cdots, X_n \varepsilon L_\infty$.

LEMMA 5. *Every $T \varepsilon O(G, L_\infty)$ is the C, o limit and the C_0, s limit of a sequence of simple operators.*

PROOF. Let

$$h_{nk}(x) = \begin{cases} 1 - |nx - k| & \text{for} \quad (k-1)/n \leqq x \leqq (k+1)/n \\ 0 & \text{for} \quad |x - (k/n)| \geqq (1/n), \end{cases}$$

and given $T \varepsilon O(G, L_\infty)$, set $T_n = \sum_{k=-n^2}^{n^2} Th_{nk} \cdot \Delta_{k/n}$. Then

$$T_n g = \sum_{k=-n^2}^{n^2} Th_{nk} \cdot g(k/n) = T \sum_{k=-n^2}^{n^2} g(k/n) h_{nk}.$$

But if $g \varepsilon C$ then $g = o - \lim \sum_{k=-n^2}^{n^2} g(k/n) h_{nk}$, and if $g \varepsilon C_0$ then the order limit becomes a limit in norm. The lemma follows from the order continuity and boundedness of T.

For functions ξ on R to G (or to L_∞) we introduce the order Riemann integral on $[a, b]$, denoted $\int_a^b \xi(x)\, dx$, and defined to be the order limit of the usual Riemann sums, provided this limit exists in G (or in L_∞). It is easily seen that the integral does exist whenever ξ is order continuous in G (or in L_∞) on the interval $[a, b]$.

LEMMA 6. *If $T \varepsilon O(G, L_\infty)$ and $g_\alpha \varepsilon G$ and is order continuous in α on $[a, b]$,* *then Tg_α is order continuous and*

$$T \int_a^b g_\alpha \, d\alpha = \int_a^b Tg_\alpha \, d\alpha.$$

PROOF. The first assertion follows from the order continuity of T. Then $\int_a^b g_\alpha \, d\alpha$ and $\int_a^b Tg_\alpha \, d\alpha$ exist, and, setting $\alpha_{nk} = a + k(b - a)/n$,

$$T \int_a^b g_\alpha \, d\alpha = To - \lim \frac{b-a}{n} \sum_{k=1}^n g_{\alpha_{nk}}$$
$$= o - \lim \frac{b-a}{n} \sum_{k=1}^n Tg_{\alpha_{nk}} = \int_a^b Tg_\alpha \, d\alpha.$$

The lemma is proved.

One of the most useful tools for studying laws of random variables is the distribution function. This method of representation of conditional distributions seems less natural, but appears as a direct generalization of the unconditional case.

A function ζ on the real line to $L_\infty(S, \mathcal{S}, \mu)$ having the properties

1. ζ is left order continuous: $x \uparrow y \Rightarrow \zeta(x) \to \zeta(y)$ in order,
2. $$\|\zeta\| = \sup_{X \in L_1, \|X\|_1 \leqq 1} \text{Total variation } \hat{X}\zeta < \infty,$$

will be called an $(S, \mathcal{S}, \mu)$ *generalized distribution function* ($(S, \mathcal{S}, \mu)$ g.d.f.). If $(S, \mathcal{S}, \mu)$ is understood, then we may omit this part of the designation of ζ.

Every $T \in O(G, L_\infty)$ determines a g.d.f. through the relation

$$\zeta(x) = TI_{(-\infty,x)}, \qquad x \in R.$$

The left order continuity of ζ follows from the order continuity of T, and the relation

$$\sum_{k=1}^n |\hat{X}(\zeta(a_k) - \zeta(a_{k-1}))| = \hat{X}T \sum_{k=1}^n s_k I_{(a_k, a_{k-1})},$$

where $X \in L_1$, $s_k = +1$ or -1 according as $\hat{X}(\zeta(a_k) - \zeta(a_{k-1})) \geqq 0$ or < 0 and we take $a_0 < a_1 < \cdots < a_n$, implies that $\|\zeta\| = \|T\| = \infty$.

On the other hand, for every g.d.f. ζ a Stieltje's integral $\int g \, d\zeta$ can be defined in the usual way, the integral existing as an order limit in L_∞ for every $g \in C$. The mapping T of C into L_∞ determined by $Tg = \int g d\zeta$ is linear and order continuous, and $\|T\| = \|\zeta\|$. Thus there is a natural correspondence between order continuous operators and g.d.f.'s. These results are in [5] for order continuous T on $C[0, 1]$ to L_∞ and require only slight modification for our situation.

Setting $\zeta = \zeta^+ - \zeta^-$, where

$$\zeta^+ = \bigvee \sum_{k=1}^n (\zeta(b_k) - \zeta(a_k)),$$

the supremum over all n, $a_1 < b_1 < \cdots < a_n < b_n < x$, it follows that ζ^+ and ζ^- are nondecreasing g.d.f.'s with max $(\|\zeta^+\|, \|\zeta^-\|) \leqq \|\zeta\|$. Thus we have

THEOREM 3. *The relation* $Tg = \int g\,d\zeta$, $g \in C$, *establishes a one to one correspondence between order continuous operators* T *and* g.d.f.'s ζ, *and* $\|T\| = \|\zeta\|$. *Moreover,* T *has the decomposition* $T = T^+ - T^-$ *where* $T^+g = \int g\,d\zeta^+$ *and* $T^-g = \int g\,d\zeta^-$ *are positive order continuous operators with* $\max(\|T^+\|, \|T^-\|) \leqq \|T\|$.

The g.d.f. yields a representation of order continuous operators on C but not directly on G. However, any $T \in O(G, L_\infty)$ defines a set function M on the Borel sets to L_∞ by $MB = TI_B$, and M is additive and order continuous: $B_n \downarrow \phi$ implies $MB_n \to 0$ in order. An integral can be defined in a natural way on G with values in L_∞ using the set function M, and a representation of T results. The theory of integration in abstract lattice spaces is developed by McShane [8].

Upon replacing any function space over the real scalar field by the corresponding space over the complex scalar field, we add a dagger to the symbol denoting that space, thus G, L_∞, G_0 are replaced by $G^\dagger$, $L_\infty^\dagger$, $G_0^\dagger$, etc. An operator $T \in O(G, L_\infty)$ has a natural extension to an operator on $G^\dagger$ to $L_\infty^\dagger$ given by the relation $Tg = T\mathcal{R}(g) + iT\mathcal{I}(g)$, $g \in G^\dagger$. We use the same notation for this extension of T and note that (bars denoting complex conjugates)

$$T\bar{g} = \overline{(Tg)}, \qquad |Tg| \leqq T^+|g| + T^-|g|.$$

Another convenient representation of operators in $O(G, L_\infty)$ is in terms of their Fourier transforms. We define the $(S, \mathcal{S}, \mu)$ *generalized characteristic function* ($(S, \mathcal{S}, \mu)$ *g.ch.f.*) of T, denoted $\tilde{T}$, through the relation $\tilde{T}(u) = Tg_u$, $-\infty < u < \infty$, where $g_u(x) = e^{iux}$. If $(S, \mathcal{S}, \mu)$ is understood, then we may omit this part of the designation of $\tilde{T}$.

Note that, if $\tilde{T}$ is the g.ch.f. of $T \in O(G, L_\infty)$ with $\|T\| = c$, then $\tilde{T}$ has the properties

1. $\tilde{T}$ is order continuous,
2. $\tilde{T}(-u) = \overline{\tilde{T}(u)}$,
3. $\|\sum a_k\tilde{T}(u_k)\|_\infty \leqq c\|\sum a_k g_{u_k}\|$,

for every finite set of numbers a_k and u_k (where, as always, $g_u(x) = e^{iux}$). In particular, $\|\tilde{T}(u)\|_\infty \leqq \|T\|$ for all u, and if $T \geqq 0$ then $\|\tilde{T}(0)\|_\infty = \|T1\|_\infty = \|T\|$. It would be interesting to know if the above three properties characterize order continuous operators.

The closure of the linear subspace generated by $\{g_u, -\infty < u < \infty\}$ in the order topology is the class of all bounded measurable complex-valued functions, hence there is a one-to-one correspondence between T and $\tilde{T}$. On C_{00} we obtain a more explicit representation of T through the inversion formula.

THEOREM 4. *For* $g \in C_{00}$,

$$\left\| Tg - \frac{1}{2\pi}\int_{-v}^{v} \tilde{T}(u) \int e^{iux}g(x)\,dx \right\| \to 0$$

as $v \to \infty$.

PROOF. Applying the definition of $\tilde{T}$ and Lemma 6,

$$\frac{1}{2\pi}\int_{-v}^{v} \tilde{T}(u)\,du \int e^{iux}g(x)\,dx = Th_v,$$

where, by Fubini's theorem and elementary computations,

$$h_v(y) = \frac{1}{2\pi}\int_{-v}^{v} e^{iuy}\,du \int e^{iux} g(x)\,dx = \int g\left(y + \frac{z}{v}\right)\frac{\sin z}{\pi z}\,dz.$$

Since $g \varepsilon C_{00}$ is uniformly continuous, $\|h_v - g\| \to 0$ as $v \to \infty$, and the theorem follows since T is order continuous, hence bounded.

2. Convergence of order continuous operators. Using the g.d.f. and g.ch.f. we can obtain conditions for various types of convergence of the corresponding operators.

Associated with every g.d.f. ζ is a function of bounded variation, F, given by $F(x) = \int \zeta(x)\,d\mu$. We say that a sequence of g.d.f.'s ζ_n converge strongly (weakly; in order; boundedly in neasure; in the L_1 topology) to a g.d.f. ζ, denoted $\zeta_n \to \zeta \quad s(w; o; b - \mu; L_1)$, if for every pair x, y of continuity points of F,

$$\zeta_n(x) - \zeta_n(y) \to \zeta(x) - \zeta(y)$$

in norm (weakly; in order; boundedly in measure; in the L_1 topology). The generalization of the Helley-Bray theorem is then

THEOREM 5. *Let ζ_n, ζ be, respectively, the* g.d.f.'s *of T_n, $T \varepsilon O(G, L_\infty)$. Then*

1. $\zeta_n \to \zeta \quad s(w; o; b - \mu; L_1) \Rightarrow T_n \to T \quad C_0, s(w; o; b - \mu; L_1)$. *If, in addition,* $\int d\zeta_n \to \int d\zeta \quad s(w; o; b - \mu; L_1)$, *then C_0 is to be replaced by C_∞ in the above implication.*

2. *If the T_n are positive, then* $\zeta_n \to \zeta \quad o(b - \mu; L_1) \Leftrightarrow T_n \to T \quad C_0, o(b - \mu; L_1)$. *If, in addition* $\int d\zeta_n \to \int d\zeta \quad o(b - \mu; L_1)$, *then C_0 is to be replaced by C in the above implication.*

PROOF.

1. The function $F(x) = \int \zeta\,d\mu$ is of bounded variation, hence has at most a countable number of discontinuities. Then the first assertion is immediate since it is possible to approximate functions $g \varepsilon C_0$ uniformly by step functions g_n that are identically 0 outside compact sets K_n and that have no discontinuities at the discontinuity points of F. The second assertion then follows since functions in C_∞ can be decomposed into the sum of a constant function and a function in C_0.

2. To prove the first assertion in 2, observe that if $T_n \geqq 0$ and $T_n \to T$ C_0, L_1 then necessarily $T \geqq 0$ and ζ is nondecreasing. But ζ is continuous in the L_1 topology at continuity points of F, hence, by the Corollary to Theorem 1, ζ is also continuous in order at these points. For $\epsilon > 0$ let

$$h_{x,\epsilon}(y) = \begin{cases} 1 & \text{for } y \leqq x - \epsilon, \\ (x - y)/\epsilon & \text{for } x - \epsilon \leqq y \leqq x, \\ 0 & \text{for } y \geqq x, \end{cases}$$

and, given x, y continuity points of F with $x < y$, let $d_\epsilon = h_{y+\epsilon,\epsilon} - h_{x,\epsilon}$ and $d'_\epsilon = h_{y,\epsilon} - h_{x+\epsilon,\epsilon}$. If $T_n \to T \quad C_0, s(w; o; b - \mu; L_1)$, then

$$T\,d'_\epsilon \leftarrow T_n\,d'_\epsilon \leqq \zeta_n(y) - \zeta_n(x) \leqq T_n\,d_\epsilon \to T\,d_\epsilon$$

in the given mode of convergence as $n \to \infty$. But $T d'_\epsilon \leqq \zeta(y) - \zeta(x) \leqq T d_\epsilon$, and y and x are order continuity points of ζ, hence, by letting $\epsilon \to 0$ it follows that $\zeta_n \to \zeta$ in the given mode of convergence.

Using the second assertion in part 1, the second assertion in part 2 can be proven by showing that, if $T_n \geqq 0$ and $T_n \to T$ on C_∞, then $T_n \to T$ on C in the given mode of convergence. But $T \geqq 0$ so $\|T\| = \|T1\|_\infty$. Now choose $h_k \, \varepsilon \, C_0$ so that $0 \leqq h_k \leqq 1$ and $1 = o\text{-lim}\, h_k$. Then, given $g \, \varepsilon \, C$, $T_n g h_k \to T g h_k$ as $n \to \infty$ for each k and

$$|T_n g(1 - h_k)| \leqq \|g\| T_n(1 - h_k) \to \|g\| T(1 - h_k)$$

in the given mode of convergence. The assertion follows by letting $k \to \infty$ since T is order continuous so $T(1 - h_k) \to 0$ in order. The theorem is proved.

Given $h \, \varepsilon \, C_{00}^\dagger$, we define a functional $h \circ (\cdot)$ on $C^\dagger$ by

$$h \circ g = \int h(x) g(x) \, dx, \qquad g \, \varepsilon \, C^\dagger,$$

and for order continuous functions ξ on R to $L_\infty^\dagger$ we define $h \circ \xi$ through the order integral $h \circ \xi = \int h(x) \xi(x) \, dx$.

For some purposes the g.ch.f. $\tilde{T}$ is not sufficiently smooth, but this difficulty is avoided by regarding $\tilde{T}$ as a functional on C_{00} defined by $\tilde{T}(h) = h \circ \tilde{T}$, $h \, \varepsilon \, C_{00}$. When confronted with this problem, a more traditional approach in probability theory is to introduce the integral characteristic function (see Loève [7, p. 189]). We may mention that this latter approach will work just as well here, but we choose the former method because it seems less artificial.

To each $h \, \varepsilon \, C_{00}^\dagger$ we correspond the Fourier transform $\tilde{h}$ given by

$$\tilde{h}(u) = h \circ g_u = \int h(x) e^{iux} \, dx,$$

and we let $H = \{\tilde{h} : h \, \varepsilon \, C_{00}^\dagger\}$. The next lemma verifies that the family H is rich enough for our purposes.

Lemma 7. *The closure of H in the normed topology is $C_0^\dagger$.*

Proof. Since $H \subset C_0^\dagger$ and the closure of $C_{00}^\dagger$ in the normed topology is $C_0^\dagger$, it suffices to show that $C_{00}^\dagger$ is contained in the closure of H.

Given $g \, \varepsilon \, C_{00}^\dagger$, set

$$f_n(x) = \begin{cases} 1 & \text{for } |x| \leqq n \\ n^2 + 1 - n \cdot |x| & \text{for } n \leqq |x| \leqq n + 1/n \\ 0 & \text{for } |x| \geqq n + 1/n \end{cases}$$

and set $g_n(x) = \tilde{g}(-x) f_n(x)/2\pi$. Then $g_n \, \varepsilon \, C_{00}^\dagger$ and

$$\left| \tilde{g}_n(u) - \frac{1}{2\pi} \int_{-n}^{n} \tilde{g}(-x) \, e^{iux} \, dx \right| \leqq \| \tilde{g} \| / n.$$

But, upon replacing $\tilde{g}$ by its definition, it follows by elementary computation that

$$\frac{1}{2\pi}\int_{-n}^{n} \tilde{g}(-x)\, e^{iux}\, dx = \int g\left(u + \frac{z}{n}\right) \frac{\sin z}{\pi z}\, dz.$$

Singe g is uniformly continuous, it follows that $\|\tilde{g}_n - g\| \to 0$, and the lemma is proved.

In the following theorems we denote the convergence in L_∞ of X_n to X in norm (weakly; in order; boundedly in measure; in the L_1 topology) by $X_n \to X \quad s(w;\, o;\, b - \mu,\, L_1)$.

THEOREM 6. *Let T_n, $T \varepsilon O(G, L_\infty)$. Then*

1. $T_n \to T \quad C_0, s(w; o; b - \mu; L_1) \Rightarrow h \circ \tilde{T}_n \to h \circ \tilde{T} \quad s(w; o; b - \mu; L_1)$ *for every $h \varepsilon C_{00}^\dagger$ and* $\sup_n \|T_n\| < \infty$.

2. $h \circ \tilde{T}_n \to X_h \quad s(w; o; b - \mu; L_1)$ *for every $h \varepsilon C_{00}^\dagger$ and* $\sup_n\|T_n\| < \infty \Rightarrow T_n \to V \quad C_0, s(w; o; b - \mu; L_1)$ *for some $V \varepsilon O(G, L_\infty)$ and $X_h = h \circ \tilde{V}$, $h \varepsilon C_{00}^\dagger$.*

PROOF. As usual, $g_u(x) = e^{iux}$. The first assertion follows from Lemma 6 since

$$h \circ \tilde{T}_n = \int h(u) T_n\, g_u\, du = T_n \int h(u)\, g_u\, du = T_n \tilde{h}$$

and $\tilde{h} \varepsilon C_0^\dagger$. And $\sup \|T_n\| < \infty$ by the uniform boundedness theorem.

The hypothesis in 2 then states that $T_n\tilde{h} \to X_h$ for every $h \varepsilon C_{00}^\dagger$. Since $\sup \|T_n\| < \infty$, Lemma 7 implies that $T_n g \to Vg$ for every $g \varepsilon C_0^\dagger$ and some linear operator V on $C_0^\dagger$ to $L_\infty^\dagger$ in the given mode of convergence. In particular, $T_n g \to Vg$ in the L_1 topology for every $g \varepsilon C_0$. It then follows by Theorem 2 that the restriction of V to C_0 is in $O(C_0, L_\infty)$. Finally, since V is order continuous, $h \circ \tilde{V} = V\tilde{h} = X_h$, and the theorem is proved.

The generalization of the continuity theorem of P. Lévy becomes

THEOREM 7. *Let T_n, $T \varepsilon O(G, L_\infty)$. Then*

1. $T_n \to T \quad AP_\infty, s(w; o; b - \mu; L_1) \Rightarrow \tilde{T}_n \to \tilde{T} \quad s(w; o; b - \mu; L_1)$.

2. $\tilde{T}_n \to \xi \quad s(o; b - \mu; L_1)$ *with ξ continuous in the L_1 topology and* $\sup \|T_n\| < \infty \Rightarrow T_n \to V \quad AP_\infty, s(o; b - \mu; L_1)$ *for some $V \varepsilon O(G, L_\infty)$, and $\xi = \tilde{V}$.*

3. $\tilde{T}_n \to \xi \quad s(o; b - \mu; L_1)$ *with ξ continuous at 0 in the L_1 topology and* $\sup \|T_n\| < \infty \Rightarrow T_n \to V \quad C_\infty, s(o; b - \mu; L_1)$ *for some $V \varepsilon O(G, L_\infty)$, and $\xi(0) = \tilde{V}(0)$.*

4. *If the T_n are positive, then $\tilde{T}_n \to \xi \quad o(b - \mu; L_1)$ with ξ continuous at 0 in the L_1 topology $\Rightarrow T_n \to V \quad C, o(b - \mu; L_1)$ for some $V \varepsilon O(G, L_\infty)$, and $\xi = \tilde{V}$.*

PROOF.

1. The first assertion is immediate since the continuous functions g_u are periodic, hence in $AP_\infty^\dagger$.

2. To prove the second assertion observe that if $\tilde{T}_n \to \xi$ strongly then for

every $h \varepsilon C_{00}$,

$$\left\| \int h(x) (\tilde{T}_n(x) - \xi(x)) \, dx \right\| \leqq \int | h(x) | \cdot \| \tilde{T}_n(x) - \xi(x) \| \, dx \to 0,$$

hence $h \circ \tilde{T}_n \to h \circ \xi$ in norm. If $\tilde{T}_n \to \xi$ in order, then

$$\int d\mu \left(\bigvee_{k \geqq n} \left| \int (\tilde{T}_k(x) - \xi(x)) h(x) \, dx \right| \right)$$

$$\leqq \int d\mu \int \bigvee_{k \geqq n} | \tilde{T}_k(x) - \xi(x) | \cdot | h(x) | \, dx$$

$$= \int | h(x) | \, dx \int \bigvee_{k \geqq n} | \tilde{T}_k(x) - \xi(x) | \, d\mu \to 0.$$

It follows by Lemma 1 that $\bigvee_{k \geqq n} | h \circ (\tilde{T}_k - \xi) | \to 0$ in order, hence $h \circ \tilde{T}_n \to h \circ \xi$ in order. Similar arguments establish that $\tilde{T}_n \to \xi \quad b - \mu(L_1)$ entails $h \circ \tilde{T}_n \to h \circ \xi \quad b - \mu(L_1)$ for all $h \varepsilon C_{00}$. It follows by Theorem 6 that $T_n \to V$ on C_0 in the given mode of convergence for some $V \varepsilon O(G, L_\infty)$ and that $h \circ V = h \circ \xi$ for every $h \varepsilon C_{00}$.

Now let

$$h_n(x) = \begin{cases} n(1 - |n(x - u)|) & \text{for } |x - u| \leqq 1/n \\ 0 & \text{for } |x - u| \geqq 1/n. \end{cases}$$

Then for every $X \varepsilon L_1$, applying Fubini's theorem,

$$\hat{X}\xi(u) = \lim h_n \circ \hat{X}\xi = \lim \hat{X}h_n \circ \xi = \lim \hat{X}h_n \circ \tilde{V}$$

$$= \lim h_n \circ \hat{X}\tilde{V} = \hat{X}\tilde{V}(u).$$

It follows that $\xi(u) = \tilde{V}(u)$ in L_∞, hence $T_n g_u \to V g_u$ in the given mode of convergence for all u. Now if $f \varepsilon AP_\infty$, then for every $\epsilon > 0$ there exists a continuous periodic function, f_ϵ, and a function in C_0, f'_ϵ, such that $\|f - (f_\epsilon + f'_\epsilon)\| \leqq \epsilon$. But f_ϵ is the uniform limit of finite linear combinations of the g_u's, hence $T_n(f_\epsilon + f'_\epsilon) \to V(f_\epsilon + f'_\epsilon)$. Since $\sup \|T_n\| < \infty$, it follows by letting $\epsilon \to 0$ that $T_n f \to Vf$ in the given mode of convergence.

3. Under the hypothesis in 3, it follows by arguments analogous to those given in part 2 of the proof that $h \circ \tilde{T}_m - h \circ \tilde{T}_n \to 0$ in the given mode of convergence as $m, n \to \infty$. Then Theorem 6 again applies and for some $V \varepsilon O(G, L_\infty)$, $T_n \to V$ on C_0 in the given mode of convergence, and arguing as in 2, $\tilde{V}(0) = \xi(0)$, hence $T_n 1 \to V1$ in the given mode of convergence, and the assertion about C_∞ follows.

4. The assertions in 4 follow by 3 and Theorem 5, and the theorem is proved.

3. Convolutions. A multiplication is introduced in $O(G, L_\infty)$ by the convolution operation. Given two simple operators

$$T = \sum_{j=1}^{m} X_j \Delta_{a_j}, \qquad U = \sum_{k=1}^{n} Y_k \Delta_{b_k}$$

we define the *convolution* of T and U to be

$$T * U = \sum_{j=1}^{m} \sum_{k=1}^{n} X_j Y_k \Delta_{a_j + b_k} .$$

It follows immediately that the g.ch.f. of $T * U$, $\widetilde{T * U} = \tilde{T}\tilde{U}$, and it is easily verified that $\|T * U\| \leqq \|T\| \cdot \|U\|$. Then given any operators T, $U \in O(G, L_\infty)$, we can choose sequences of simple operators T_n and U_n as in Lemma 5 so that $T_n \to T$ and $U_n \to U$ $\quad C, o$ and C_0, s and with $\|T_n\| \leqq \|T\|$, $\|U_n\| \leqq \|U\|$. But then $\widetilde{T_n * U_n} = \tilde{T}_n \tilde{U}_n \to \tilde{T}\tilde{U}$ in order, $\tilde{T}\tilde{U}$ is order continuous and $\|T_n * U_n\| \leqq \|T\| \cdot \|U\|$. It follows by Theorem 7 that $\tilde{T}\tilde{U}$ is the g.ch.f. of some operator in $O(G, L_\infty)$. We then define the *convolution* of any two operators T, $U \in O(G, L_\infty)$ to be the operator $T * U$ determined by the g.ch.f. $\tilde{T}\tilde{U}$. To simplify notation we will write TU for $T * U$ whenever convenient. Note that, using the sequence of simple operators we have constructed, $T_n U_n \to TU$ $\quad AP_\infty$, o and C_0, s upon applying Theorems 6 and 7. Also, the convolution of positive simple operators is positive, and it follows that the convolution of arbitrary positive operators is positive.

The foregoing discussion yields immediately

THEOREM 8. *Convolution is an associative and commutative operation, and the space $O(G, L_\infty)$ is an algebra under the convolution multiplication and has the multiplicative unit $I = 1.\ \Delta_0$, where 1 denotes the unit function in L_∞. The subspace of positive operators is invariant under convolution. Moreover, convolution is related to the norm through the inequality $\|TU\| \leqq \|T\| \cdot \|U\|$.*

Corresponding to each $g \in G$ is a *reflected function* g' defined by $g'(x) = g(-x)$. Then given $T \in O(G, L_\infty)$, we define the *reflected operator* T' of T by $T'g = Tg'$. An operator T will be called *symmetric* if $T = T'$, and given any operator $T \in O(G, L_\infty)$, we define the *symmetrized operator* T^s of T by $T^s = T * T'$. One readily verifies that an operator is symmetric if and only if its g.ch.f. is real. In particular, the symmetrized operator T^s is symmetric and has g.ch.f. $|\tilde{T}|^2$.

A positive operator T will be said to be *infinitely divisible* if for every integer $n \geqq 1$ there exists a positive $T_n \in O(G, L_\infty)$ such that $T_n^n = T$.

LEMMA 8. *The class of infinitely divisible operators is closed under convolutions, and, if operators T_n are infinitely divisible and $T_n \to T$ $\quad C_\infty$, $b - \mu$ (a fortiori, in any stronger sense), then T is infinitely divisible.*

PROOF. The first assertion is immediate since the convolution of positive operators is positive. If $T_n \to T$ $\quad C_\infty$, $b - \mu$, then $T_n \to T$ $\quad C$, $b - \mu$ by Theorem 5 since the T_n are positive, hence $\tilde{T}_n \to \tilde{T}$ $\quad b - \mu$. For every integer $m \geqq 1$, $\tilde{T}_n^{1/m}$ is the g.ch.f. of a positive operator $T_{n,m}$ with $\|T_{n,m}\| \leqq \|T_n\|^{1/m} = \|T_n 1\|_\infty^{1/m}$ and, by hypothesis these norms are uniformly bounded. Moreover $\tilde{T}_{n,m} \to \tilde{T}^{1/m}$ $\quad b - \mu$, and $\tilde{T}^{1/m}$ is order continuous, hence is a g.ch.f. of an operator V_m. But then $T_{n,m} \to V_m \quad C, b - \mu$ and the $T_{n,m}$ are positive, hence V_m is positive, and the lemma is proved.

Note that Lemma 8 is not true if C_∞, $b - \mu$ is replaced by C, L_1. For example, let $(S, \mathcal{S}, \mu)$ be the Lebesgue interval [0, 1], let $A_n = \bigcup_{k=0}^{2^{n-1}-1} [2k/2^n, (2k+1)/2^n]$ and define T_n by

$$T_n g = g(1) I_{A_n} + g(0) I_{A_n^c}.$$

Then $T_n \geqq 0$, and setting

$$T_{n,m} g = g(1/m) I_{A_n} + g(0) I_{A_n^c}$$

we have that $T_{n,m} \geqq 0$ and $T_n = T_{n,m}^m$. It follows that the T_n are infinitely divisible. Moreover, $T_n \to T \quad C, L_1$ given by

$$Tg = \tfrac{1}{2}(g(1) + g(0)).$$

Thus T has g.ch.f. corresponding to the characteristic function of the binomial distribution and is not infinitely divisible.

Furthermore, the convergences $T_n \to T \quad C, L_1$, $V_n \to V \quad C, L_1$ do not imply that $T_n * V_n \to T * V$ in any sense. In fact, let $T_n = V_n$ be defined as in the above example. Then

$$T_n * V_n g = T_n^2 g = g(2) I_{A_n} + g(0) I_{A_n^c} \to \tfrac{1}{2}(g(2) + g(0))$$

in the L_1 topology for all $g \in C$, but

$$T * Vg = T^2 g = \tfrac{1}{4}(g(2) + 2g(1) + g(0)).$$

4. Probabilistic representation of order continuous operators. The operators we have been studying are closely connected to conditional probability distributions. Thus let $(\Omega, \mathcal{A}, P)$ be a probability space and $\mathcal{A}^*$ be a sub-σ-field of $\mathcal{A}$. Then every random variable X on $(\Omega, \mathcal{A}, P)$ to the Borel line induces an operator T_X on G to $L_\infty(\Omega, \mathcal{A}^*, P)$ through the relation $T_X g = E(g(X) \mid \mathcal{A}^*)$, or, equivalently, $T_X g$ is the $\mathcal{A}^*$ measurable function determined by

$$\int_A T_X g = \frac{1}{\mu \mathcal{S}} \int_A g(X), \qquad A \in \mathcal{A}^*.$$

We then call T_X the $\mathcal{A}^*$ *conditional distribution* or $\mathcal{A}^*$ *conditional operator* of X. The operator T_X is order continuous (see Loève [7, p. 348]), hence in $O(G, L_\infty(\Omega, \mathcal{A}^*, P))$. Moreover, T_X is positive and $T_X 1 = 1$.

Let $O_P(G, L_\infty(S, \mathcal{S}, \mu))$ denote the subspace of $O(G, L_\infty(S, \mathcal{S}, \mu))$ of all positive operators T with $T1 = 1$, and, given the probability space $(\Omega, \mathcal{A}, P)$ with sub-σ-field $\mathcal{A}^*$, let $O(\mathcal{A}^*)$ denote the subspace of $O(G, L_\infty(\Omega, \mathcal{A}^*, P))$ of all $\mathcal{A}^*$ conditional operators of random variables.

THEOREM 9. *Let $(S, \mathcal{S}, \mu))$ be a finite positive measure space. Then there is a probability space $(\Omega, \mathcal{A}, P)$ and a sub-σ-field $\mathcal{A}^*$ of $\mathcal{A}$ such that $O_P(G, L_\infty(S, \mathcal{S}, \mu))$ is homeomorphic to $O(\mathcal{A}^*)$ in the relative normed topologies of these spaces.*

PROOF. Let $(\Omega, \mathcal{A}, P)$ be the product measure space with the coordinate spaces $(S, \mathcal{S}, \mu)$ and the Lebesgue interval $[0, 1/\mu S]$, and let $\mathcal{A}^*$ be the sub-σ-field of cylinders with base in $\mathcal{S}$. Given $T \varepsilon O_P(G, L_\infty(S, \mathcal{S}, \mu))$, define a function ξ on R to $L_\infty(\Omega, \mathcal{A}^*, P)$ by

$$\int_{\mathcal{C}(A)} \xi(x)\, dP = \frac{1}{\mu \mathcal{S}} \int_A TI_{(-\infty, x)}\, d\mu, \qquad A \varepsilon \mathcal{S}.$$

where $\mathcal{C}(A)$ denotes the cylinder in Ω with base A. Then ξ is a nondecreasing g.d.f. with $\lim_{x\to-\infty}\xi(x) = 0$ and $\lim_{x\to+\infty}\xi(x) = 1$.

But then ξ has the properties of a conditional probability distribution function, and proceeding as in [7, pp. 361–362], we can find a regular conditional distribution function $\eta(x, \omega)$. That is, η on $R \times \Omega$ to $[0, 1]$ is a distribution function in x for each fixed $\omega \varepsilon \Omega$, and, for each fixed $x \varepsilon R$, $\eta(x, \cdot)$ is $\mathcal{A}^*$ measurable and in the equivalence class of $\xi(x)$. Since η is $\mathcal{A}^*$ measurable for each x, we can consider it to be a function on $R \times S$. Then we define a random variable X_T on $(\Omega, \mathcal{A}, P)$ by setting $[X_T < x]_s = [0, \eta(x, s)/\mu S]$ for all $x \varepsilon R$, $s \varepsilon S$, where $[X_T < x]_s$ denotes the section at s of the inverse image under X_T of the interval $(-\infty, x)$. We then define the mapping τ of $O_P(G, L_\infty(S, \mathcal{S}, \mu))$ into $O(\mathcal{A}^*)$: $T \to T_X$ where T_X is the $\mathcal{A}^*$ conditional distribution of X_T. Then

$$\int_{\mathcal{C}(A)} T_X\, g\, dP = \int_{\mathcal{C}(A)} g(X_T)\, dP = \frac{1}{\mu \mathcal{S}} \int_A Tg\, d\mu, \qquad A \varepsilon \mathcal{S}, g \varepsilon G$$

and every random variable X on $(\Omega, \mathcal{A}, P)$ determines an operator in $O_P(G, L_\infty(S, \mathcal{S}, \mu))$ through this same relation. We obtain that τ is a homeomorphism, and the theorem is proved.

With the above result it follows readily that any order continuous operator T can be represented in the form $T = Z_1 T_{X_1} - Z_2 T_{X_2}$, where T_{X_1} and T_{X_2} are $\mathcal{A}^*$ conditional operators of random variables and Z_1 and Z_2 are nonnegative $\mathcal{A}^*$ measurable random variables. Moreover, operators of this more general type may be useful in probability theory. For example, in Section 5 use is made of operators that are the differences of two conditional probability operators.

If random variables X and Y, with $\mathcal{A}^*$ conditional operators T and U, respectively, are conditionally independent given $\mathcal{A}^*$, then

$$E^{\mathcal{A}^*} e^{iu(X+Y)} = E^{\mathcal{A}^*} e^{iuX} \cdot E^{\mathcal{A}^*} e^{iuY} \quad \text{a.s.}$$

for all u. Thus, if V is the $\mathcal{A}^*$ conditional operator of $X + Y$, then $\tilde{V} = \tilde{T}\tilde{U}$, hence $V = T * U$. Moreover, given operators T, $U \varepsilon O_P(G, L_\infty(S, \mathcal{S}, \mu))$, it is possible to modify the probability space constructed in Theorem 9 so that T and U are the conditional operators of random variables X and Y, respectively, where X and Y are conditionally independent given $\mathcal{A}^*$. Then it follows that $T * U$ is the conditional operator of $X + Y$. To summarize, *convolution of $\mathcal{A}^*$ conditional operators of random variables corresponds to addition of $\mathcal{A}^*$ conditionally independent random variables.*

5. Application to a mixing problem—central structure. In this section the basic frame of reference will be a probability space $(\Omega, \mathcal{A}, P)$ together with a sub-σ-field $\mathcal{A}_0$ of $\mathcal{A}$ and a group of measure preserving set translations Λ_t on $\mathcal{A}$ (that is, $\Lambda_{s+t} = \Lambda_s\Lambda_t$, Λ_0 is the identity, and for each t, Λ_t maps $\mathcal{A}$ into $\mathcal{A}$, commutes with countable set operations, $\Lambda_t\Omega = \Omega$ and $P\Lambda_t A = PA$, $A \varepsilon \mathcal{A}$), where either t takes all real values (continuous case) or $t = 0, \pm 1, \pm 2, \cdots$ (discrete case). Intuitively, $\mathcal{A}_0$ represents the events observable at time 0, and $\mathcal{A}_t = \Lambda_t\mathcal{A}_0$ represents the events observable at time t. We let $\mathcal{P}_t$ and $\mathcal{F}_t$ denote the σ-fields generated by $\bigcup_{s \leqq t}\mathcal{A}_s$ and $\bigcup_{s \geqq t}\mathcal{A}_s$, respectively. We will call $\mathcal{P}_t$ and $\mathcal{F}_t$ the σ-fields of past and future at time t, respectively, and let $\mathfrak{J}_P = \bigcap \mathcal{P}_t$ and $\mathfrak{J}_F = \bigcap \mathcal{F}_t$ denote, respectively, the tail σ-fields of past and future. An event A is invariant if $\Lambda_t A = A$ for all t, and we let $\mathcal{C}$ denote the σ-field of invariant events. Finally, we let $\mathfrak{D}_t$ denote the sub-σ-field of $\mathcal{P}_t$ generated by the events

$$\{[P^{\mathcal{P}_t}A < a], \qquad A \varepsilon \mathcal{F}_t, \qquad -\infty < a < +\infty\}.$$

Thus, $\mathfrak{D}_t$ is the smallest σ-field for which the conditional probabilities $P^{\mathcal{P}_t}A$ are measurable for all $A \varepsilon \mathcal{F}_t$, and we call $\mathfrak{D}_t$ the σ-field of dependence at time t.

The set translations Λ_t extend to the class of all random variables X on $(\Omega, \mathcal{A}, P)$ through the relation $[\Lambda_t X \varepsilon B] = \Lambda_t[X \varepsilon B]$ for all Borel sets B. *We assume throughout this section that the random functions $\Lambda_t X$, $-\infty < t < \infty$, are measurable in the continuous case*, and for $t \neq 0$ we set

$$\bar{\Lambda}_t X = \frac{1}{t}\int_0^t \Lambda_s X \, ds$$

whenever the integral exists (in particular, the integral will exist if X is bounded). In the discrete case we set

$$\bar{\Lambda}_t X = \frac{1}{|t| + 1}\sum_{k=0}^{t} \Lambda_k X$$

for the integer-valued $t \neq 0$.

Let Y_n, Y, Z_n be random variables on $(\Omega, \mathcal{A}, P)$ with $\mathcal{A}^*$ conditional operators T_n, T, U_n, respectively, for some sub-σ-field $\mathcal{A}^*$ of $\mathcal{A}$. Then we say that the *$\mathcal{A}^*$ conditional law of Y_n converges to the $\mathcal{A}^*$ conditional law of Y*, denoted $\mathcal{L}^{\mathcal{A}^*}(Y_n) \to \mathcal{L}^{\mathcal{A}^*}(Y)$, if $T_n \to T \quad C, b - P$ (note that $b - \mu$ becomes $b - P$ since the range of T_n is now $L_\infty(\Omega, \mathcal{A}^*, P)$). Since the T_n are positive, an equivalent condition is that $T_n \to T \quad C_\infty, b - P$. We say that the $\mathcal{A}^*$-conditional laws of Y_n and Z_n are *asymptotically equivalent*, denoted $\mathcal{L}^{\mathcal{A}^*}(Y_n) \sim \mathcal{L}^{\mathcal{A}^*}(Z_n)$, if $T_n - U_n \to 0 \quad C_\infty, b - P$, or, equivalently, if $T_n - U_n \to 0 \quad C_0, b - P$, since $T_n 1 = U_n 1 = 1$.

The law of a random variable Y will be said to be $\mathcal{A}^*$ *infinitely divisible* if the $\mathcal{A}^*$ conditional operator of Y is infinitely divisible.

Note that, if $\mathcal{L}^{\mathcal{A}^*}(Y_n) \to \mathcal{L}^{\mathcal{A}^*}(Y)$ and $\mathcal{L}^{\mathcal{A}^*}(Y_n) \sim \mathcal{L}^{\mathcal{A}^*}(Z_n)$, then $U_n \to T \quad C_\infty, b - P$ and the U_n are positive, hence $U_n \to T \quad C, b - P$, and $\mathcal{L}^{\mathcal{A}^*}(Z_n) \to$

$\mathcal{L}^{\mathcal{A}^*}(Y)$. In particular, if the Z_n have $\mathcal{A}^*$ infinitely divisible laws, then by Lemma 10, Y has an $\mathcal{A}^*$ infinitely divisible law.

For sequences of random variables that are asymptotically invariant in a weak sense, $\mathcal{A}^*$ conditional limit laws reduce to $\mathcal{C}$ conditional limit laws provided $\mathcal{A}^* \supset \mathcal{C}$.

LEMMA 9. *If a sequence of uniformly bounded random variables Y_n satisfy $\Lambda_t Y_n - Y_n \to 0$ in probability for every fixed t as $n \to \infty$, then for every integrable Z,*

$$EY_nZ - EY_nE^{\mathcal{C}}Z \to 0.$$

PROOF. Since the Y_n are uniformly bounded, the ergodic theorem yields

$$\sup_n EY_n(\bar{\Lambda}_tZ - E^{\mathcal{C}}Z) \to 0$$

as $t \to \infty$. By the dominated convergence theorem

$$EY_n(Z - \Lambda_sZ) = EZ(Y_n - \Lambda_{-s}Y_n) \to 0$$

as $n \to \infty$ for each fixed s, hence $EY_n(Z - \bar{\Lambda}_tZ) \to 0$, and

$$EY_nZ - EY_nE^{\mathcal{C}}Z = EY_n(Z - \bar{\Lambda}_tZ) + EY_n(\bar{\Lambda}_tZ - E^{\mathcal{C}}Z) \to 0$$

as $n \to \infty$ and then $t \to \infty$. The lemma is proved.

THEOREM 10. *Let Y_n, Y be random variables with $\mathcal{A}^*$ conditional operators T_n, T, respectively, and $\mathcal{C}$ conditional operators U_n, U, respectively, where the σ-field $\mathcal{A}^* \supset \mathcal{C}$* a.s., *and let $\Lambda_tY_n - Y_n \to 0$ in probability for every fixed t as $n \to \infty$. Then $T_ng \to Tg$ in the L_1 topology of $L_\infty(\Omega, \mathcal{A}^*, P)$, $g \varepsilon C$, implies that Tg is invariant and $Tg = Ug$* a.s. *Moreover, $U_ng \to Ug$ in the L_1 topology of $L_\infty(\Omega, \mathcal{C}, P)$. In particular, if $\mathcal{L}^{\mathcal{A}^*}(Y_n) \to \mathcal{L}^{\mathcal{A}^*}(Y)$, then $\mathcal{L}^{\mathcal{A}^*}(Y) = \mathcal{L}^{\mathcal{C}}(Y)$ and $\mathcal{L}^{\mathcal{C}}(Y_n) \to \mathcal{L}^{\mathcal{C}}(Y)$.*

PROOF. For $g \varepsilon C$, the hypothesis implies that $\Lambda_tg(Y_n) - g(Y_n) = g(\Lambda_tY_n) - g(Y_n) \to 0$ in probability for each fixed t as $n \to \infty$. Then if $T_ng \to Tg$ in the L_1 topology and Z is integrable and $\mathcal{A}^*$ measurable, Lemma 9 yields

$$\begin{aligned} E(ZE^{\mathcal{A}^*}g(Y)) &= \lim E(ZE^{\mathcal{A}^*}g(Y_n)) = \lim E(g(Y_n)Z) \\ &= \lim E(g(Y_n)E^{\mathcal{C}}Z) = \lim E(E^{\mathcal{C}}Z\cdot E^{\mathcal{A}^*}g(Y_n)) = E(E^{\mathcal{C}}Z\cdot E^{\mathcal{A}^*}g(Y)). \end{aligned}$$

Hence

$$\begin{aligned} E(E^{\mathcal{A}^*}g(Y) - E^{\mathcal{C}}g(Y))Z &= E\{(E^{\mathcal{A}^*}g(Y) - E^{\mathcal{C}}g(Y))E^{\mathcal{C}}Z\} \\ &= E\{E^{\mathcal{C}}Z\cdot E^{\mathcal{C}}(E^{\mathcal{A}^*}g(Y) - E^{\mathcal{C}}g(Y))\} = 0. \end{aligned}$$

It follows that $E^{\mathcal{A}^*}g(Y) = E^{\mathcal{C}}g(Y)$ a.s. The assertions in the theorem are then immediate.

We say that X is an *additive random function* (of intervals) if for every $s < t$, $X(s, t)$ is a random variable and if for every $s < t < u$,

$$X(s, t) + X(t, u) = X(s, u).$$

An additive random function is *structure preserving on* $(\Omega, \mathcal{A}, \mathcal{A}_0, P)$ if for every

$s < t$ and every u,

1. $X(s, t)$ is measurable on $\mathcal{P}_t \cap \mathcal{F}_s$,
2. $X(s + u, t + u) = \Lambda_u X(s, t)$,

and we let $\mathfrak{X}$ denote the class of all structure preserving additive random functions. As examples of random functions in $\mathfrak{X}$ we may take X_0 measurable on $\mathcal{P}_0 \cap \mathcal{F}_0$ and set

$$X(s,t) = \int_s^t \Lambda_u X_0 \, du \quad \text{or} \quad X(s,t) = \sum_{s \leqq k < t} \Lambda_k X_0$$

in the continuous and discrete cases, respectively.

We say that a sequence of random functions $X_n \varepsilon \mathfrak{X}$ are *uniformly asymptotically locally negligible* (u.a.l.n.) if for every $\epsilon > 0$ and every finite $l > 0$,

$$\sup_{0 < t-s \leqq l} P[|X_n(s, t)| \geqq \epsilon] \to 0$$

as $n \to \infty$.

Loève [7, pp. 378–383], has introduced the weighted limit problem, that is, the problem of finding conditions under which the $\mathcal{A}^*$ conditional laws of a sequence of random variables converge to some type of weighted law. As a natural extension of the central limit problem, we may ask when $\mathcal{A}^*$ conditional limit laws of sequences $X_n(s_n, t_n)$ of u.a.l.n. random functions $X_n \varepsilon \mathfrak{X}$ are $\mathcal{A}^*$ infinitely divisible. But u.a.l.n. random functions $X_n \varepsilon \mathfrak{X}$ satisfy the condition

$$\Lambda_t X_n(s_n, t_n) - X_n(s_n, t_n) \to 0$$

in probability for each fixed t as $n \to \infty$ however the sequences $s_n < t_n$ of numbers are chosen. It follows from Theorem 10 that we can reduce $\mathcal{A}^*$ to $\mathcal{C}$ whenever $\mathcal{A}^* \supset \mathcal{C}$ a.s., and it is easily seen that the $\mathcal{A}^*$ infinitely divisible property of limit laws mentioned above cannot hold unless $\mathcal{A}^* \supset \mathcal{C}$ a.s. Thus we confine our attention to the invariant σ-field. We say that $(\Omega, \mathcal{A}, \mathcal{A}_0, P)$ has *central structure* if, for every sequence of u.a.l.n. random functions $X_n \varepsilon \mathfrak{X}$ and numbers $s_n < t_n$, letting T_n denote the $\mathcal{C}$-conditional operator of $X_n(s_n, t_n)$, there exists a sequence of infinitely divisible operators U_n with range in $L_\infty(\Omega, \mathcal{C}, P)$ such that

$$T_n - U_n \to 0 \qquad C_\infty, b - P.$$

Clearly, if $(\Omega, \mathcal{A}, \mathcal{A}_0, P)$ has central structure and $\mathcal{L}^{\mathcal{C}}(X_n(s_n, t_n)) \to \mathcal{L}^{\mathcal{C}}$ for u.a.l.n. random functions $X_n \varepsilon \mathfrak{X}$ and some $\mathcal{C}$-conditional law, then $\mathcal{L}^{\mathcal{C}}$ is $\mathcal{C}$-infinitely divisible. In particular, if the structure is ergodic ($\mathcal{C} = \{\Omega, \phi\}$ a.s.), then the conditional laws become unconditioned laws and the limits become infinitely divisible.

We will study conditions for central structure in the remainder of this section and will show that central structure is closely related to a kind of mixing: let

$$d(P, \mathcal{A}_0, t) = \sup_{A \varepsilon \mathcal{F}_0} E\, |E^{\mathcal{P}_0} \bar{\Lambda}_t I_A - P^{\mathcal{C}} A|.$$

The quantity $d(P, \mathcal{A}_0, t)$ measures the rate of mixing within the ergodic sets and we say that $(\Omega, \mathcal{A}, \mathcal{A}_0, P)$ is *uniformly ergodic* on $\mathcal{C}$ if $d(P, \mathcal{A}_0, t) \to 0$ as $t \to \infty$.

We have introduced a stochastic structure in this section depending on the four elements $(\Omega, \mathcal{A}, \mathcal{A}_0, P)$ and the group of set translations Λ_t. If the σ-field $\mathcal{A}_0$ is replaced by any other sub-σ-field $\mathcal{A}^*$ of $\mathcal{A}$, then a related but different structure is obtained. Note that, if $\mathcal{A}^* \supset \mathcal{A}_0$, then the past and future σ-field induced by $\mathcal{A}^*$ will be at least as large as the past and future σ-fields induced by $\mathcal{A}_0$, hence $d(P, \mathcal{A}^*, t) \geqq d(P, \mathcal{A}_0, t)$. We will be particularly interested in the structures obtained by replacing $\mathcal{A}_0$ by $\mathcal{D}_0$ or $\mathcal{P}_t \cap \mathcal{F}_{-t}$. To avoid confusion, *the symbols $\mathcal{P}_t$ and $\mathcal{F}_t$ will always refer to the past and future σ-fields induced by $\mathcal{A}_0$.*

LEMMA 10. *Let Y, Z be real- or complex-valued random variables measurable on $\mathcal{P}_t$, $\mathcal{F}_t$, respectively, and let $|Y| \leqq 1$, $|Z| \leqq 1$. Then for $l > 0$,*

$$E\,|E^{\mathcal{C}} Y \bar{\Lambda}_l Z - E^{\mathcal{C}} Y E^{\mathcal{C}} Z| \leqq 4d(P, \mathcal{A}_0, l).$$

PROOF. Let $\mathcal{A}'$ denote the σ-field generated by $\bigcup \mathcal{A}_t$, the union over all t. and $\mathcal{C}' = \mathcal{C} \cap \mathcal{A}'$. Then it follows (see [1], lemma 2) that $\mathcal{C}' \subset \mathcal{P}_t \cap \mathcal{F}_t$ a.s, Moreover, for any $\mathcal{A}'$ measurable function W, we have $E^{\mathcal{C}} W = E^{\mathcal{C}'} W$ a.s. The ensuing computations are based on the smoothing properties of conditional expectations and the stationarity relations valid for any two bounded random variables $W_1, W_2, EW_1 \bar{\Lambda}_l W_2 = EW_2 \bar{\Lambda}_{-l} W_1$.

Assume first that Z is real-valued and let $A = [E^{\mathcal{P}_t} \bar{\Lambda}_l Z - E^{\mathcal{C}} Z \geqq 0]$. Then

$$\begin{aligned} E\,|E^{\mathcal{C}} Y \bar{\Lambda}_l Z - E^{\mathcal{C}} Y E^{\mathcal{C}} Z| &= E\,|E^{\mathcal{C}'} Y (E^{\mathcal{P}_t} \bar{\Lambda}_l Z - E^{\mathcal{C}} Z)| \\ &\leqq E\,|E^{\mathcal{P}_t} \bar{\Lambda}_l Z - E^{\mathcal{C}} Z| = 2EI_A(E^{\mathcal{P}_t} \bar{\Lambda}_l Z - E^{\mathcal{C}} Z) \\ &= 2E(Z \bar{\Lambda}_{-l} I_A - P^{\mathcal{C}} A E^{\mathcal{C}} Z) = 2EZ(E^{\mathcal{F}_t} \bar{\Lambda}_{-l} I_A - P^{\mathcal{C}} A) \\ &\leqq 4 \sup_{B \in \mathcal{F}_t} EI_B(E^{\mathcal{F}_t} \bar{\Lambda}_{-l} I_A - P^{\mathcal{C}} A) = 4 \sup_{B \in \mathcal{F}_t} EI_A(E^{\mathcal{P}_t} \bar{\Lambda}_l I_B - P^{\mathcal{C}} B) \\ &\leqq 2d(P, \mathcal{A}_0, l). \end{aligned}$$

The case that Z is complex then follows by applying this bound separately to the real and complex parts of Z, and the lemma is proved.

LEMMA 11. *Let $X \in \mathcal{X}$, and for $\epsilon, l > 0$ let*

$$\sup_{0 < t-s \leqq l} P[|X(s, t)| > \epsilon] = \delta,$$

and for $r < s < t$ let T, U, V denote the $\mathcal{C}$-conditional operators of $X(r, s)$, $X(s, t)$, $X(r, t)$, respectively. Then

$$E|Tg_u \cdot Ug_u - Vg_u| \leqq 2(\delta + |u|\epsilon + 2d(P, \mathcal{A}_0, l))$$

where $g_u(x) = e^{iux}$, u real.

PROOF. If $|x - y| \leqq \epsilon$, then $|g_u(x) - g_u(y)| \leqq |u|\epsilon$. For $0 < h \leqq l$, the hypothesis then implies that $E|\Lambda_h g_u(X(s, t)) - g_u(X(s, t))| = E|g_u(\Lambda_h X(s, t)) - g_u(X(s, t))| \leqq 2(\delta + |u|\epsilon)$, hence

$$\begin{aligned} &E|E^{\mathcal{C}} g_u(X(r, s)) \{\bar{\Lambda}_l g_u(X(s, t)) - g_u(X(s, t))\}| \\ &\quad = E|E^{\mathcal{C}}\{g_u(X(r, s)) \cdot \bar{\Lambda}_l g_u(X(s, t))\} - E^{\mathcal{C}} g_u(X(r, t))| \leqq 2(\delta + |u|\epsilon). \end{aligned}$$

The proof is completed by applying Lemma 10 and the definition of T, U, V.

LEMMA 12. *For every $g \varepsilon C_\infty$ there exist constants a_k depending only on g and k such that $a_k \to 0$ as $k \to \infty$ and such that for every conditioning σ-field $\mathcal{A}^*$ and every $\mathcal{A}^*$ conditional operator T of a random variable Y and every k, there exists an infinitely divisible operator U satisfying $\|T^k g - Ug\| \leqq a_k$.*

PROOF. Let $F(\omega, x)$ be a regular conditional distribution function of Y given $\mathcal{A}^*$. That is, $F(\omega, x)$ is a distribution function in x for each ω and $\mathcal{A}^*$-measurable in ω for each x and $F(\cdot, x) = P^{\mathcal{A}^*}[Y < x]$ a.s. For integers $1 \leqq m < n$ and integers j, set

$$j_{nm}(\omega) = \max \{j: F(\omega, (j/n)) \leqq (m/n)\}$$

and set $j_{n0} = -\infty$ and

$$F_n(\omega, x) = \begin{cases} m/n & \text{for } \; j_{nm}(\omega) < nx \leqq j_{n,m+1}(\omega), \quad m = 0, \cdots, n-2, \\ 1 & \text{for } \; nx > j_{n,n-1}(\omega). \end{cases}$$

The F_n are regular conditional distribution functions and we let $F_n^{(k)}$, $F^{(k)}$ denote the kth convolution of F_n, F, respectively. For any two regular conditional distribution functions $G_1(\omega, x)$ and $G_2(\omega, x)$, let $d(G_1, G_2)$ denote the least upper bound (as ω varies) of the Paul Lévy distances between the distribution functions $G_1(\omega, \cdot)$ and $G_2(\omega, \cdot)$ (for a discussion of Paul Lévy distance see [3]). One easily verifies that, given $g \varepsilon C_\infty$ and $\epsilon > 0$, there exists a $\delta > 0$ such that $d(G_1, G_2) < \delta$ implies

$$\left| \int g(x) G_1(\omega, dx) - \int g(x) G_2(\omega, dx) \right| < \epsilon$$

for all ω. Now $T^k g$ is the equivalence class in $L_\infty(\Omega, \mathcal{A}^*, P)$ determined by $\int g(x) F^{(k)}(\cdot, dx)$, and it follows that to prove the lemma it will be sufficient to show that, given $\epsilon > 0$, there exists a k_ϵ such that for all $k \geqq k_\epsilon$ and all kth convolutions $F^{(k)}$ of regular conditional distribution functions, there exist regular conditional distribution functions $G(\omega, x)$ that are infinitely divisible for each ω and satisfying $d(F^{(k)}, G) < \epsilon$.

But $d(F_n^{(k)}, F^{(k)}) \to 0$ as $n \to \infty$, hence we can confine our attention to regular conditional distribution functions of the type $F_n^{(k)}$. The $F_n^{(k)}$ are nondecreasing and can assume only the values j/n^k, $j = 0, \cdots, n^k$ and can have discontinuities only at points that are multiples of $1/n$. It follows that for any n there are only a countable number of different distribution functions $F_n^{(k)}(\omega, \cdot)$ possible. Let H_i, $i = 1, 2, 3, \cdots$, be an enumeration of these possibilities.

We now appeal to a theorem of Kolmogorov [6]. According to this theorem, there exists a constant c, not depending on the distribution functions H_i (all of which are kth convolutions of distribution functions), and infinitely divisible distribution functions H_i' such that

$$\sup_x |H_i(x) - H_i'(x)| \leqq ck^{-1/5},$$

hence the Paul Lévy distance is also bounded by $\sqrt{2}\, ck^{-1/5}$.

Let G_n be defined by $G_n(\omega, x) = H_i'(x)$ for those ω such that $F_n^{(k)}(\omega, \cdot) = H_i$, $i = 1, 2, 3, \cdots$. Then G_n is a regular conditional distribution function and $d(F_n^{(k)}, G_n) \leqq ck^{-1/5}$. Moreover, the operator U mapping g into the equivalence class in $L_\infty(\Omega, \mathfrak{A}^*, P)$ determined by $\int gG_n(\cdot, dx)$ is infinitely divisible. The lemma is proved.

LEMMA 13. *Given $0 < a < b$ and $\epsilon > 0$, there exists a $\delta > 0$ such that $P[Y \geqq a] < \epsilon$ for every random variable Y having a symmetric infinitely divisible distribution and satisfying $P[Y \geqq b] < \delta$.*

PROOF. Let $Z_1, \cdots, Z_n$ be independent and identically distributed and $\sum_{k=1}^n Z_k = Y$. Then the Z_k are symmetrically distributed and $P[Z_k \geqq b/n] \leqq P^{1/n}[Y \geqq b]$, hence

$$P\left[\max_{1 \leqq k \leqq n} | Z_k | \geqq \frac{b}{n}\right] \leqq 2nP^{1/n}[Y \geqq b].$$

Moreover, a regular version of the conditional distribution of Y given $|Z_1|, \cdots, |Z_n|$ is obtained by letting $W_1, \cdots, W_n$ be independent random variables with $P[W_k = \pm|Z_k|] = \frac{1}{2}$. It follows by Chebychev's inequality that

$$P[Y \geqq a \mid | Z_1 |, \cdots, | Z_n |] \leqq \frac{1}{a^2} \sum_{k=1}^n | Z_k |^2 \text{ a.s.}$$

But then

$$P[Y \geqq a] \leqq P\left[\max_{1 \leqq k \leqq n} |Z_k| \geqq \frac{b}{n}\right] + \frac{b^2}{na^2} \leqq 2nP^{1/n}[Y \geqq b] + \frac{b^2}{na^2},$$

and the lemma follows by selecting first n and then δ.

LEMMA 14. *Let Y_n be random variables with infinitely divisible $\mathfrak{A}^*$ conditional operators T_n, and let $T_n g \to 0 \quad b - P$ for some $g \in C_\infty$ with $g \geqq 0$ and $g(\infty) > 0$ and $T_n h \to h(0) \quad b - P$ for some $h \in C$ such that $h(x) \neq h(0)$ for $x \neq 0$. Then $T_n \to I \; C, b - P$.*

PROOF. Let Y_n^s denote the symmetrization of Y_n ($Y_n^s = Y_n - Y_n'$ where Y_n' is independent of Y_n with the same distribution as Y_n). By hypothesis, the $\mathfrak{A}^*$ conditional distribution of Y_n, hence of Y_n^s, is infinitely divisible. Also, the hypothesis implies that $P^{\mathfrak{A}^*}[|Y_n| \geqq b] \to 0$ in probability for some finite b, hence $P^{\mathfrak{A}^*}[|Y_n^s| \geqq 2b] \to 0$ in probability. Applying lemma 13 to the conditional distribution given $\mathfrak{A}^*$ of Y_n^s, it follows that $P^{\mathfrak{A}^*}[|Y_n^s| \geqq a] \to 0$ in probability for every $a > 0$. Thus $Y_n^s \to 0$ in probability, and, letting m_n denote the median of Y_n, $Y_n - m_n \to 0$ in probability (see Loéve [7, p. 245]). It follows that $T_n f - f(m_n) \to 0 \quad b - P$ for every $f \in C$. But then $g(m_n) \to 0$ and $h(m_n) \to h(0)$. The first convergence implies that the m_n are bounded, and the second then implies that $m_n \to 0$, hence $T_n \to I \quad C, b - P$. The lemma is proved.

The structure $(\Omega, \mathfrak{A}, \mathfrak{A}_0, P)$ is said to be *Markov dependent* if $\mathcal{P}_0$ and $\mathfrak{F}_0$ are conditionally independent given $\mathfrak{A}_0$, or, equivalently, if $\mathfrak{D}_0 \subset \mathfrak{A}_0$ a.s.

LEMMA 15. *The structure $(\Omega, \mathfrak{A}, \mathfrak{D}_0, P)$ is always Markov dependent.*

This lemma is proved in [1] for the discrete case. Using the Markov Equivalence Theorem [7, p. 565], the proof for the discrete case extends directly to the continuous case.

THEOREM 11.

1. *For any sub-σ-field* $\mathfrak{A}_0$ *of* $\mathfrak{A}$, *if the structure* $(\Omega, \mathfrak{A}, \mathfrak{A}_0, P)$ *is uniformly ergodic on* $\mathfrak{C}$, *then* $(\Omega, \mathfrak{A}, \mathfrak{A}_0, P)$ *has central structure.*

2. *Let* $\mathcal{P}_t$ *and* $\mathfrak{F}_t$ *denote, respectively, the past and future σ-fields at time t induced by* $\mathfrak{A}_0$. *Then for every* $t > 0$, $(\Omega, \mathfrak{A}, \mathfrak{A}_0, P)$ *is uniformly ergodic on* $\mathfrak{C}$ *if and only if* $(\Omega, \mathfrak{A}, \mathcal{P}_t \cap \mathfrak{F}_{-t}, P)$ *is uniformly ergodic on* $\mathfrak{C}$.

3. *If* $(\Omega, \mathfrak{A}, \mathfrak{A}_0, P)$ *has central structure and is Markov dependent, then both* $(\Omega, \mathfrak{A}, \mathfrak{A}_0, P)$ *and* $(\Omega, \mathfrak{A}, \mathfrak{D}_0, P)$ *are uniformly ergodic on* $\mathfrak{C}$, *where* $\mathfrak{D}_0$ *denotes the dependence σ-field of* $(\Omega, \mathfrak{A}, \mathfrak{A}_0, P)$ *at time* 0.

4. *The structure* $(\Omega, \mathfrak{A}, \mathfrak{D}_0, P)$ *is uniformly ergodic on* $\mathfrak{C}$ *if and only if it has central structure, and in this case* $(\Omega, \mathfrak{A}, \mathfrak{A}_0, P)$ *is also uniformly ergodic on* $\mathfrak{C}$, *hence has central structure.*

PROOF.

1. Let $d(P, \mathfrak{A}_0, t) \to 0$ as $t \to \infty$. Given a sequence of u.a.l.n. random functions $X_n \, \varepsilon \, \mathfrak{X}$ and $s_n < t_n$ and a fixed integer $k > 1$, set $s_{nj} = s_n + j(t_n - s_n)/k$, $j = 0, \cdots, k$, and let $Y_{nj} = X_n(s_{n,j-1}, s_{nj})$, $j = 1, \cdots, k$. Let T_n, $T_{n,j}$ and U_n denote the $\mathfrak{C}$-conditional operators of $X_n(s_n, t_n)$, $\sum_{i=1}^{j} Y_{ni}$ and Y_{n1}, respectively. Since X_n is structure preserving, U_n is also the $\mathfrak{C}$-conditional operator of Y_{nj} for all j. Given ϵ, $l > 0$, choose n large enough so that

$$\sup_{0<t-s\leqq l} P[|X_n(s, t)| > \epsilon] \leqq \epsilon,$$

and let $g_u(x) = e^{iux}$. Then by Lemma 11,

$$E|T_n g_u - U_n^k g_u| \leqq \sum_{j=2}^{k} E|T_{n,j} g_u - T_{n,j-1} g_u U_n g_u|$$
$$\leqq 2(k-1)(\epsilon + |u|\epsilon + 2d(P, \mathfrak{A}_0, l)).$$

It follows by letting $n \to \infty$ and then $\epsilon \to 0$ and $l \to \infty$, that $\tilde{T}_n - \tilde{U}_n^k \to 0$ boundedly in probability for each fixed k and as $k = k(n) \to \infty$ sufficiently slowly. By Theorem 7, $T_n - U_n^k \to 0 \quad AP_\infty, b - P$, and by Lemma 12, there exist infinitely divisible operators V_n such that $U_n^k - V_n \to 0 \quad C_\infty, b - P$ as $k = k(n) \to \infty$. Hence $T_n - V_n \to 0 \quad C_\infty, b - P$, and the first assertion is proved.

2. Since $\mathfrak{A}_0 \subset \mathcal{P}_t \cap \mathfrak{F}_{-t}$ for $t > 0$,

$$d(P, \mathcal{P}_t \cap \mathfrak{F}_{-t}, t) \geqq d(P, \mathfrak{A}_0, t),$$

and it suffices to prove that, if $(\Omega, \mathfrak{A}, \mathfrak{A}_0, P)$ is uniformly ergodic on $\mathfrak{C}$, then the same is true of $(\Omega, \mathfrak{A}, \mathcal{P}_t \cap \mathfrak{F}_{-t}, P)$. But the past (future) σ-field at time 0 of $(\Omega, \mathfrak{A}, \mathcal{P}_t \cap \mathfrak{F}_{-t}, P)$, is contained in $\mathcal{P}_t(\mathfrak{F}_{-t})$. Thus, if A is in the future σ-field at time 0 of $(\Omega, \mathfrak{A}, \mathcal{P}_t \cap \mathfrak{F}_{-t}, P)$, then $\Lambda_{2t}A$ is in $\mathfrak{F}_t$, and for $s > 0$,

$$E|E^{\mathcal{P}_t}\bar{\Lambda}_s I_A - P^{\mathfrak{C}}A| \leqq E|\bar{\Lambda}_s I_A - \bar{\Lambda}_s \Lambda_{2t} I_A| + E|E^{\mathcal{P}_t}\bar{\Lambda}_s \Lambda_{2t} I_A - P^{\mathfrak{C}}A|$$
$$\leqq (2t/s) + d(P, \mathfrak{A}_0, s).$$

It follows that

$$d(P, \mathcal{P}_t \cap \mathfrak{F}_{-t}, s) \leqq (2t/s) + d(P, \mathfrak{A}_0, s),$$

and, if, as $s \to \infty$, $d(P, \mathfrak{A}_0, s) \to 0$, then $d(P, \mathcal{P}_t \cap \mathfrak{F}_{-t}, s) \to 0$.

3. Let $(\Omega, \mathfrak{A}, \mathfrak{A}_0, P)$ be Markov dependent and not uniformly ergodic on $\mathfrak{C}$. Then $\mathfrak{D}_0 \subset \mathfrak{A}_0$ and there is a sequence of events $A_n \varepsilon \mathfrak{F}_0$ and numbers $t_n \to \infty$ and an $\epsilon > 0$ such that

$$E|E^{\mathcal{P}_0}\bar{\Lambda}_{t_n}I_{A_n} - P^{\mathfrak{C}}A_n| \geqq \epsilon.$$

Define random functions X_n by

$$X_n(s, t) = \frac{1}{t_n}\int_s^t \Lambda_u(P^{\mathcal{P}_0}A_n - P^{\mathfrak{C}}A_n)\,du$$

or

$$X_n(s, t) = \frac{1}{t_n}\sum_{s \leqq k < t} \Lambda_k(P^{\mathcal{P}_0}A_n - P^{\mathfrak{C}}A_n)$$

in the continuous and discrete cases, respectively. Clearly, the X_n are additive random functions and $P[|X_n(s, t)| > (t - s + 1)/t_n] = 0$, hence the X_n are u.a.l.n. Moreover, $P^{\mathcal{P}_0}A_n - P^{\mathfrak{C}}A_n = P^{\mathfrak{D}_0}A_n - P^{\mathfrak{C}}A$ is measurable on $\mathcal{P}_0 \cap \mathfrak{F}_0$, and it follows that the X_n are structure preserving on $(\Omega, \mathfrak{A}, \mathfrak{A}_0, P)$. Now suppose $(\Omega, \mathfrak{A}, \mathfrak{A}_0, P)$ has central structure and let T_n be the $\mathfrak{C}$-conditional operator of $X_n(0, t_n)$. Then there exist infinitely divisible operators U_n such that $T_n - U_n \to 0$ C_∞, $b - P$. Choose $g \varepsilon C_\infty$ so $g \geqq 0$, $g(\infty) > 0$ and $g(x) = 0$ for $|x| \leqq 1$, and $h \varepsilon C_\infty$ so $h(x) = x$ for $|x| \leqq 1$ and $h(x) \neq 0$ for $x \neq 0$. Then $T_n g \to 0$ and $T_n h \to 0$, hence $U_n g \to 0$ and $U_n h \to 0$. It follows by Lemma 14 that $U_n \to I$ C, $b - P$, hence $T_n \to I$ C_∞, $b - P$ (in fact, $C, b - P$). But then, letting $f \varepsilon C_\infty$ with $f(x) = |x|$ for $|x| \leqq 2$,

$$E|E^{\mathcal{P}_0}\bar{\Lambda}_{t_n}I_{A_n} - P^{\mathfrak{C}}A_n| \leqq ET_nf \to EIf = 0,$$

and it follows *ab contrario* that $(\Omega, \mathfrak{A}, \mathfrak{A}_0, P)$ does not have central structure. Thus, central structure implies uniform ergodicity for Markov dependent $(\Omega, \mathfrak{A}, \mathfrak{A}_0, P)$, and since $\mathfrak{D}_0 \subset \mathfrak{A}_0$, $(\Omega, \mathfrak{A}, \mathfrak{D}_0, P)$ is also uniformly ergodic on $\mathfrak{C}$.

4. The first assertion in 4 is immediate, since by Lemma 15, $(\Omega, \mathfrak{A}, \mathfrak{D}_0, P)$ is Markov dependent. Letting $\mathcal{P}_0$, $\mathfrak{F}_0$ be the past and future σ-fields at time 0 of $(\Omega, \mathfrak{A}, \mathfrak{A}_0, P)$, respectively, the following identity is valid for all $A \varepsilon \mathfrak{F}_0$ and $t > 0$:

$$E^{\mathcal{P}_0}\bar{\Lambda}_t I_A = E^{\mathfrak{D}_0}\bar{\Lambda}_t P^{\mathfrak{D}_0}A.$$

This relation follows easily from the measure preserving property of Λ_s and is discussed in [1]. Applying Lemma 10, it follows that

$$E|E^{\mathfrak{D}_0}\bar{\Lambda}_t P^{\mathfrak{D}_0}A - P^{\mathfrak{C}}D| \leqq 4d(P, \mathfrak{D}_0, t),$$

hence $d(P, \mathfrak{A}_0, t) \leqq 4d(P, \mathfrak{D}_0, t)$ (in fact, a slight modification of the inequalities in Lemma 10 yields $d(P, \mathfrak{A}_0, t) \leqq d(P, \mathfrak{D}_0, t)$). The last assertion follows, and the theorem is proved.

It is well known that if $(\Omega, \mathfrak{A}, \mathfrak{A}_0, P)$ is uniformly strongly mixing, that is, if

$$\sup_{A \varepsilon \mathcal{P}_0, B \varepsilon \mathfrak{F}_0} |PA\Lambda_t B - PAPB| \to 0$$

as $t \to \infty$, then the past and future tail σ-fields are $0 - 1$, that is, equivalent to $\{\Omega, \phi\}$. In the discrete time case there is an analogous connection between uniform ergodicity and the tail σ-field. We say that $(\Omega, \mathfrak{A}, \mathfrak{A}_0, P)$ is *uniformly ergodic* if

$$\sup_{A \varepsilon \mathfrak{F}_0} E|E^{\mathcal{P}_0}\bar{\Lambda}_t I_A - PA| \to 0$$

as $t \to \infty$. Letting $\mathfrak{A}'$ denote the σ-field generated by $\bigcup \mathfrak{A}_t$, the union over all t, an equivalent condition is that $\mathfrak{C} \cap \mathfrak{A}'$ be $0 - 1$ and $(\Omega, \mathfrak{A}, \mathfrak{A}_0, P)$ be uniformly ergodic on $\mathfrak{C}$. We say that a σ-field is finite if it contains only a finite number of sets.

THEOREM 12. *In the discrete time case, if $(\Omega, \mathfrak{A}, \mathfrak{A}_0, P)$ is uniformly ergodic, then the tail σ-fields $\mathfrak{J}_P$ and $\mathfrak{J}_F$ are equivalent to finite σ-fields.*

PROOF. Since $\mathfrak{J}_F \subset \mathfrak{F}_s$ for every s,

$$\sup_{A \varepsilon \mathfrak{J}_F} E|E^{\mathcal{P}_m}\bar{\Lambda}_n I_A - PA| \to 0$$

as $n \to \infty$ uniformly in m. But $E^{\mathcal{P}_m}\bar{\Lambda}_n I_A \to \bar{\Lambda}_n I_A$ a.s. as $m \to \infty$ since $\mathfrak{J}_F$ is contained in the σ-field generated by $\bigcup \mathcal{P}_m$. It follows that

$$\sup_{A \varepsilon \mathfrak{J}_F} E|\bar{\Lambda}_n I_A - PA| \to 0$$

as $n \to \infty$. Now suppose there exist sets $A_m \,\varepsilon\, \mathfrak{J}_F$ such that $0 < PA_m \to 0$. Because of ergodicity, $P \bigcup_{k=1}^{\infty} \Lambda_k A_m = 1$ for every m, hence there exist numbers k_n such that, setting

$$B_m = \bigcup_{k=0}^{k_n} A_k,$$

$PB_m \to \frac{1}{2}$. But $PB_m^c \Lambda_n B_m \leqq n\, PA_m \to 0$ as $m \to \infty$ for each fixed n. It follows that $E|\bar{\Lambda}_n I_{B_m} - PB_m| \to \frac{1}{2}$ as $m \to \infty$ for each fixed n. This contradiction then establishes that there is an $\epsilon > 0$ such that for every $A \,\varepsilon\, \mathfrak{J}_F$, either $PA \geqq \epsilon$ or $PA = 0$, and it follows easily that $\mathfrak{J}_F$ is equivalent to a finite σ-field. The assertion for $\mathfrak{J}_P$ follows by interchanging past and future and negative and positive time in the foregoing argument, since the hypothesis also entails

$$\sup_{A \varepsilon \mathcal{P}_0} E|E^{\mathfrak{F}_0}\bar{\Lambda}_{-n} I_A - PA| \to 0$$

as $n \to \infty$. The theorem is proved.

The following example will serve to show that the preceding theorem is not valid in the continuous time case: let Ω be the unit circle in the plane, $\mathfrak{A}_0 = \mathfrak{A}$ be the Borel subsets of Ω, P be the probability on $(\Omega, \mathfrak{A})$ invariant under rotations (Lebesgue measure up to a constant factor), and let Λ_t be the rotation through angle t. Then $\mathfrak{A}_0 = \mathcal{P}_0 = \mathfrak{F}_0 = \mathfrak{J}_P = \mathfrak{J}_F = \mathfrak{A}$, and it is easily verified that $(\Omega, \mathfrak{A}, \mathfrak{A}_0, P)$ is uniformly ergodic, contradicting the assertion in the theorem for the continuous case.

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