F. N. David and D. E. Barton, *Combinatorial Chance*. Hafner Publishing Co. New York, 1962. $10.25 ix + 356 pp.

Review by John Riordan

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The attractive but unusual title asks for explanation. What is combinatorial chance? The authors say in the preface that its doctrine "was firmly established by the great triumvirate Pierre-Raymond Montmort, James Bernoulli, and Abraham de Moivre," and given nineteenth and twentieth century development by Laplace and MacMahon. On page 2 they say that "exhaustive enumeration ... lies at the root of combinatorial probability, and is therefore the main theme of this present book." The mention of MacMahon and exhaustive enumeration clearly justifies "combinatorial", while Laplace inevitably suggests probability, and thus chance. Hence the subject may be regarded as either combinatorial analysis in probability form, or probability in its combinatorial aspect. Or again, it is combinatoric without number theory or probability without (mathematical) analysis.

The book is intended as a "synthesis of the old and the new—but principally the old" and as "a delineation of such combinatorial methods as have been found useful by the statistician." Since it arose in editing *Symmetric Function and Allied Tables*, a book by the authors and M. G. Kendall, which is to appear, it is presumably both the companion and expansion of the tables.

The development is in eighteen chapters with many familiar mathematical objects and themes. Naturally, there are generating functions of many kinds: probability, ordinary and central moment, factorial moment, cumulant and factorial cumulant (a useful addition to the armory). There are card problems and dice problems treated first in the style of their eighteenth century originators and later in more modern dress, occupancy problems (what MacMahon calls *distribution*) both in old and new forms, run problems in various aspects, and extreme-value distributions (the hard part of combinatorial theory). For run problems, circular (ring) arrangements as well as the usual arrangements on a line are treated. Permutations are considered in many familiar and unfamiliar ways, by what Netto called *sequences* (rises and falls when the elements are numbers and are plotted against position), by numbers in forbidden positions (derangements or matching), by successions (pairs of consecutive elements, 12, 23, $\cdots$), by records (an element of a permutation is an upper record if it is larger than any of its predecessors, a lower if it is smaller; the first element is ignored), and by cycles (curiously in a chapter on Generalized Bernoulli Numbers). As expected, there are partitions, symmetric functions, the indispensable Stirling numbers, and a number of other numbers. (One of the nice features of the book is an extensive collection of numerical tables; the Stirling numbers themselves are tabled up to and including $n = 15$). A considerable space is given to limiting distributions.

Besides the usual combinatorial methods of argument by recurrence and of inclusion and exclusion (which the authors prefer to call the Halley-de Moivre Theorem, though it also appears as Whitworth's Theorem) the authors make systematic use of what they call a characteristic random variable (a variable assuming only two values, 1 and 0, with respective probabilities $p$ and $1 - p$). The idea, but not the term surely now must bear the trademark Feller! This usage gives many of their treatments a refreshing novelty, though it sometimes leads to amnesia relative to the theme of "exhaustive enumeration."

There is an index and a glossary of notation and usage. Thus, the book contains a mass of material tailored to the interest of the mathematical statistician and attractively presented. Nevertheless it contains numerous sins against unity, coherence and emphasis. One example is the Stirling numbers. Those of the second kind appear first on page 14 in the notation $C_k = \Delta^k 0^n/k!$ and without their identity papers. They reappear on page 35 in continuation of the same occupancy problem still unidentified. On page 47, they are identified, having received a new notation $C_{k,n-1}$ on page 46. They go back into hiding throughout the occupancy chapter (fourteen) and finally reappear on page 292 where they are given the notation $\Delta_{kn}$, preferred in the glossary but unused elsewhere. The signless numbers of the first kind appear as $d_{kn}$ on page 47, as $(-1)^{n+k}S_n^k$ on page 74, as $T_{n,u}$ on page 180, and as $D_{kn}$ on page 292, also preferred in the glossary. A second example is the treatment of the simplest occupancy problem, the distribution of unlike objects into unlike cells. This appears first on page 13 (exercise 14 of Chapter 1), next on page 35 with a reference backward, next on page 41 with a reference to its second appearance only, next on page 49, again with reference only to second appearance, and finally on page 251 without reference. Moreover in the last, the enumeration is by number of empty cells, giving the simplest formula for factorial moments, and the authors fail to note the simple formula for change of origin, relating these moments to those for enumeration by number of occupied cells.

The authors seem allergic to the use of generating functions in an algebraic way. For instance, on page 45 they verify the relation

$$\kappa_{r+1} = pq \frac{\partial}{\partial p} \kappa_r$$

for cumulants of the binomial distribution by multiple differentiations, although the result is a direct consequence of the generating function relation

$$\frac{\partial}{\partial t} \mathrm{K}(t) = np + pq \frac{\partial}{\partial p} \mathrm{K}(t)$$

with $\mathrm{K}(t) = \sum \kappa_r t^r/r! = \log (q + pe^t)^n$. On page 52 they examine the distribution of an $n$-sided die and give its ordinary moment generating function as $n^{-1}(e^{nt} - 1)e^t(e^t - 1)^{-1}$ and remark that "there is clearly no ease of approach in using this generating function." But the generating function may be rewritten $\exp t(b + 1) (\exp nbt)^{-1}$ where $t(e^t - 1)^{-1} = \exp bt$, $b^r \equiv b_r$, the Bernoulli numbers, employing the usual convention for exponential generating functions. Then since $\exp t(b+1) = t + \exp bt$, $(\exp nbt)^{-1} = \exp \beta t$, $\beta^r \equiv \beta_r = n^r(r+1)^{-1}$, it follows at once that the $r$th ordinary moment is given by

$$\mu_r' = n^{r-1} + \sum_{j=0}^{r} \binom{r}{j} b_{r-j} n^j (j+1)^{-1}$$

the formula the authors write down "from first principles," that is, without derivation.

Again, the same allergy seems to have kept the authors from the use of the Bell multivariable polynomials, $Y_n(fg_1, fg_2, \cdots, fg_n)$ in the notation of the reviewer's book [*An Introduction to Combinatorial Analysis*, Wiley, New York, 1958], although the relations of moments to cumulants are most concisely expressed through them, namely by

$$\mu_n' = Y_n(\kappa_1, \kappa_2, \cdots, \kappa_n), \qquad \kappa_n = Y_n(f\mu_1', \cdots, f\mu_n'),$$
$$\mu_n = Y_n(0, \kappa_2, \cdots, \kappa_n), \qquad \quad = Y_n(0, f\mu_2, \cdots, f\mu_n),$$

with $\mu_n$ the $n$th central moment, $\kappa_n$ the $n$th cumulant, and

$$f^k \equiv f_k = (-1)^{k-1}(k-1)!, \qquad k = 1, 2, \cdots.$$

Finally, it should be noticed that a famous theorem due to George Pólya, which N. G. de Bruijn has called his "fundamental theorem in enumerative combinatorial analysis," is nowhere mentioned, although it is the natural tool for runs in a ring, where the authors are content with E. Jablonski (1892).

Many of these slips and incoherencies are possibly due to the well-known difficulties of co-authorship. And despite all criticisms, it may be that the working statistician on some occasion will find in this book exactly what he needs.