Further, we notice from (3.4) that for odd p there is a single factor in the central position of the factors of Λ_p. In case p is even there will be two factors in this central position and the result (3.7) will be slightly changed.

CASE 2. For k and p both even the terms of the second factor of (3.2) may be grouped two by two by applying the duplication formula for the Gamma functions and the case reduces to Case 1.

CASE 3. For k even and p odd the terms may be grouped two by two as in Case 2 and the last term may either be expanded by a known formula (see, e.g., [10], p. 260) for the expansion of $\Gamma(x + r)/\Gamma(x)$ when r is not an integer, or else it may be approximated by the formula

$$\Gamma(x + r)/\Gamma(x) \doteq x^r \tag{3.8}$$

and the case reduces to Case 1.

We wish to consider the computational aspects of the result (3.7) in a future communication.

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ON THE PARAMETERS AND INTERSECTION OF BLOCKS OF BALANCED INCOMPLETE BLOCK DESIGNS

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1. Summary. In this investigation we derive a few properties of the intersection of blocks in a balanced incomplete block (b.i.b. for conciseness) design with

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parameters v, b, r, k, λ and discuss the divisibility of v by k. The last section deals with certain characteristic properties of symmetric and affine resolvable b.i.b. designs.

2. Divisibility of v by k in a b.i.b. design. Let v, b, r, k, λ be the parameters of a b.i.b. design. We will suppose here and below that it is a nontrivial design so $v > k \geqq 3$ and $r > \lambda \geqq 1$. One knows that the parameters are subject to the arithmetical constraints $vr = bk$ and $\lambda(v - 1) = r(k - 1)$. A well known inequality due to Fisher states that we must also have $b \geqq v$. It is easy to determine five positive integers v, b, r, k, λ with $v > k \geqq 3$, $r > \lambda \geqq 1$ and satisfying the arithmetical constraints given above and simultaneously having $b < v$. Fisher's inequality then guarantees the nonexistence of a b.i.b. design with these parameters.

Concerning the divisibility of v by k we establish the following theorem.

THEOREM 1. *A necessary and sufficient condition for v to be divisible by k in a* b.i.b. *design with parameters v, b, r, k, λ is that $b = (v - 1)m + r$ where m is a positive integer and $r - m$ is divisible by λ.*

PROOF. More generally, let v, b, r, k, λ with $v > k$ be any five positive integers satisfying the conditions (i) $vr = bk$ and (ii) $\lambda(v - 1) = r(k - 1)$.

First, suppose that v is divisible by k. Let $v = nk$, n an integer. As $v > k$, $r > \lambda$ by (ii). Since $v = nk$, by (ii) we have $r - \lambda = rk - \lambda v = (r - n\lambda)k = mk$ (say) where m is an integer and it is positive as $r > \lambda$. From (i) we have $b = vr/k$ and so $b = v(mk + \lambda)/k = vm + n\lambda = vm + r - m = (v - 1)m + r$. Clearly $r - m$ is divisible by λ.

Next, suppose that $b = (v - 1)m + r$ where m is an integer and $r - m$ is divisible by λ. Put $r - m = n\lambda$, n an integer. We have now, using (i) and (ii), $bk - b = vr - (v - 1)(r - n\lambda) - r = n\lambda(v - 1) = nr(k - 1)$; consequently $b = nr$ and then by (i) $v = nk$. Plainly m is positive as $(v - 1)m = b - r = (n - 1)r$, $n > 1$.

On remembering the arithmetical constraints on the parameters of a b.i.b. design, the theorem follows immediately from the preceeding. This completes the proof.

One sees from the above theorem that, for the type of b.i.b. designs in which $v = nk$ with n an integer, we must have $b = (v - 1)m + r$, m a positive integer. Trivially then $b \geqq v + r - 1$. In fact the above proof shows that it is not possible to determine five positive integers v, b, r, k, λ satisfying the relations $v > k$, $vr = bk$, $\lambda(v - 1) = r(k - 1)$, and $v = nk$ with n an integer and simultaneously having $b < v + r - 1$. The inequality $b \geqq v + r - 1$ for this type of b.i.b. designs was obtained by Roy [1] (and independently by Mikhail [2]). As resolvable b.i.b. designs constitute a subset of this type of designs, the same inequality holds for them. This inequality for resolvable b.i.b. designs was established by Bose [3].

3. An upper bound for the number of blocks disjoint with a given block. In this section we give an upper bound for the number of blocks in a b.i.b. design each of which has no variety common with a given block. Our result is contained in the following theorem.

THEOREM 2. *A given block in a* b.i.b. *design with parameters* v, b, r, k, λ *can never have more than* $b - 1 - (r-1)^2k/(r - \lambda - k + k\lambda)$ *blocks disjoint with it. If some block has that many, then* $(r - \lambda - k + k\lambda)/(r-1)$ *is a positive integer and each of the nondisjoint blocks has* $(r - \lambda - k + k\lambda)/(r-1)$ *varieties common with it.*

PROOF. Take any block in the design. Let it have m blocks disjoint with it. We are not supposing that these latter blocks are mutually disjoint. If the ith of the remaining $b - m - 1$ blocks has x_i varieties common with the arbitrarily chosen block, then considering its varieties singly and pairwise we have

$$\sum x_i = (r-1)k, \tag{3.1}$$

$$\sum x_i(x_i - 1) = (\lambda - 1)(k-1)k, \tag{3.2}$$

where in these two sums and the sum below the index of summation runs from 1 to $b - m - 1$. Now (3.1) and (3.2) give

$$\sum x_i^2 = (r - \lambda - k + k\lambda)k. \tag{3.3}$$

Forming the ratio $(\sum x_i)^2/(\sum x_i^2)$ and applying the inequality

$$(\sum_{i=1}^{n} a_i)^2/(\sum_{i=1}^{n} a_i^2) \leqq n,$$

wherein a_i are real numbers, we get

$$(r-1)^2k/(r - \lambda - k + k\lambda) \leqq b - m - 1, \tag{3.4}$$

whence

$$m \leqq b - 1 - (r-1)^2k/(r - \lambda - k + k\lambda). \tag{3.5}$$

If the equality sign holds in (3.4), all the x_i's are equal and then (3.1) gives

$$x_i = (r - \lambda - k + k\lambda)/(r-1), \qquad i = 1, 2, \cdots, b - m - 1.$$

In this case each of the $b - m - 1$ nondisjoint blocks has $(r - \lambda - k + k\lambda)/(r-1)$ varieties common with the arbitrarily chosen block. This completes the proof.

As an application of this theorem consider Bhattacharya's b.i.b. design, $v = 16$, $b = 24$, $r = 9$, $k = 6$, $\lambda = 3$. We infer that no block can have more than one block disjoint with it. The solution given by Bhattacharya [4] for this design has a block which is disjoint with another block.

A companion to Theorem 2 is Theorem 3.

THEOREM 3. *If in a* b.i.b. *design with parameters* v, b, r, k, λ, $v = nk$ *(n denoting an integer greater than* 1) *and* $b = v + r - 1$, *and if there exists a block which has* $n - 1$ *blocks disjoint with it, then* k/n *is an integer and each of the nondisjoint blocks has* k/n *varieties common with it.*

PROOF. As $b = v + r - 1$ we have $r = k + \lambda$ by virtue of the arithmetical constraints on the parameters. Further $\lambda(v - 1) = r(k - 1)$ gives $\lambda(nk - 1) = rk - r = rk - k - \lambda$ so that $n\lambda = r - 1$. Therefore

$$b - 1 - (r - 1)^2k/(r - \lambda - k + k\lambda) = b - 1 - n^2\lambda.$$

But $b - 1 - n^2\lambda = b - 1 - n(r - 1) = n - 1$ as $b = nr$. So Theorem 3 follows from Theorem 2.

Consider as an example the existent b.i.b. design $v = 28, b = 36, r = 9, k = 7, \lambda = 2$. Here $n = v/k = 4$ and $v + r - 1 = 28 + 9 - 1 = 36 = b$. By Theorem 2 no block of this design can have more than $m = 3$ blocks disjoint with it. By Theorem 3 if there does exist such a block then k/n must be an integer whereas $k/n = \frac{7}{4}$—a contradiction. Consequently in no solution of this design can there exist a block which has as many as 3 blocks disjoint with it. *A fortiori* no resolvable solution can exist for this design. In a similar manner it can be shown that the following b.i.b. designs cannot have resolvable solutions.

$$v = 6, \quad b = 10, \quad r = 5, \quad k = 3, \quad \lambda = 2$$

$$v = 10, \quad b = 18, \quad r = 9, \quad k = 5, \quad \lambda = 4$$

$$v = 21, \quad b = 30, \quad r = 10, \quad k = 7, \quad \lambda = 3.$$

4. Characteristic properties of symmetric and affine resolvable b.i.b. designs. Our next theorem gives a characteristic property of symmetric b.i.b. designs.

THEOREM 4. *A necessary and sufficient condition that a* b.i.b. *design be symmetric is that it has a block which has the same number of varieties common with each of the other blocks.*

PROOF. Let the b.i.b. design have the parameters v, b, r, k, λ. If it has a block which has the same number, say c, of varieties common with each of the other $b - 1$ blocks, then (3.1) and (3.2) (now with $m = 0$ and $x_i = c$ for all i) give

$$c(b - 1) = (r - 1)k \tag{4.1}$$

$$c^2(b - 1) = (r - \lambda - k + k\lambda)k. \tag{4.2}$$

These two relations directly give

$$c = (r - 1)k/(b - 1) = (r - \lambda - k + k\lambda)/(r - 1) \tag{4.3}$$

and so

$$\begin{aligned} c &= [(r - 1)k - (r - \lambda - k + k\lambda)]/[(b - 1) - (r - 1)] \\ &= [(k - 1)r - (k - 1)\lambda]/(b - r) \\ &= [\lambda(v - 1) - (k - 1)\lambda]/(r(v - k)/k) \\ &= k\lambda/r, \end{aligned} \tag{4.4}$$

where we used the arithmetical constraints on the parameters. From (4.3) and (4.4) we get

$$(r - \lambda - k + k\lambda)/(r - 1) = k\lambda/r. \tag{4.5}$$

From this we obtain $(r - k)(r - \lambda) = 0$ and as $r - \lambda > 0$, we infer that $r = k$ (and equivalently $b = v$) i.e., the design is symmetric.

Next, let us suppose that the design is symmetric, so that $b = v$ and $r = k$. We expand the expression

$$\sum_{i=1}^{b-1}(x_i - \lambda)^2 = \sum_{i=1}^{b-1}x_i^2 - 2\lambda\sum_{i=1}^{b-1}x_i + (b - 1)\lambda^2,$$

which by (3.1) and (3.3) gives

$$\sum_{i=1}^{b-1}(x_i - \lambda)^2 = (r - k - \lambda + k\lambda)k - 2\lambda k(r - 1) + (b - 1)\lambda^2.$$

Applying $k = r$ and $\lambda(b - 1) = \lambda(v - 1) = r(k - 1)$ to this expression we quickly see that $\sum_{i=1}^{b-1}(x_i - \lambda)^2 = 0$. So $x_i = \lambda$ for all i. Consequently if the b.i.b. design is symmetric, any pair of blocks has λ varieties in common—a property of b.i.b. designs due to Bose.

In an analogous manner the following property is true for affine resolvable b.i.b. designs.

THEOREM 5. *A necessary and sufficient condition that a resolvable* b.i.b. *design be affine resolvable is that it has a block which has the same number of varieties common with each of the blocks not belonging to its own replication.*

PROOF. Consider a block in a resolvable b.i.b. design with the parameters $v = nk$, $b = nr$, r, k, λ so that each replication of the varieties consists of n mutually disjoint blocks. Let the ith of the remaining $b - n$ blocks not belonging to its own replication have x_i varieties common with the considered block. Then

$$\sum x_i = (r - 1)k, \tag{4.6}$$

$$\sum x_i(x_i - 1) = (\lambda - 1)(k - 1)k, \tag{4.7}$$

where the index of summation here and below runs from 1 to $b - n$. Suppose that the design has a block which has the same number, say c, of varieties common with each of the blocks not belonging to its own replication. Assuming the considered block to be this special one we get

$$c(b - n) = (r - 1)k, \tag{4.8}$$

$$c^2(b - n) = (r - \lambda - k + k\lambda)k. \tag{4.9}$$

From these

$$c = (r - \lambda - k + k\lambda)/(r - 1) = (r - 1)k/(b - n). \tag{4.10}$$

As $b = nr$, $(r - 1)k/(b - n) = k/n$ and thus $(r - \lambda - k + k\lambda)/(r - 1) = k/n$. From this we get

$$(r - 1)k = (r - \lambda - k)n + \lambda v = (r - \lambda - k)n + r(k - 1) + \lambda,$$

that is $(n - 1)(r - \lambda - k) = 0$. So $r = k + \lambda$ and equivalently $b = v + r - 1$; thus the b.i.b. design is affine resolvable by definition.

Suppose now that the design is affine resolvable. One has $b = nr$ and $r = k + \lambda$, then $n\lambda = r - 1$ as in Theorem 3. Using these and (4.6) and (4.7),

$$\sum (x_i - k/n)^2 = \lambda k^2 + k^2(b - n)/n^2 - 2k(r - 1)k/n$$
$$= k^2(n\lambda - (r - 1))/n = 0.$$

Thus any two blocks not belonging to the same replication have k/n common varieties—a result due to Bose. This completes the proof.

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