ON THE TRANSIENT BEHAVIOR OF A QUEUEING SYSTEM WITH BULK SERVICE AND FINITE CAPACITY

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1. Introduction. We consider the following queueing system. Customers arrive at a single service station and are served in groups of exactly r-members ($r \geqq 1$). The service times of successive groups are identically and independently distributed random variables with common distribution function

$$B(x) = 1 - e^{-\mu x}, \qquad x \geqq 0.$$

The system is of finite capacity, that is not more than $Nr + r$ customers can be present at any time, Nr customers waiting for service and r customers being served. If a customer arrives to find $Nr + r - 1$ customers present the input process stops and does not restart until Nr customers only are present, that is until the current service period is completed. In the terminology of Foster [4] the 1-input process, that is the arrival of customers, is a triggered input process, and the 0-input process, that is the service mechanism, is also a triggered process. The 1-input is triggered after an arrival if there are then $Nr + r - 1$ or less customers present. If after an arrival $Nr + r$ customers are present the 1-input is stopped until the service then going on is completed, when it is retriggered. When the 1-input is triggered the time to the next arrival is called the 1-input time. We suppose that the successive 1-input times are identically and independently distributed non-negative random variables with common distribution function $A(x) = 1 - e^{-\lambda x}$, $x \geqq 0$.

The service mechanism, or 0-input process is triggered by the presence of r or more customers. Groups of customers are served in the order of their arrival and it is never the case that r or more customers are in the system and the server is idle. If less than r customers are in the system the server is idle and service starts as soon as r customers are present, service being given to those r customers as a single group.

Some of the results of this paper apply equally to a queueing system such as the above where either (a) the 1-input process is untriggered, that is customers who arrive to find the system full depart never to return, or (b) service is performed on groups of not more than r customers. The fundamental equations (2) apply equally to these systems and hence also do the deductions from them. Theorem 6 does not however apply to these systems.

A queueing system of the type (b) with infinite capacity, that is $N = \infty$, has been considered by Bailey [1]. In Section 7 we relate the queueing system defined above to the queueing system $E_r/M/1$. We remark that general formula for the queueing system $GI/M/1$ have been given recently by Takács [7]. The results of

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Section 6 are a particular case of Takács general formulae but are new in the sense that we obtain explicit formulae.

2. The fundamental equations. Let the system be said to be in the state E_j, $j = 0, 1, \cdots, Nr + r$, if j customers are present. Define random variables η_m, ζ_m as follows. Let η_m be the number of customers in the system just before the mth arrival, write $Q_j^m = P(\eta_m = j), j = 0, 1, \cdots, Nr + r - 1$. Let ζ_m be the number of customers left in the system just after the mth service period, write $R_j^m = P(\zeta_m = j)$, $R_j = lim_{m\to\infty} R_j^m$, $j = 0, 1, \cdots Nr$.

Denote by k_n the probability that there are n successive potential 1-input times during a service period and write $K_n = \sum_{j=n}^{\infty} k_j$. Then

$$k_n = a^k b, \qquad K_n = a^n, \tag{1}$$

where $a = \lambda(\lambda + \mu)^{-1}$ and $b = \mu(\lambda + \mu)^{-1}$.

It is easy to see that the probabilities R_j^m satisfy the following recurrence relations

$$
\begin{aligned}
R_j^{m+1} &= \sum_{s=1}^{j} R_{r+s}^m k_{j-s} + (R_r^m + R_{r-1}^m + \cdots + R_0^m)k_j, \\
&\qquad\qquad\qquad\qquad\qquad\qquad j = 0, 1, \cdots, Nr - r, \\
R_j^{m+1} &= \sum_{s=1}^{Nr-r} R_{r+s}^m k_{j-s} + (R_r^m + R_{r-1}^m + \cdots + R_0^m)k_j, \\
&\qquad\qquad\qquad\qquad\qquad\qquad j = Nr - r, \cdots, Nr - 1 \\
R_{Nr}^{m+1} &= \sum_{s=1}^{Nr-r} R_{r+s}^m K_{Nr-s} + (R_r^m + R_{r-1}^m + \cdots + R_0^m)K_{Nr}.
\end{aligned}
\tag{2}
$$

Substitution from (1) into (2) gives after simplification

$$
\begin{aligned}
R_0^{m+1} &= (R_r^m + R_{r-1}^m + \cdots + R_0^m)b \\
R_j^{m+1} &= bR_{j+r}^m + aR_{j-1}^{m+1}, \qquad j = 1, 2, \cdots, Nr - r, \\
a^r R_{Nr-r+s}^{m+1} &= a^s b R_{Nr}^{m+1}, \qquad s = 0, 1, \cdots, r - 1.
\end{aligned}
\tag{3}
$$

The process, $\{\zeta_m\}$ is a finite, irreducible, aperiodic Markov chain. It follows that the limiting distribution $\{R_j\}$ exists and is uniquely determined by equations (2) and hence (3) with superscripts $m, m + 1$, suppressed.

In the next section we determine the limiting distribution $\{R_j\}$ and in Section 4 we determine the distribution $\{R_j^m\}$. We do not determine the distribution $\{Q_j^m\}$ explicitly since we show in Section 5 how this distribution may be derived from the distribution $\{R_j^m\}$.

3. The limiting distribution $\{R_j\}$. Write $X_j = R_{Nr-j}$, $j = 0, 1, \cdots, Nr$.

Then from (3), with the superscripts suppressed, we obtain

$$
\begin{aligned}
X_{Nr} &= (X_{Nr} + X_{Nr-1} + \cdots + X_{Nr-r})b, \\
X_j &= bX_{j-r} + aX_{j+1}, \qquad j = r, r+1, \cdots, Nr-1, \\
a^j X_j &= bX_0, \qquad j = 1, 2, \cdots, r.
\end{aligned} \tag{4}
$$

It is easily verified that the first of equations (4) is implied by the second and third equations together with the normality condition $\sum_{j=0}^{Nr} X_j = \sum_{j=0}^{Nr} R_j = 1$. Thus the X_j can be obtained successively from (4) in terms of X_0 which can be obtained by normalization.

We prove

THEOREM 1. *The limiting distribution* $\{R_j\}$ *is given by*

$$
\begin{aligned}
R_j &= (F_{Nr-j} - F_{Nr-j-1})\,[F_{Nr}]^{-1}, \qquad j = 0, 1, \cdots, Nr-1, \\
R_{Nr} &= [F_{Nr}]^{-1},
\end{aligned} \tag{5}
$$

where $F_0 = 1$ *and*

$$
F_k = \sum_{s=0}^{[k/(r+1)]} (-)^s \binom{k - sr}{s} b^s a^{sr-k}, \qquad k \geqq 1, \tag{6}
$$

and $[k/(r+1)]$ *is the integral part of* $k/(r+1)$.

PROOF. Define an infinite sequence $\{Y_j\}$ by the equations

$$
\begin{aligned}
a^j Y_j &= b, \qquad j = 1, 2, \cdots, r, \\
aY_{j+1} &= Y_j - bY_{j-r}, \ (Y_0 = 1), \qquad j \geqq r,
\end{aligned} \tag{7}
$$

then

$$
X_j = Y_j \left[\sum_{i=0}^{Nr} Y_i \right]^{-1}, \qquad j = 0, 1, \cdots, Nr. \tag{8}
$$

Write $Y(z) = \sum_{j=0}^{\infty} Y_j z^j$. From (7) we obtain

$$
Y(z) = (1 - z)[1 - za^{-1}(1 - bz^r)]^{-1}. \tag{9}
$$

Expanding the right-hand side of (9) in powers of z for a suitable domain of z, for example $|z| < a$, we obtain

$$
Y(z) = (1 - z) \sum_{k=0}^{\infty} F_k z^k, \tag{10}
$$

where the coefficients F_k are given by (6). Thus $Y_j = F_j - F_{j-1}$, $j \geqq 1$, $F_0 = 1$ and from (8) we obtain (5).

EXAMPLE. In the case $r = 1$ it is easily verified that

$$
\sum_{s=0}^{[k/2]} (-)^s \binom{k - s}{s} b^s a^{s-k} = 1 + \rho^{-1} + \rho^{-2} + \cdots + \rho^{-k}, \qquad k \geqq 1,
$$

where $\rho = a/b = \lambda/\mu$. Hence $F_k = (1 - \rho^{-k-1})(1 - \rho)^{-1}$ and equations (5) give

$R_j = (1 - \rho)\rho^j/(1 - \rho^{N+1}), j = 0, 1, \cdots, N$. This is the solution obtained by a number of authors, for example, Morse [5] and Finch [3].

It is of interest to examine the behavior of the distribution $\{R_j\}$ as $N \to \infty$. To do so write $a - z + bz^{r+1} = a(1 - z)(1 - \gamma_1^{-1}) \cdots (1 - \gamma_r^{-1}z)$. It is easily verified that the roots of $a - z + bz^{r+1} = 0$ are distinct except when $\lambda = r\mu$, in which case $z = 1$ is a double root. When $\lambda = r\mu$ we shall suppose that $\gamma_1 = 1$. From (9) we have

$$Y(z) = \sum_{j=1}^{r} A_j(1 - z\gamma_j^{-1})^{-1}, \tag{11}$$

where $A_j = \prod_{i \neq j}(1 - \gamma_j\gamma_i^{-1})^{-1}$. Thus $Y_k = \sum_{j=1}^{r} A_j\gamma_j^{-k}$ and

$$R_k = \left[\sum_{j=1}^{r} A_j\, \gamma_j^{-Nr+k}\right]\left[\sum_{i=1}^{r} A_i(1 - \gamma_i^{-Nr-1})(1 - \gamma_i^{-1})^{-1}\right]^{-1}, \tag{12}$$

for $k = 0, 1, \cdots, Nr$ and $\lambda \neq r\mu$. If $\lambda = r\mu$ the same expression is valid provided we replace the indeterminate ratio $(1 - \gamma_1^{-Nr-1})(1 - \gamma_1^{-1})^{-1}$ by its limit at $\gamma_1 = 1$, namely, $(Nr + 1)$.

It is easily shown by means of Rouché's Theorem that the equation $a - z + bz^{r+1} = 0$ has only one root inside the unit circle $|z| = 1$, if $\lambda < r\mu$ and no root inside the unit circle if $\lambda \geqq r\mu$. Thus from (12) we obtain (i) $\lim_{N\to\infty} R_k = 0$, if $\lambda \geqq r\mu$, and (ii) $\lim_{N\to\infty} R_k = (1 - \gamma)\gamma^k$, if $\lambda < r\mu$, where γ is the only root of $a - z + bz^{r+1} = 0$ inside the unit circle.

4. The distribution $\{R_j^m\}$. The probabilities $P\{\zeta_m = j \mid \zeta_1 = i\} = P\{\zeta_m = j \mid \zeta_1 = 0\}$, $m > 1, j \geqq 0$, for $i = 1, \cdots, r - 1$, since when $\zeta_1 < r$ the second service commences as soon as r customers are present and the subsequent values of ζ_m are independent of the history of the process up to the instant the second service commences. We shall determine the distribution $\{R_j^m\}$ subject to the initial condition $\zeta_1 < r$, equivalently we can write

$$\sum_{j=0}^{r-1} R_j^1 = 0, \qquad R_j^1 = 0, \qquad j \geqq r.$$

Because of the above remark it is sufficient to determine the distribution R_j^m subject to the initial condition $R_0^1 = 1$ and we shall suppose that this is so throughout the present section.

Write $R_j(w) = \sum_{m=1}^{\infty} R_j^m w^{m-1}$. Then from equations (3) with $R_0^1 = 1$ we obtain

$$\begin{aligned} R_0(w) &= 1 + bw\{R_0(w) + R_1(w) + \cdots + R_r(w)\}, \\ R_j(w) &= bwR_{j+r}(w) + aR_{j-1}(w) - a\delta_{i,j}, \qquad 1 \leqq j \leqq Nr - r, \\ a^r R_{Nr-r+s}(w) &= a^s b R_{Nr}(w), \qquad 0 \leqq s < r, \end{aligned} \tag{13}$$

where $\delta_{i,j}$ is the Kronecker delta. Write

$$X_j^m = R_{Nr-j}^m, \; X_j(w) = R_{Nr-j}(w).$$

Then

$$X_{Nr}(w) = 1 + bw \sum_{k=0}^{r} X_{Nr-k}(w),$$

$$(14) \qquad X_j(w) = bw\, X_{j-r}(w) + aX_{j+1}(w) - a\delta_{Nr-1,j}, \quad r \leqq j < Nr,$$

$$a^s X_s(w) = bX_0(w).$$

It is easily verified that the first of equations (14) is implied by the second and third equation together with the normality condition $\sum_{j=0}^{Nr} X_j(w) = \sum_{j=0}^{Nr} R_j(w) = 1/(1-w)$. Thus the $X_j(w)$ can be obtained successively from (14) in terms of $X_0(w)$ which can be obtained by normalization.

We prove

Theorem 2. *If $R_j^1 = 0$, $j \geqq r$, then the generating functions $R_j(w)$ are given by*

$$R_0(w) = 1 + w(1-w)^{-1}[F_{Nr}(w) - F_{Nr-1}(w)][F_{Nr}(w)]^{-1},$$

$$(15) \qquad R_j(w) = w(1-w)^{-1}[F_{Nr-j}(w) - F_{Nr-j-1}(w)][F_{Nr}(w)]^{-1},$$

$$R_{Nr}(w) = w(1-w)[F_{Nr}(w)]^{-1},$$

where $F_0(w) = 1$ and

$$(16) \qquad F_k(w) = \sum_{s=0}^{[k/(r+1)]} (-)^s \binom{k - sr}{s} b^s a^{sr-k} w^s, \qquad k \geqq 1,$$

and $[k/(r+1)]$ is the integral part of $k/(r+1)$.

Proof. Introduce a sequence $\{Y_j(w)\}$ defined by the equations

$$Y_0(w) \equiv 1,$$

$$(17) \qquad a^j Y_j(w) = bY_0(w), \qquad 1 \leqq j \leqq r,$$

$$aY_{j+1}(w) = Y_j(w) - bwY_{j-r}(w), \qquad j \geqq r.$$

Then $X_j(w) = Y_j(w)X_0(w)$, $0 \leqq j < Nr$, $X_{Nr}(w) = Y_{Nr}(w)X_0(w) + 1$. The normality condition $\sum X_j(w) = 1/(1-w)$ gives

$$(18) \qquad X_0(w) = w(1-w)^{-1}\left[\sum_{i=0}^{Nr} Y_i(w)\right]^{-1},$$

hence

$$(19) \qquad X_j(w) = w(1-w)^{-1}Y_j(w)\left[\sum_{i=0}^{Nr} Y_i(w)\right]^{-1}.$$

Write $Y(w, z) = \sum_{j=0}^{\infty} Y_j(w)z^j$; then from (17) we obtain, for $|z| < a$, $|w| < 1$,

$$(20) \quad Y(w, z) = (1-z)[1 - za^{-1}(1 - bwz^r)]^{-1} = (1-z)\sum_{k=0}^{\infty} F_k(w)w^k,$$

where $F_k(w)$ is given by (16). Thus $Y_j(w) = F_j(w) - F_{j-1}(w)$, $j \geqq 1$, and from (19) we obtain (15).

The special case of this theorem, when $r = 1$, will be studied in more detail later in this section, (Theorem 5).

The generating functions $R_j(w)$ can be expressed also in terms of the roots of the equation $a - z + bwz^{r+1} = 0$ and it is convenient to do so in order to obtain limiting formulae for the probabilities.

Thus we prove

THEOREM 3. *If $R_j^1 = 0$, $j \geqq r$, then the generating functions $R_k(w)$, $0 < w < 1$, are given by*

$$R_0(w) = 1 - w(1-w)^{-1}\left[\sum_{j=0}^{r} A_j(w)\{1-\gamma_j(w)\}\gamma_j^{-Nr}(w)\right] \cdot \left[\sum_{i=0}^{r} A_i(w)\gamma_i(w)\{1-\gamma_i^{-Nr-1}(w)\}\right]^{-1},$$

$$(21)\qquad R_k(w) = -w(1-w)^{-1}\left[\sum_{j=0}^{r} A_j(w)\{1-\gamma_j(w)\}\gamma_j^{-Nr+k}(w)\right] \cdot \left[\sum_{i=0}^{r} A_i(w)\gamma_i(w)\{1-\gamma_i^{-Nr-1}(w)\}\right]^{-1}, \qquad 1 \leqq k \leqq Nr,$$

where $\gamma_j(w)$, $j = 0, 1, \cdots, r$, are the roots of $a - z + bwz^{r+1} = 0$, and

$$A_j(w) = \prod_{i \neq j} \{1 - \gamma_j(w)\gamma_i^{-1}(w)\}^{-1}.$$

PROOF. Let

$$a - z + bwz^{r+1} = a(1 - z\gamma_0^{-1}(w))(1 - z\gamma_1^{-1}(w)) \cdots (1 - z\gamma_r^{-1}(w)).$$

Then it is easily seen that the roots $\gamma_i(w)$ are distinct (for a repeated root implies that $w = (ra^{-1})^r\{b(r+1)^{r+1}\}^{-1} \geqq 1$).

Thus from (20) we obtain

$$Y(w,z) = (1-z)\sum_{j=0}^{r} A_j(w)\{1 - z\gamma_j^{-1}(w)\}^{-1}.$$

Thus $Y_0(w) = 1$ and

$$(22)\qquad Y_k(w) = \sum_{j=0}^{r} A_j(w)\{1-\gamma_j(w)\}\gamma_j^{-k}(w).$$

Substituting from (22) into (19) we obtain (21).

We prove now the following lemma.

LEMMA 1. *For $0 < w < 1$, the equation*

$$(23)\qquad a - z + bwz^{r+1} = 0$$

has only one root $z = \gamma(w)$ within the unit circle $|z| = 1$. Further this root is given explicitly by

$$(24)\qquad \{\gamma(w)\}^j = a^j + j\sum_{n=1}^{\infty} n^{-1}\binom{nr+j+n-1}{n-1} b^n a^{nr+j} w^n, \qquad j \geqq 1.$$

PROOF. On the unit circle $|z| = 1$ we have $|bwz^{r+1}| < b$ for $0 < w < 1$. But $|z - a| \geqq 1 - a = b$, $|z| = 1$ and thus $|bwz^{r+1}| < |z - a|$ on the unit circle $|z| = 1$. It follows from Lagrange's theorem (Whittaker and Watson [8]) that equation (23) has only one root within the unit circle and that (24) is the case.

We prove next

THEOREM 4. *When $N = \infty$, $R_j^1 = 0$, $j \geqq r$, the limiting generating functions $R_k(w)$, $k \geqq 0$, are given by*

$$(25) \qquad \begin{aligned} R_0(w) &= 1 + w(1 - w)^{-1}\{1 - \gamma(w)\}, \\ R_k(w) &= w(1 - w)^{-1}\{1 - \gamma(w)\}\{\gamma(w)\}^k, \qquad k \geqq 1, \end{aligned}$$

where $\{\gamma(w)\}^j$, $j \geqq 1$ is given by (24). Further we have the following explicit formulae for the limiting probabilities R_k^m.

$$(26) \qquad \begin{aligned} R_0^m &= 1 - \sum_{j=0}^{m-2} C_0^j, \qquad & m \geqq 2, \\ R_k^m &= \sum_{j=0}^{m-2} C_k^j, \qquad & m \geqq 2, k > 0, \end{aligned}$$

where $C_0^0 = a$, $C_k^0 = a^k b$, and

$$C_0^n = n^{-1} \binom{nr + n}{n - 1} b^n a^{nr+1}, \qquad n > 0$$

$$(27) \quad C_k^n = n^{-1} b^n a^{nr+k} \left[k \binom{nr + k + n - 1}{n - 1} - (k + 1)a \binom{nr + k + n}{n - 1} \right], \qquad n \geqq 1, \quad k \geqq 1$$

PROOF. Letting $N \to \infty$ in equation (21), we obtain (25) where $\gamma(w)$ is that root of (23) with smallest modulus within the unit circle. By Lemma 1 there is only one root within the unit circle for $0 < w < 1$ and this is given explicitly by (24). Expanding (25) in powers of w by means of (24) we obtain (26). That the coefficients of powers of w in these expansions are in fact the probabilities corresponding to the case $N = \infty$ follows from the fact that the generating functions $R_k(w)$ are then uniquely determined by the equations (13) with $N = \infty$. It is easily verified that the generating functions given by (25) do in fact satisfy equations (13). Thus the R_k^m given by (26) satisfy the recurrence relations (2) with $R_j^1 = 0$, $j \geqq r$, and $N = \infty$. They are therefore the required probabilities.

EXAMPLE. When $r = 1$ we obtain from (24)

$$\gamma(w) = a \left[1 + \sum_{n=1}^{\infty} (n + 1)^{-1} \binom{2n}{n} (abw)^n \right].$$

In virtue of the binomial expansion

$$(1 - x)^{\frac{1}{2}} = 1 - 2 \sum_{n=0}^{\infty} (n + 1)^{-1} \binom{2n}{n} (x/4)^{n+1},$$

we obtain

$$\gamma(w) = (2bw)^{-1}\{1 - (1 - 4abw)^{\frac{1}{2}}\}, \tag{28}$$

and equations (25) become those obtained in Finch [3]. Formula (28) is the root in the unit circle of (23) with $r = 1$. When $r = 1$ and $N < \infty$ it is possible to obtain expressions for the generating functions $R_j(w)$ in terms of the function $\gamma(w)$ given by (28). These expressions are simpler than those given by Theorem 2 and give explicit formulae for the probabilities R_j^m.

We prove

THEOREM 5. *If $r = 1$, $N < \infty$, and $R_j' = 0$, $j \geqq r$, then*

$$\sum_{k=j}^{N} R_k(w) = w(1 - w)^{-1}\{\gamma(w)\}^j[1 - (a^{-1}bw)^{N+1-j}\{\gamma(w)\}^{2N+2-j}] \cdot [1 - (a^{-1}bw)^{N+1}\{\gamma(w)\}^{2N+2}]^{-1}, \tag{29}$$

where $\gamma(w)$ is given by (28) *and $\{\gamma(w)\}^j$, $j \geqq 1$, is given by the series expansion* (24) *with $r = 1$. The probabilities R_j^m can be obtained from the equations*

$$\begin{aligned} \sum_{k=j}^{N} R_k^m &= T_j^0 + T_j^1 + \cdots + T^{m-2}, \quad m \geqq 2, 1 \leqq j \leqq N, \\ R_0^m &= 1 - \sum_{k=1}^{N} R_k^m, \qquad m \geqq 2, \end{aligned} \tag{30}$$

where

$$\begin{aligned} T_j^k = &\sum_{s=0}^{[k/N+1]} (a^{-1}b)^{Ns+s}\, \Gamma_{k-s-Ns}^{2s(N+1)+j} \\ &- \sum_{s=1}^{[k+j/N+1]} (a^{-1}b)^{Ns+s-j}\, \Gamma_{k+j-s-Ns}^{2s(N+1)}, \qquad k \geqq 0, \quad j \geqq 1, \end{aligned} \tag{31}$$

and the second sum in (35) *is zero if $k + j < N + 1$ and the Γ_n^j are given by*

$$\Gamma_n^j = jn^{-1}\binom{2n + j - 1}{n + j} a^{n+j}b^n, \qquad j \geqq 1, \quad n \geqq 0. \tag{32}$$

PROOF. From equations (15), with $r = 1$, we have

$$\sum_{k=j}^{N} R_k(w) = w(1 - w)^{-1}F_{N-j}(w)[F_N(w)]^{-1}, \qquad 1 \leqq j \leqq N.$$

In order to prove (29) it will be sufficient therefore to prove (33).

$$F_{N-j}(w)[F_N(w)]^{-1} = \{\gamma(w)\}^j[1 - (a^{-1}bw)^{N+1-j}\{\gamma(w)\}^{2N+2j}] \cdot [1 - (a^{-1}bw)^{N+1}\{\gamma(w)\}^{2N+2}]^{-1}, \tag{33}$$

where $F_k(w)$ is given by (16) with $r = 1$.

By the definition of the $F_k(w)$ we have from (16) with $r = 1$, with $x = (a^{-1}bw)^{\frac{1}{2}}z$,

$$\sum_{k=0}^{\infty} F_k(w)(a^{-1}bw)^{k/2}x^k = (1 - 2x\cosh\theta + x^2)^{-1},$$

where $(abw)^{-\frac{1}{2}} = 2\cosh\theta$. But

$$x\sinh\theta(1 - 2x\cosh\theta + x^2) = \sum_{k=0}^{\infty} x^k \sinh k\theta, \qquad |x|<1,$$

and hence

$$F_k(w) = (a^{-1}bw)^{k/2}(\sinh\theta)^{-1}\sinh(k+1)\theta.$$

Thus

$$(34) \quad F_{N-j}(w)[F_N(w)]^{-1} = (a^{-1}bw)^{-j/2}e^{-j\theta}[1 - e^{-2(N+1-j)\theta}]\,[1 - e^{-2(N+1)\theta}]^{-1}.$$

But $e^{-\theta} = \cosh\theta - (\cosh^2\theta - 1)^{\frac{1}{2}}$. Substituting $\cosh\theta = (abw)^{-\frac{1}{2}}/2$ and using (28) we obtain $e^{-\theta} = (a^{-1}bw)^{\frac{1}{2}}\gamma(w)$. Substituting for $e^{-\theta}$ in (34) gives (33) and hence (29). Expanding (29) as a power series in $\gamma(w)$ and using (24) with $r = 1$ we obtain (30).

5. The relationship between the distributions $\{Q_j^m\}$, $\{R_j^m\}$. In this section we prove the following

THEOREM 6. *For the queueing system of Section* 1 *we have*

$$\begin{aligned} &Q_{j-1}^{mr+j} = R_0^m + R_1^m + \cdots + R_{j-1}^m, && m \geqq 1, 1 \leqq j \leqq r,\\ (35)\quad &Q_{nr+j-1}^{mr+j} = \sum_{s=(n-1)r+j}^{Nr} R_s^{m-n+1} - \sum_{s=nr+j}^{Nr} R_s^{m-n}, && j = 1, 2, \cdots, r-1, r,\ m > n = 1, 2, \cdots, N-1,\\ &Q_{Nr+j-1}^{mr+j} = \sum_{s=(N-1)r+j}^{Nr} R_s^{m-N+1}, && n > N, j = 1, 2, \cdots, r, \end{aligned}$$

PROOF. Denote by w_m waiting time of the mth customer and by ϕ_m the length of the time interval after the mth arrival that $(Nr + r)$ customers are present. If the mth arrival finds fewer than $(Nr + r - 1)$ customer in the system then $\phi_m = 0$. Let s_m be the duration of the mth service period.

Consider the inequality

$$(36) \quad w_{(m-n)r} + s_{m-n} \geqq \phi_{(m-n)r} + \tau_{(m-n)r+1} + \cdots + \phi_{mr+j-1} + \tau_{mr+j},$$

where the τ_j are successive 1-input times, and $m > n \geqq 0$, $j = 1, 2, \cdots, r$, and $1 \leqq nr + j \leqq Nr$. If (36) is the case, then on the $(m - n)$th departure there are at least $nr + j$ customers present and on the $(nr + j)$th arrival there are at least $(n + 1)r + j - 1$ customers present. Conversely if either of these events occurs, so does the other and (36) is the case. Noting that $Q_k^{mr+j} = 0$ unless $k \equiv (j - 1) \bmod r$, we have

$$(37) \quad \sum_{s=n+1}^{N} Q_{sr+j-1}^{mr+j} = \sum_{s=nr+j}^{Nr} R_s^{m-n}, \qquad m > n \geqq 0, \ j - 1, 2, \cdots, r.$$

From (37) we deduce the second and third equations (35). The first equation is established as follows:

$$Q_{j-1}^{mr+j} = 1 - \sum_{s=1}^{N} Q_{sr+j-1}^{mr+j},$$
$$= 1 - \sum_{s=j}^{Nr} R_s^m,$$

because of (37) with $n = 0$.

This proves the theorem. We remark that the proof applies equally to the bulk service queue of Section 1 with general distribution of 1-input and 0-input times. It is necessary however when $N < \infty$ that the 1-input be triggered, that is stops as soon as $(Nr + r)$ customers are present.

Write $Q_{nr+j-1}^{*j} = \lim_{m\to\infty} Q_{nr+j-1}^{mr+j}$, $1 \leqq j \leqq r$, $0 \leqq n \leqq N$, then from (35) we have

$$Q_{j-1}^{*j} = \sum_{s=0}^{n-1} R_s,$$

$$(38)\qquad Q_{nr+j-1}^{*j} = \sum_{s=0}^{r} R_{(n-1)r+j+s},$$

$$Q_{Nr+j-1}^{*j} = \sum_{s=0}^{r-j} R_{(N-1)r+j+s}.$$

In virtue of Theorem 6 the distribution $\{Q_j^m\}$ can be obtained from the distribution $\{R_j^m\}$ and the results of the previous sections can be formulated in terms of the distribution $\{Q_j^m\}$ in an obvious way.

6. The queueing system $E_r/M/1$. The queueing system of Section 1 can be interpreted as the queueing system $E_r/M/1$ with finite capacity and triggered 1-input process. Thus if we consider every rth customer entering the system the input process can be regarded as a triggered E_r process; that is, in the terminology of Foster [4]. the 1-input time has an Erlang E_r distribution with mean value r/λ, and the 1-input process stops as soon as $(N + 1)$ customers are present and restarts as soon as N customers are present. This process we call the imbedded E_r queueing process. We remark that this queueing system differs from that studied by Takács [6] who considered the process $GI/M/s$ with finite capacity and untriggered 1-input process.

Denote by R_j^{*m} the probability that the mth departure in the imbedded E_r queueing process leaves j customers in the system. Then the following lemma is self-evident.

LEMMA 2.

$$(39)\qquad R_j^{*m} = \sum_{k=0}^{r-1} R_{jr+k}^m, \qquad 0 \leqq j < N,$$
$$R_N^{*m} = R_{Nr}^m.$$

Thus, in particular, $\sum_{k=0}^{r-1} R_k^1 = 1$ *implies* $R_0^{*1} = 1$, *and conversely.*

Denote by Q_j^{*m} the probability that the mth arrival in the imbedded E_r queueing process finds j customers in the system. Then we have

$$Q_j^{*m} = Q_{jr+r-1}^{mr}, \qquad 0 \leqq j \leqq N. \tag{40}$$

We remark that Theorem 6 is valid also for a queueing system such as that of Section 1 with general distribution of 1-input and 0-input times. It applies therefore to the imbedded E_r queueing process which is a special case obtained by putting $r = 1$ in equations (35). Thus we have

$$\begin{aligned} Q_0^{*m+1} &= R_0^{*m}, \\ Q_j^{*m+1} &= \sum_{s=j}^{N} R_s^{*m+1-j} - \sum_{s=j+1}^{N} R_s^{*m-j}, \qquad m > j, \\ Q_N^{*m+1} &= R_N^{*m+1-N}. \end{aligned} \tag{41}$$

Equations (41) may be obtained also by direct substitution from (39) and (40) into equation (35).

Write $Q_j^* = \lim_{m\to\infty} Q_j^{*m}$, $R_j^* = \lim_{m\to\infty} R_j^{*m}$, then from (41) we obtain

$$Q_j^* = R_j^*, \qquad 0 \leqq j \leqq N. \tag{42}$$

We state now some theorems concerning the distribution $\{R_j^{*m}\}$.

THEOREM 7. *If* $R_0^{*1} = 1$, *limiting distribution* $\{R_j^*\}$ *for the imbedded* E_r *queueing process is given by*

$$\begin{aligned} R_j^* &= [F_{(N-j)r} - F_{(N-j-1)r}][F_{Nr}]^{-1} \qquad 0 \leqq j < N, \\ R_N^* &= [F_{Nr}]^{-1}, \end{aligned} \tag{43}$$

where F_k *is given by* (6).

THEOREM 8. *If* $R_0^{*1} = 1$, *then the generating function* $R_j^*(w) = \sum_{m=1}^{\infty} R_j^{*m} w^{m-1}$ *for the imbedded* E_r *queueing process is given by*

$$\begin{aligned} R_0^*(w) &= 1 + w(1-w)^{-1}[F_{Nr}(w) - F_{(N-1)r}(w)][F_{Nr}(w)]^{-1}, \\ R_j^*(w) &= w(1-w)^{-1}[F_{(N-j)r}(w) - F_{(N-j-1)r}(w)][F_{Nr}(w)]^{-1}, \\ R_N^*(w) &= w(1-w)^{-1}[F_{Nr}(w)]^{-1}, \end{aligned} \tag{44}$$

where $F_k(w)$ *is given by* (16).

Theorems 7 and 8 are immediate consequences of Lemma 2 and Theorems 1 and 2.

Similarly from Theorem 4 we obtain

THEOREM 9. *If* $N = \infty$ *and* $R_0^{*1} = 1$, *then the generating functions* $R_k^*(w)$ *of the imbedded* E_r *queueing process are given by*

$$\begin{aligned} R_0^*(w) &= 1 + w(1-w)^{-1}[1 - \{\gamma(w)\}^r], \\ R_j^*(w) &= w(1-w)\{\gamma(w)\}^r[1 - \{\gamma(w)\}^r], \qquad j \geqq 1, \end{aligned} \tag{45}$$

where $\{\gamma(w)\}^j$ *is given by* (24). *Further probabilities* R_j^{*m} *are given explicitly by*

$$(46)\quad \begin{aligned} R_0^{*m} &= 1 - \sum_{n=0}^{m-2} rn^{-1}\binom{nr+n+r-1}{n-1} b^n a^{nr+r}, & m \geqq 2,\\ R_j^{*m} &= \sum_{n=0}^{m-2} D_j^n, & j \geqq 1, m \geqq 2, \end{aligned}$$

where $D_j^0 = a^{jr}(1 - a^r)$, $j \geqq 1$, *and*

$$(47)\quad \begin{aligned} D_j^n = n^{-1} r b^n a^{nr+jr}\Big[j &\binom{nr+jr+n-1}{n-1} \\ &-(j+1)\binom{nr+jr+r+n-1}{n-1} a^r \Big], \qquad n \geqq 1, j \geqq 1. \end{aligned}$$

PROOF. Equations (45) follow immediately from equations (39) and (25). Expanding the expressions (45) are power series in $\gamma(w)$ and using (24) we obtain (46). A general expression for $R^*(w, z) = \sum_{j=0}^{\infty} R_j^*(w) z^j$ for the queueing system $GI/M/1$ has been given by Takács.

We remark (c.f., Foster [4]) that the queuing process dual to the imbedded E_r queueing process is the queueing system $M/E_r/1$ with finite capacity. If $\bar{R}_j^m$ denote the probability that the mth departure in the dual of the imbedded E_r queueing process leaves j customers waiting, then $\bar{R}_j^m = Q_{N-j}^{*m}$. Similarly if $\bar{Q}_j^m$ denotes the probability that the mth arrival in the dual of the imbedded E_r queueing processes finds j customers present, then $\bar{Q}_j^m = R_{N-j}^{*m}$. Thus Theorem 7 gives the limiting distribution $\bar{Q}_j$ (and also $\bar{R}_j$ in virtue of (42)). Theorem 8 gives the generating functions $\bar{Q}_j(w) = \sum_{m=1} \bar{Q}_j^m w^{m-1}$ under the initial conditions $\bar{Q}_N^1 = 1$, that is the system is full just after the first arrival. It is possible to obtain an analogue of Theorem 9 for the dual of the imbedded queueing process when $N = \infty$ under the initial condition $\bar{Q}_0^1 = 1$, but the expressions for the generating functions $\bar{Q}_j(w)$ depend on all the roots $\gamma_k(w)$, $k = 0, 1, \cdots, r$ of the equation (28). These expressions are very complicated and will not be given here. We remark that the transient behavior of $M/G/1$ with infinite capacity is studied in Finch [3].

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REFERENCES

[1] BAILEY, N. T. J. (1954). On queueing processes with bulk service. *J. Roy. Statist. Soc. Ser. B.* **16** 80–87.

[2] FINCH, P. D. (1958). The effect of the size of the waiting room on a simple queue. *J. Roy. Statist. Soc., Ser. B.* **20** 182–186.

[3] FINCH, P. D. (1960). On the transient behavior of a simple queue. *J. Roy. Statist. Soc., Ser. B.* **22** 277–284.

[4] FOSTER, F. G. (1959). A unified theory of stock, storage and queue control. *Operations Res. Quart.* **10** 121–130.

[5] MORSE, P. M. (1958). *Queues, Inventories and Maintenance.* Wiley, New York.

[6] TAKÁCS, L. (1958). A combined waiting and loss problem concerning telephone traffic. *Ann. Univ. Sci. Budapest* **1** 73–82.

[7] TAKÁCS, L. (1960). Transient behavior of single server queueing processes with recurrent input and exponentially distributed service times. *Operations Res. Quart.* **8** 231–245.

[8] WHITTAKER, E. T. and WATSON, G. N. (1950). *Modern Analysis,* 4th Ed. Cambridge Univ. Press.