NOTES

A FLUCTUATION THEOREM FOR CYCLIC RANDOM VARIABLES[1]

By Meyer Dwass[2]

University of Minnesota

1. Statement of theorem. We shall call $X_1, \cdots, X_n$ a *cyclic* set of random variables (or simply *cyclic*) if $P(X_1 \leqq t_1, \cdots, X_n \leqq t_n)$ is constant for all n cyclic permutations of the sequence $t_1, \cdots, t_n$. Loosely speaking, the random variables are cyclic if their distribution law is invariant under cyclic permutations. Similarly, the set is called exchangeable (or symmetrically dependent) if their distribution law is invariant under *all* permutations. Exchangeable sets of random variables are cyclic, but the converse is not true. Let

$$S_k = X_1 + \cdots + X_k, \qquad k = 1, \cdots, n,$$

$$M = \max(S_1, \cdots, S_n),$$

$$x^+ = \begin{cases} x & \text{if } x \geqq 0, \\ 0 & \text{if } x \leqq 0, \end{cases}$$

$$x^- = \begin{cases} x & \text{if } x \leqq 0, \\ 0 & \text{if } x \geqq 0. \end{cases}$$

The main purpose of this note is to prove the following.

Theorem 1. *Suppose that $X_1, \cdots, X_n$ is a cyclic set of random variables. Then*

$$E(M^- \mid S_n = s) = s^-/n. \tag{1.1}$$

The proof will be given in Section 2.

Remark. By the conditional expectation in (1.1) is meant a measurable function of s, which, for any measurable set A on the real line satisfies

$$\int_A E(M^- \mid S_n = s)\, dP(S_n \leqq s) = E(M; S_n \text{ in } A),$$

whenever this expectation exists. The assertion of the theorem is then that s^-/n is one possible version of this function. Of course, any other version must agree with s^-/n, except possibly on a linear set of probability zero.

An interesting special case of (1.1) arises where the X_i assign all their mass to $-1, 0, 1, 2, \cdots$. Then under the condition that $S_n = u$, a negative integer, and M is negative, then M *must equal* -1. Hence, in that case, (1.1) says,

$$P(M < 0 \mid S_n = u) = -u/n. \tag{1.2}$$

A version of (1.2) that has been proved before, but only for exchangeable

Received December 15, 1961; revised May 2, 1962.

[1] The study was supported by the Air Force Office of Scientific Research.

[2] Now at Northwestern University.

random variables, is the following. If $Y_1, \cdots, Y_n$ are cyclic, and each Y_i assigns all its mass to 0, 1, 2, $\cdots$, then

$$(1.3)\quad P(Y_1 + \cdots + Y_k < k, k = 1, \cdots, n \mid Y_1 + \cdots + Y_n = r) = 1 - r/n,$$

if r is a possible value of $Y_1 + \cdots + Y_n$, $0 \leqq r \leqq n$.

It is easy to verify that (1.2) and (1.3) are equivalent. The relation (1.3) was proved before for exchangeable random variables in [1]. However, the method of proof there uses the exchangeability in an essential way, and is not adaptable to the proof of (1.1) or even to the special cases (1.2) and (1.3). The special case of X_i assigning all mass to -1 and $+1$, and exchangeable is discussed by Feller under the subject of Bertrand's "Ballot Problem," ([2], pp. 66, 70.)

2. Proof of theorem. In order to motivate the proof, we give first a numerical example which illustrates the method. Suppose $(X_1, \cdots, X_6)$ is equally likely to be any of the six cyclic permutations of $(-7, 1, -3, 2, -2, -3)$. (We choose integer values only for simplicity.) Table I lists all possible values of the rele-

TABLE I

X_1	X_2	X_3	X_4	X_5	X_6	S_1	S_2	S_3	S_4	S_5	S_6	M^-		
−7	1	−3	2	−2	−3	−7	−6	−9	−7	−9	−12	−6	$k_1 = 2$	$-7 + 1 = -6$
1	−3	2	−2	−3	−7	1	−2	0	−2	−5	−12	0		
−3	2	−2	−3	−7	1	−3	−1	−3	−6	−13	−12	−1	$k_2 = 2$	$-3 + 2 = -1$
2	−2	−3	−7	1	−3	2	0	−3	−10	−9	−12	0		
−2	−3	−7	1	−3	2	−2	−5	−12	−11	−14	−12	−2	$k_3 = 1$	$-2 = -2$
−3	−7	1	−3	2	−2	−3	−10	−9	−12	−10	−12	−3	$k_4 = 1$	$-3 = -3$

vant vaiiables. First observe that $EM^- = -12/6 = S_6/6$, which is the assertion of the theorem. In the first permutation M^- is achieved in the second (k_1) position. The value of M^- in the permutations that follow up to but not including the third ($k_1 + 1$) is zero. In the third ($k_1 + 1$) permutation, M^- is achieved in the second (k_2) position, and the value of M^- in the permutations that follow up to but not including the fifth ($k_1 + k_2 + 1$) is 0. In the fifth ($k_1 + k_2 + 1$) permutation, M^- is achieved in the 1st (k_3) position. In the sixth ($k_1 + k_2 + k_3 + 1$) permutation, M^- is achieved in the 1st position ($k_4 = 1$). Since $k_1 + k_2 + k_3 + k_4 = 6 = n$ we stop the process. Thus $6EM^- = (-7 + 1) + (-3 + 2) + (-2) + (-3) = S_6$.

We now proceed to the formal proof. It will be sufficient to prove the theorem assuming all the mass is concentrated on the n cyclic permutations of a given set of numbers, $x_1, \cdots, x_n$ (as in the example), since the general situation follows easily from this special case. Denote $x_1 + \cdots + x_k = s_k$, $k = 1, \cdots, n$. The cyclically permuted sequence $x_k, \cdots, x_{k-1}$, will be denoted by $T(k)$, and the maximum of the partial sums in $T(k)$ will be denoted by $m(k)$. *The idea of the proof will be to show that each x_i occurs exactly once in some negative $m(k)$.* We first remark that there is no loss of generality in supposing that $m(1)$ is negative;

for it is easy to verify that since s_n is negative then $m(k)$ must be negative for some k, and $T(k)$ could become the initial sequence, $T(1)$ under an appropriate relabelling.

(a) For any k, the partial sums of $T(k + 1)$ are

$$s_{k+1} - s_k, \cdots, s_n - s_k, s_n - s_k + s_1, \cdots, s_n - s_k + s_k. \tag{2.1}$$

Since $s_n - s_k > s_n - s_k + s_i$, $i = 1, \cdots, k$, this means that $m(k)$ is achieved for the last time by one of the $n - k$ terms, $s_{k+1} - s_k, \cdots, s_n - s_k$.

(b) Consider now the position in $T(1)$ where $m(1)$ is last achieved. Call this position k_1. Since $s_{k_1} \geqq s_k$ for $k < k_1$, and $s_{k_1} > s_k$ for $k > k_1$, it follows from examining (2.1) for $k = k_1$ that

$$m(k_1 + 1) < 0, \quad \text{and} \quad m(k) \geqq 0, \quad \text{if} \quad 1 < k \leqq k_1.$$

(c) Let k_2 be the position in $T(k_1 + 1)$ where $m(k_1 + 1)$ is last achieved. From (a), $k_2 \leqq n - k_1$. Applying the analysis in (b) once again, we have that

$$m(k_1 + k_2 + 1) < 0, \quad \text{and} \quad m(k) \geqq 0, \quad \text{if} \quad k_1 + 1 < k \leqq k_1 + k_2.$$

(d) The above procedure is continued for a finite number of steps until $k_1 + \cdots + k_r = n$.

(e) Finally we have

$$\begin{aligned} m(k_1) &= s_{k_1}, \\ m(k_2) &= s_{k_1+k_2} - s_{k_1}, \\ &\vdots \\ m(k_r) &= s_{k_1+\cdots+k_r} - s_{k_1+\cdots+k_{r-1}}, \end{aligned}$$

and so $nEM^- = m(k_1) + \cdots + m(k_r) = s_n$, which completes the proof.

3. Additional results for independent random variables. If $X_1, \cdots, X_n$ are independent and identically distributed then of course they are cyclic and (1.1) holds. But in this case somewhat more precise information can be given. Suppose (X_n) is an infinite sequence of independent and identically distributed random variables. We now write $M_n = \max(S_1, \cdots, S_n)$ to emphasize the dependence of M on n.

THEOREM 2. *Under the above assumptions,*

$$Et^{M_n^+} = Et^{X_1}Et^{M_{n-1}^+} + P(M_n < 0) - E(t^{M_n}; M_n < 0),$$

$|t| \leqq 1$.

PROOF. The proof follows by an elementary calculation using the facts that $M_n = M_n^- + M_n^+$, $t^{M_n^+} + t^{M_n^-} = t^{M_n} + 1$, and $Et^{M_n} = Et^{X_1}Et^{M_{n-1}^+}$.

COROLLARY 1. *If* $\lim_{n\to\infty} P(M_n < 0) > 0$, *then* $\lim_{n\to\infty} M_n = M$ *is a well-defined random variable and Theorem 2 implies, on going to the limit, that*

$$Et^{M^+} = \frac{P(M < 0)(1 - E(t^M \mid M < 0))}{1 - Et^{X_1}}. \tag{3.2}$$

COROLLARY 2. *Suppose EX_1 exists and is negative. Since the left side of (3.2) is 1 for $t = 1$, then evaluation of the indeterminacy in the right hand of (3.2) gives that*

$$E(M; M < 0) = EX_1 . \tag{3.3}$$

This result can easily be obtained from (1.1) by a limiting argument. As before, if the X_i assign all their mass to $-1, 0, 1, \cdots$, then

$$P(S_k < 0, k = 1, 2, \cdots) = -EX_1 .$$

4. Relation to a known formula. It has been proved by Kac [3] and Spitzer [4] that if the random variables $X_1, \cdots, X_n$ are exchangeable, then

$$EM_n^+ - EM_{n-1}^+ = ES_n^+/n. \tag{4.1}$$

We wish to point out here that (4.1) is true also in the cyclic case, and is easily deduced from (1.1). Conversely, (4.1) follows from (1.1). To verify these assertions, notice that $M_n^+ - M_{n-1}^+$ is zero unless S_n exceeds 0 and $S_1, \cdots, S_{n-1}$, and therefore equals $[\min (S_n, S_n - S_1, \cdots, S_n = S_{n-1})]^+$. Hence,

$$E(M_n^+ - M_{n-1}^+ \mid S_n) = E[(\min S_n, S_n - S_1, \cdots, S_n - S_{n-1})^+ \mid S_n]$$
$$= E[(\min T_1, T_2, \cdots, T_n)^+ \mid T_n],$$

where $T_1 = X_n$, $T_2 = X_n + X_{n-1}, \cdots, T_n - S_n$. Since the cyclicity of $(X_1, X_2, \cdots, X_n)$ and $(X_n, X_{n-1}, \cdots, X_1)$ are equivalent, then the last expectation equals S_n^+/n by Theorem 1, (replacing the original random variables by their negatives). Thus (4.1) holds if (1.1) does, and conversely, if (4.1) holds, the same argument shows that (1.1) does.

The form of (4.1) that was proved in [3] and [4] is

$$EM_n^+ = ES_1^+/1 + ES_2^+/2 + \cdots + ES_n^+/n. \tag{4.2}$$

For exchangeable random variables (4.1) and (4.2) are obviously equivalent, because a subset of exchangeable variables are still exchangeable. On the other hand, if $X_1, \cdots, X_n$ are cyclic, then $X_1, \cdots, X_k$ for $k < n$, are in general not cyclic, and thus (4.2) will in general not hold for cyclic random variables, though (4.1) will be true.

A numerical illustration might be in order at this point. Suppose the random vector (X_1, X_2, X_3, X_4) assumes for its values the four cyclic permutations of

TABLE II

Possible values of				Possible values of					
X_1	X_2	X_3	X_4	S_1	S_2	S_3	S_4	M_4^+	M_3^+
−6	−3	10	11	−6	−9	1	12	12	1
−3	10	11	−6	−3	7	18	12	18	18
10	11	−6	−3	10	21	15	12	21	21
11	−6	−3	10	11	5	2	12	12	11

$(-6, -3, 10, 11)$. Table II provides the necessary data. $EM_4^+ - EM_3^+ = 3 = S_4/4$. The computation for the right side of (4.2) gives $(1/4)\ (21/1 + 33/2 + 36/3 + 48/4) \neq 63/4 = EM_4^+$.

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DETERMINING BOUNDS ON EXPECTED VALUES OF CERTAIN FUNCTIONS

BY BERNARD HARRIS

University of Nebraska

1. Introduction and summary. Let $\mathfrak{F}$ be the collection of cumulative distribution functions on $(-\infty, \infty)$ and $\mathfrak{F}_{[a,b]}$ that subset of $\mathfrak{F}$ all of whose elements have $F(a - 0) = 0$ and $F(b) = 1$.

Let $\mathfrak{F}^{(\mu_1,\mu_2,\cdots,\mu_k)}$ $(\mathfrak{F}_{[a,b]}^{(\mu_1,\mu_2,\cdots,\mu_k)})$ be the class of cumulative distribution functions on $(-\infty, \infty)$ $([a, b])$ whose first k moments are $\mu_1, \mu_2, \cdots, \mu_k$ respectively. We will suppose throughout that $\mu_1, \mu_2, \cdots, \mu_k$ is a legitimate moment sequence, i.e., that there exists a cumulative distribution function $F(x) \varepsilon \mathfrak{F}(\mathfrak{F}_{[a,b]})$ whose first k moments are $\mu_1, \mu_2, \cdots, \mu_k$.

Let $g(x)$ be a continuous and bounded function on $[a, b]$. Then, we wish to determine $F^*(x) \varepsilon \mathfrak{F}_{[a,b]}^{(\mu_1,\mu_2,\cdots,\mu_k)}$ with

$$(1) \qquad \int_a^b g(x)\, dF^*(x) = \min_{F \varepsilon \mathfrak{F}_{[a,b]}^{(\mu_1,\mu_2,\cdots,\mu_k)}} (\max) \int_a^b g(x)\, dF(x).$$

Any $F^*(x)$ satisfying (1) will be called an extremal distribution with respect to $g(x)$.

Let $\mathcal{G}_{[a,b]}^{(k)}$ be the set of continuous, bounded, and monotonic functions on $[a, b]$, whose first k derivatives exist and are monotonic in (a, b). In addition, we further require that $\mathcal{G}_{[a,b]}^{(k)}$ contain only functions not linearly dependent on the monomials $1, x, x^2, \cdots, x^k$.

This paper characterizes the extremal distributions for $g(x) \varepsilon \mathcal{G}_{[a,b]}^{(k)}$. The results are extended to $\mathfrak{F}_{[0,\infty)}^{(\mu_1,\mu_2,\cdots,\mu_k)}$ and $\mathfrak{F}^{(\mu_1,\mu_2,\cdots,\mu_k)}$, in that we investigate

Received May 16, 1961; revised March 19, 1962.