TESTING APPROXIMATE HYPOTHESES IN THE COMPOSITE CASE[1]

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1. Introduction. One of the chief reasons for not using the Kolmogorov-Smirnov tests is that, as originally presented, they were only suitable for testing the simple hypothesis $F = F_0$ against all alternatives. In a previous paper [3], the author investigated these tests, and extended them to eliminate the paradox of almost sure rejection of the null hypothesis when too much data is observed. Also Kac, Kiefer and Wolfowitz in [1], investigated extensions of the Kolmogorov-Smirnov tests for testing "larger" null hypotheses by means of minimum distance methods. Mention was made in [1] of the difficulty of computing the test statistic. Due to this latter difficulty they suggested a test of normality in which the composite null hypothesis is essentially reduced to a simple one by replacing μ and σ^2 by their estimates $\bar{X}_n$ and s_n^2. Such a test suffers from the disadvantage that distributions which are distance-wise "close" to being normal can lead to rejection of the hypothesis of normality with high probability (since closeness of distribution does not imply closeness of their corresponding moments).

In this paper the basic theory for such tests is briefly developed, and then attention is turned to the practical problem of performing the tests, with round-off error taken into account. Two classes of tests of translation-scale parameter families are presented. They can be performed in a finite number of operations, and have the property that distributions in a "neighborhood" of some member of the family will lead to acceptance of the null hypothesis with at least a specified probability, while distributions at least a specified distance from all such neighborhoods will lead to rejection of the null hypothesis with at least a given probability. Though not done explicitly in this paper, it is clear that the methods developed could be extended to n-parameter families in certain cases.

Essentially, this paper is an extension of the work originated in [3] (which did for the Kolmogorov-Smirnov tests what the paper [2], of Hodges and Lehmann did for the chi-square test), to cases of richer null hypotheses.

2. Some theory of testing hypotheses based on the use of a metric. Assume $X_1, \cdots, X_n$ to be independent random variables with a common distribution function, and let F_n be the random process whose value is the empirical distribution function formed from the observed values of $X_1, \cdots, X_n$. Let $\mathfrak{D}$ be

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the set of all one-dimensional distribution functions. Let d be a metric on $\mathfrak{D} \times \mathfrak{D}$ such that for some sequences $\lim_{n\to\infty} h_{\alpha,n} = h_\alpha$, we have

$$P_F\{d(F_n, F)/c(n) \geqq h_{\alpha,n}\} \leqq \alpha\{c(n)\}$$

and $\{h_{\alpha,n}\}$ with $\lim_{n\to\infty} c(n) = 0$ and for every $F \varepsilon \mathfrak{D}$. (Note that $c(n)$ and $h_{\alpha,n}$ are not to depend on F.) Such conditions are satisfied for example, if $d = d_1$ (the uniform metric), $c(n) = n^{-\frac{1}{2}}$, and $h_{\alpha,n} = h_{1,\alpha,n}$ as in [1]; i.e., $d_1(F_1G) = \sup_x |F(x) - G(x)|$, and for F continuous

$$P_F\{n^{\frac{1}{2}} d_1(F_n, F) \leqq h_{1,\alpha,n}\} = 1 - \alpha.$$

For $\mathfrak{H}_o^*$ any given subset of $\mathfrak{D}$, a test of $\mathfrak{H}_o^*$ with useful properties is:

Rej $\mathfrak{H}_o^*$ when $X_1(\omega), \cdots, X_n(\omega)$ are observed

$$\Leftrightarrow \inf_{H\varepsilon\mathfrak{H}_o^*} d(F_n, H)(\omega)/c(n) \geqq h_{\alpha,n}.$$

Just how to carry out this procedure, which constitutes the major portion of this paper, depends both on $\mathfrak{H}_o^*$ and on the nature of the metric d.

THEOREM 2.1. *P_F*{rej $\mathfrak{H}_o^*$} $\leqq \alpha$ *for all F in* $\mathfrak{H}_o^*$.

The proof is obvious.

THEOREM 2.2. *Let $l \varepsilon (0, 1)$ and n be an integer for which*

$$l/c(n) - h_{\alpha,n} \geqq h_{\beta,n}.$$

Then P_F{rej $\mathfrak{H}_o^*$} $\geqq 1 - \beta$ *for all F for which* $\inf_{H\varepsilon\mathfrak{H}_o^*} d(F, H) \geqq l$.

PROOF. We first note that under the assumptions previously made there is an n satisfying the hypothesis of this theorem. Again from the previous assumptions, and the hypothesis,

$$\begin{aligned}1 - \beta &\leqq P_F\{d(F_n, F)/c(n) < h_{\beta,n}\} \\ &\leqq P_F\{d(F_n, F)/c(n) < l/c(n) - h_{\alpha,n}\}.\end{aligned} \tag{2.1}$$

It can easily be seen from the triangle inequality that

$$d(F, H) \geqq a, \qquad d(G, F) < b \Rightarrow d(G, H) \geqq a - b. \tag{2.2}$$

For $\inf_{H\varepsilon\mathfrak{H}_o^*} d(F, H) \geqq l$, applying (2.2), (identifying $Fn(\ )(\omega)$ with G),

$$\begin{aligned}P_F\{d(F_n, F) < l - c(n)h_{\alpha,n}\} &\leqq P_F\{\inf_{H\varepsilon\mathfrak{H}_o^*} d(F_n, H) \geqq c(n)h_{\alpha,n}\} \\ &= P_F\{\inf_{H\varepsilon\mathfrak{H}_o^*} d(F_n, H)/c(n) \geqq h_{\alpha,n}\} = P_F\{\text{rej } \mathfrak{H}_o^*\}.\end{aligned} \tag{2.3}$$

Combining (2.3) and (2.1) proves the desired result. If n is sufficiently large so that we almost have $P_F\{d(F_n, F)/c(n) \geqq h_\alpha\} \leqq \alpha$, [or if $h_\alpha > h_{\alpha,n}$ for all n], we may replace the suggested test with the simpler, rej $\mathfrak{H}_o^* \Leftrightarrow \inf_{H\varepsilon\mathfrak{H}_o^*} d(F_n, H)(\omega)/c(n) \geqq h_\alpha$, with n chosen to satisfy $l/c(n) - h_\alpha \geqq h_\beta$. In this case, Theorems 2.1 and 2.2 hold asymptotically, [or conservatively]. The significance of asymptotic results on power is that as $l \to 0$, the asymptotic results tend to exact ones.

In the next two sections we construct some workable tests of the hypothesis

$\mathcal{H}_o^*$ that F is "approximately" in the translation-scale parameter family $\mathcal{H}_o$. Here $\mathcal{H}_o^* \supset \mathcal{H}_o$ is large enough for it to be "actually possible" that F is in $\mathcal{H}_o^*$.

3. A test based on the uniform metric d_1.

DEFINITION.

$$\mathcal{H}_o = \{G: G(x) = F_c[(x - \mu)/\sigma] \text{ all } x, \text{ some } \mu, \sigma > 0, F_c \text{ a given continuous d.f.}\}$$
$$\mathcal{H}_o^* = \{F \in \mathfrak{D}: d_1(F, G) \leqq k, G \in \mathcal{H}_o\}.$$

The number k is determined from realistic considerations external to the mathematics. For each $G \in \mathcal{H}_o$ let $\mathcal{G}_G = \{H \in \mathfrak{D}: d_1(G, H) \leqq k\}$. If $F \in \mathfrak{D}$ and $F \notin \mathcal{G}_G$, we have, say, $d_1(F, G) = k + l, l > 0$.

LEMMA. *If $F^* \in \mathcal{G}_G$ satisfies*

$$\inf_{H \in \mathcal{G}_G} d_1(F, H) = d_1(F, F^*),$$

then $d_1(F^*, G) = k$, $d_1(F, F^*) = l$.

PROOF. We note that in [3] the existence of such an F^* was shown. The proof is geometrically obvious from the construction of F^* in [1]; since at the point x_0 at which $|F(x_0) - G(x_0)| = k + l$, we have $|F^*(x_0) - G(x_0)| = k$. Thus since $d_1(F^*, G) \leqq k$, we have $d_1(F^*, G) = k$. The final assertion, $d_1(F^*, F) = l$, then follows from the triangle inequality.

The proposed test of $\mathcal{H}_o^*$ is:

Rej $\mathcal{H}_o^*$ when $X_1(\omega), \cdots, X_n(\omega)$ are observed

$$\Leftrightarrow \inf_{H \in \mathcal{H}_o} d_1(F_n, H)(\omega) \geqq k + h_{1,\alpha,n}/n^{\frac{1}{2}},$$

where n is an integer for which

$$n^{\frac{1}{2}}l - h_{1,\alpha,n} \geqq h_{\beta,n}. \tag{3.1}$$

It is clear from the previous lemma and the theory of Section 2 that this test satisfies

$$P_F\{\text{rej } \mathcal{H}_o^*\} \leqq \alpha \text{ when } F \in \mathcal{H}_o^*; \quad \text{and}$$
$$P_F\{\text{rej } \mathcal{H}_o^*\} \geqq 1 - \beta \text{ when } \inf_{H \in \mathcal{H}_o^*} d_1(F, H) \geqq l. \tag{3.2}$$

The only problem which arises is how to determine for each $q, X_1(\omega), \cdots, X_n(\omega)$, in a finite number of steps, whether or not $\inf_{H \in \mathcal{H}_o} d_1(F_n, H)(\omega) < q$. The idea for the computational technique employed is due to Professor T. W. Anderson.

DEFINITION. Let $a_{j,q}$ and $b_{j,q}$ be any numbers for which $F_c(a_{j,q}) = j/n - q$, $F_c(b_{j,q}) = (j - 1)/n + q$ for $j = 1, \cdots, n$. Let $X_{[j]}(\omega)$ stand for that $X_i(\omega)$ which is jth in order of magnitude, i.e., if $X_{[j]}(\omega) = X_i(\omega)$, then $X_i(\omega)$ is greater than or equal to at least j of the values $X_1(\omega), \cdots, X_n(\omega)$, but does not exceed more than $j - 1$ of these values. We may have $X_{[j]}(\omega) = X_{[j+1]}(\omega), \cdots$ where $X_{[j]}(\omega) = X_i(\omega), X_{[j+1]}(\omega) = X_{i'}(\omega), i \neq i'$.

THEOREM 3.1. $\inf_{H \in \mathcal{H}_o} < q$ *if and only if for at least one* (μ, σ), $\sigma > 0$,

$$(X_{[j]}(\omega) - \mu)/\sigma < b_{j,q}, \quad (X_{[j]}(\omega) - \mu)/\sigma > a_{j,q} \quad \text{for all } j = 1, \cdots, n. \tag{3.3}$$

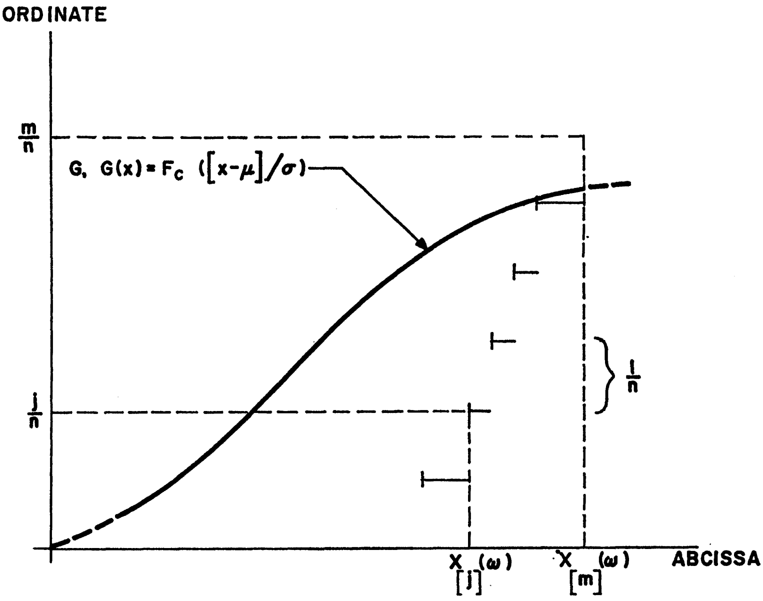

FIGURE 1

PROOF. In order that there be a $G \varepsilon \mathcal{H}_o$ for which $d_1(F_n, G) < q$ we must have for at least one (μ, σ), $\sigma > 0$

$$F_c[(X_{[j]}(\omega) - \mu)/\sigma] - (j-1)/n < q, \qquad j/n - F_c[(X_{[j]}(\omega) - \mu)/\sigma] < q$$

for all $j = 1, \cdots, n$ as illustrated in Figure 1. But this is equivalent to (3.3).

The significance of this theorem is that for each q, $X_1(\omega), \cdots, X_n(\omega)$ we can determine whether or not there is a (μ, σ), $\sigma > 0$ for which the inequalities (3.3) are all satisfied, in a finite number of operations. This is accomplished geometrically by looking at each inequality in (3.3) separately and blocking out those points in the (μ, σ) plane for which each inequality can not be satisfied. Only a straight-edge and graph paper are required for this. If there are any points (μ, σ), $\sigma > 0$, not blocked out for at least one of these inequalities, then we know that there is a G in $\mathcal{H}_o$ for which $d_1(F_n, G)(\omega) < q$. For $q = k + h_{1,\alpha,n}/n^{\frac{1}{2}}$ if there is a (μ, σ), $\sigma > 0$ for which (3.3) are satisfied we accept $\mathcal{H}_o^*$.

Determining whether or not there is a (μ, σ), $\sigma > 0$, satisfying (3.3) can also be programmed on a digital computer. However there is one difficulty which has been glossed over, namely errors in any such procedure. (In the geometric solution these are measurement errors. On a digital computer they arise from the restriction to a finite number of arithmetic operations.) In practice any such procedure can usually at best guarantee that for any given $\epsilon > 0$ if

$$\text{(3.4)} \qquad \inf_{H \varepsilon \mathcal{H}_o} d_1(F_n, H)(\omega) \geqq q,$$

no (μ, σ), $\sigma > 0$, will be found satisfying (3.3), and if

$$\inf_{H \varepsilon \mathcal{H}_o} d_1(F_n, H)(\omega) < q - \epsilon, \tag{3.5}$$

at least one such (μ, σ) will be found.

It can be shown that if F_c satisfies a Lipschitz condition with known parameter, then for given $X_1(\omega), \cdots, X_n(\omega), \epsilon > 0$, one can satisfy (3.4) and (3.5) using a suitable digital procedure. Intuitively we feel that for ϵ sufficiently small, such errors will have little effect on the test. The following theorem shows rigorously how the desired results can be achieved even in the presence of (a sufficiently small) round-off error.

THEOREM 3.2. *Let* $\delta \varepsilon (0, \alpha)$,

$$n^{\frac{1}{2}}l - h_{1,\alpha-\delta,n} \geqq h_{1,\beta,n}, \qquad q = k + h_{1,\alpha-\delta,n}/n^{\frac{1}{2}}, \qquad \epsilon = (h_{1,\alpha-\delta,n} - h_{1,\alpha,n})/n^{\frac{1}{2}}.$$

Then a test procedure satisfying (3.4), (3.5) *satisfies* (3.2). This theorem states that if round-off and other errors can be kept sufficiently small, their undesirable effects can be overcome by increasing the sample size. We shall not prove this theorem, because of the presence of similar more difficult ones in the next section.

If for all γ, $h_{1,\gamma,n} \leqq h_{1,\gamma}$, then all of our results hold with $h_{1,\alpha,n}$, $h_{1,\alpha-\delta,n}$ and $h_{1,\beta,n}$ replaced by $h_{1,\alpha}$, $h_{1,\alpha-\delta}$ and $h_{1,\beta}$ respectively (for $\alpha, \beta < \frac{1}{2}$).

Anderson's computational technique can easily be generalized to the case $\mathcal{H}_o^* = \{F \varepsilon \mathfrak{D}: H_1([x - \mu]/\sigma) \leqq F(x) \leqq H_2([x - \mu]/\sigma)$ all x, some μ, some $\sigma > 0$, $\lim_{x \to -\infty} H_1(x) = 0, \lim_{x \to \infty} H_2(x) = 1\}$. Similarly to the previous case we define $a_{j,q}^*$ and $b_{j,q}^*$ by

$$H_2(a_{j,q}^*) = j/n - q, \qquad H_1(b_{j,q}^*) = (j-1)/n + q.$$

In order for there to be a G in $\mathcal{H}_o^*$ for which $d_1(F_n, G)(\omega) < q$, we must have for at least one (μ, σ), $\sigma > 0$,

$$H_1[(X_{[j]}(\omega) - \mu)/\sigma] - (j-1)/n < q, \qquad j/n - H_2[(X_{[j]}(\omega) - \mu)/\sigma] < q$$

for all $j = 1, \cdots, n$. That is, for at least one pair (μ, σ), $\sigma > 0$, we must have

$$(X_{[j]}(\omega) - \mu)/\sigma < b_{j,q}^*, \ (X_{[j]}(\omega) - \mu)/\sigma > a_{j,q}^*, \quad \text{all } j = 1, \cdots, n.$$

Then as before, it is easily seen geometrically, or digitally whether or not there is at least one (μ, σ), $\sigma > 0$, satisfying the above inequalities.

4. A test based on the metric d_2. Here $d_2(F, G) = \sup_I |P_F(I) - P_G(I)|$, when I is an interval. We let $\mathcal{H}_o = \{G \varepsilon \mathfrak{D}: G(x) = F_c([x - \mu]/\sigma)$, all x, some $\mu, \sigma > 0$, F_c continuous in $\mathfrak{D}\}$, and

$$\mathcal{H}_o^* = \{F \varepsilon \mathfrak{D}: d_2(F, G) \leqq k, G \varepsilon \mathcal{H}_o\}.$$

From [4] and Section 2 of this paper we know that the test of $\mathcal{H}_o^*$

Rej H_o^* when $X_1(\omega), \cdots, X_n(\omega)$ are observed

$$\Leftrightarrow \inf_{H \varepsilon \mathcal{H}_o} d_2(F_n, H)(\omega) \geqq k + h_{2,\alpha,n}/n^{\frac{1}{2}},$$

where $l \varepsilon (0, 1)$ and n and $h_{2,\alpha,n}$ satisfy

$$n^{\frac{1}{2}}l - h_{2,\alpha,n} \geqq h_{2,\beta,n}, \quad \text{and} \quad P_F\{n^{\frac{1}{2}} d_2(F_n, F) \leqq h_{2,\alpha,n}\} = 1 - \alpha \tag{4.1}$$

for continuous F, satisfies (3.2) (with d_1 replaced by d_2). Unfortunately we know of no way to compute $\inf_{H \varepsilon \mathcal{H}_o} d_2(F_n, H)(\omega)$, or even to determine whether or not $\inf_{H \varepsilon \mathcal{H}_o} d_2(F_n, H)(\omega) \geqq k + h_{2,\alpha,n}/n^{\frac{1}{2}}$, in a finite number of performable operations. However, as we shall now demonstrate, a performable test of $\mathcal{H}_o^*$ with the desired properties can be constructed. The test will demand more observations than (4.1) would indicate.

DEFINITION. Let $\mathcal{K}(r) = \{G \varepsilon \mathfrak{D}: \inf_{H \varepsilon \mathcal{H}_o} d_2(G, H) < r\}$.

DEFINITION. Let $\mathcal{P}(r, \epsilon)$ be a mapping on the set of possible empirical distribution functions based on $X_1, \cdots, X_n$ to {"rej $\mathcal{H}_o^*$," "acc $\mathcal{H}_o^*$"};

$$\mathcal{P}(r, \epsilon)(F_n[\,](\omega)) = \text{"rej } \mathcal{H}_o^* \text{" when } F_n(\,)(\omega) \;\varepsilon\; \mathcal{K}^C(r + \epsilon),$$
(4.2)
$$\mathcal{P}(r, \epsilon)(F_n[\,](\omega)) = \text{"acc } \mathcal{H}_o^* \text{" when } F_n(\,)(\omega) \;\varepsilon\; \mathcal{K}(r).$$

Let $\delta \;\varepsilon\; (0, \alpha)$, let n be an integer satisfying

$$n^{\frac{1}{2}} l - h_{2,\alpha-\delta,n} \geqq h_{2,\beta,n}, \tag{4.3}$$

and let

$$r = k + h_{2,\alpha,n}/n^{\frac{1}{2}} \tag{4.4}$$

and

$$\epsilon = (h_{2,\alpha-\delta,n} - h_{2,\alpha,n})/n^{\frac{1}{2}}. \tag{4.5}$$

The proposed test (which we shall show can be carried out in a finite number of steps) is

Rej $\mathcal{H}_o^*$ when $X_1(\omega), \cdots, X_n(\omega)$ are observed

$$\Leftrightarrow \mathcal{P}(r, \epsilon)(F_n[\,](\omega)) = \text{"rej } \mathcal{H}_o^* \text{"}.$$

We shall show how to construct a mapping $\mathcal{P}(r, \epsilon)$ after the following theorems.

THEOREM 4.1. *For this test* $P_F\{\text{rej } \mathcal{H}_o^*\} \leqq \alpha$ *for all* $F \;\varepsilon\; \mathcal{H}_o^*$.

PROOF. When $F \;\varepsilon\; \mathcal{H}_o^*$

$$P_F\{\text{rej } \mathcal{H}_o^*\} = 1 - P_F\{\text{acc } \mathcal{H}_o^*\}$$
$$= 1 - P_F\{\mathcal{P}(r, \epsilon)(F_n) = \text{"acc } \mathcal{H}_o^*\text{"}\}$$
$$= 1 - P_F\{\mathcal{P}(r, \epsilon)(F_n) = \text{"acc } H_o^*\text{"} \mid F_n \;\varepsilon\; \mathcal{K}(r)\} \cdot P_F\{F_n \;\varepsilon\; \mathcal{K}(r)\}$$
$$- P_F\{\mathcal{P}(r, \epsilon)(F_n) = \text{"acc } \mathcal{H}_o^*\text{"} \mid F_n \not\varepsilon\; \mathcal{K}(r)\} \cdot P_F\{F_n \not\varepsilon\; \mathcal{K}(r)\}$$
$$\leqq 1 - P_F\{\mathcal{P}(r, \epsilon)(F_n) = \text{"acc } \mathcal{H}_o^*\text{"} \mid F_n \;\varepsilon\; \mathcal{K}(r)\} \cdot P_F\{F_n \;\varepsilon\; \mathcal{K}(r)\}$$
$$= 1 - P_F\{F_n \;\varepsilon\; \mathcal{K}(r)\}$$
$$= 1 - P_F\{\inf_{H \varepsilon \mathcal{H}_o} d_2(F_n, H) < r\}$$
$$= 1 - P_F\{\inf_{H \varepsilon \mathcal{H}_o} d_2(F_n, H) < k + h_{2,\alpha,n}/n^{\frac{1}{2}}\}$$
$$= P_F\{\inf_{H \varepsilon \mathcal{H}_o} d_2(F_n, H) \geqq k + h_{2,\alpha,n}/n^{\frac{1}{2}}\} \leqq \alpha.$$

THEOREM 4.2. *For this test* $P_F\{\text{rej } \mathcal{H}_o^*\} \geqq 1 - \beta$ *when* $\inf_{H \varepsilon \mathcal{H}_o^*} d_2(F, H) \geqq l$.

PROOF. Assume $\inf_{H \varepsilon \mathcal{H}_o^*} d_2(F, H) \geqq l$. From [2], Theorem 3.2

$$\inf_{H \varepsilon \mathcal{H}_o} d_2(F, H) \geqq k + l \Leftrightarrow \inf_{H \varepsilon \mathcal{H}_o^*} d_2(F, H) \geqq l.$$

Thus

$$\begin{aligned} &P_F\{\text{rej } \mathcal{H}_o^*\} = P_F\{\mathcal{P}(r, \epsilon)(F_n) = \text{“rej } \mathcal{H}_o^*\text{”}\} \\ &= P_F\{\mathcal{P}(r, \epsilon)(F_n) = \text{“rej } \mathcal{H}_o^*\text{”} \mid F_n \varepsilon \mathcal{K}^C(r + \epsilon)\} \cdot P_F\{F_n \varepsilon \mathcal{K}^C(r + \epsilon)\} \\ &+ P_F\{\mathcal{P}(r, \epsilon)F_n = \text{“rej } \mathcal{H}_o^*\text{”} \mid F_n \not\varepsilon \mathcal{K}^C(r + \epsilon)\} \cdot P_F\{F_n \not\varepsilon \mathcal{K}^C(r + \epsilon)\} \\ &\geqq P_F\{\mathcal{P}(r, \epsilon)(F_n) = \text{“rej } \mathcal{H}_o^*\text{”} \mid F_n \varepsilon \mathcal{K}^C(r + \epsilon)\} P_F\{F_n \varepsilon \mathcal{K}^C(r + \epsilon)\} \\ &= P_F\{F_n \varepsilon \mathcal{K}^C(r + \epsilon)\} = P_F\{\inf_{H \varepsilon \mathcal{H}_o} d_2(F_n, H) \geqq r + \epsilon\} \\ &= P_F\{\inf_{H \varepsilon \mathcal{H}_o} d_2(F_n, H) \geqq k + h_{2,\alpha-\delta,n}/n^{\frac{1}{2}}\} \\ &= P_F\{\inf_{H \varepsilon \mathcal{H}_o^*} d_2(F_n, H) \geqq h_{2,\alpha-\delta,n}/n^{\frac{1}{2}}\} \geqq 1 - \beta. \end{aligned}$$

This last step is justified by (4.3), and the results of Theorem 2.2, together with Theorem 3.2 of [2].

The only remaining step is to define a particular mapping $\mathcal{P}_o(r, \epsilon)$ of type $\mathcal{P}(r, \epsilon)$, and show that for $r > 0$, $\epsilon > 0$, $X_1(\omega), \cdots, X_n(\omega)$, we can compute $\mathcal{P}_0(r, \epsilon)(F_n[\ ](\omega))$ in a finite number of operations.

We will assume that r is a multiple of ϵ. Looking at (4.3), (4.4) and (4.5), we see that if r is not a multiple of ϵ for the α, β, l, δ and n chosen, since $h_{2,\alpha-\delta,n}$ varies continuously with δ, by decreasing δ so that $h_{2,\alpha-\delta,n}$ is sufficiently decreased, we can insure that r is a multiple of ϵ, i.e., hold α, β, l and n fixed at the value originally chosen and decrease δ until r is a multiple of ϵ. Then (4.3), (4.4), and (4.5) will still be satisfied.

Let $\mathcal{P}_o(r, \epsilon)(F_n[\ ](\omega)) = \text{“acc } \mathcal{H}_o^*\text{”}$ if and only if for some μ, $\sigma > 0$, all x, at least one of the following $r/\epsilon + 1$ pairs of inequalities holds:

$$F_c([x - \mu]/\sigma) - [r + (1 - m)\epsilon] < F_n(x)(\omega) < F_c([x - \mu]/\sigma) + m\epsilon, \quad m = 0, \cdots, r/\epsilon. \tag{4.6}$$

THEOREM 4.3. The mapping $\mathcal{P}_o(r, \epsilon)$ just defined satisfies (4.2).

PROOF. Suppose $F_n[\ ](\omega) \varepsilon \mathcal{K}^C(r + \epsilon)$, i.e., $\inf_{H \varepsilon \mathcal{H}_o} d_2(F_n, H)(\omega) \geqq r + \epsilon$. Then from the definition of d_2 and $\mathcal{H}_o$, for all μ, $\sigma > 0$,

$$\sup_x\{F_c([x - \mu]/\sigma) - F_n(x)(\omega)\} + \sup_x\{F_n(x)(\omega) - F_c([x - \mu]/\sigma)\} \geqq r + \epsilon.$$

Assuming these two suprema achieved at $y_1 = x_1(\mu, \sigma, \omega)$ and $y_2 = x_2(\mu, \sigma, \omega)$ respectively, we have

$$F_c([y_1 - \mu]/\sigma) - F_n(y_1)(\omega) + F_n(y_2)(\omega) - F_c([y_2 - \mu]/\sigma) \geqq r + \epsilon. \tag{4.7}$$

Clearly we can not have both

$$F_c([y_1 - \mu]/\sigma) - [r + (1 - m)\epsilon] < F_n(y_1)(\omega) < F_c([y_1 - \mu]/\sigma) + m\epsilon \tag{4.8}$$

and

(4.9) $F_c([y_2 - \mu]/\sigma) + [r + (1 - m)\epsilon] < F_n(y_2)(\omega) < F_c([y_2 - \mu]/\sigma) + m\epsilon$

for even one $m = 0, \cdots, r/\epsilon$, for then multiplying (4.8) by -1 and adding to (4.9) we would obtain

$$-(r + \epsilon) < F_c([y_1 - \mu]/\sigma) - F_n(y_1)(\omega) + F_n(y_2)(\omega) - F_c([y_2 - \mu]/\sigma) < r + \epsilon,$$

which would contradict (4.7).

Thus if $F_n(\ )(\omega)\ \varepsilon\ \mathcal{K}^C(r + \epsilon)$ (4.6) can not be satisfied for any $m = 0, \cdots, r/\epsilon$, i.e., *if* $F_n(\ )(\omega)\ \varepsilon\ \mathcal{K}^C(r + \epsilon)$, *we reject* $\mathcal{K}_o^*$.

Now suppose $F_n(\ )(\omega)\ \varepsilon\ \mathcal{K}(r)$. Then $\inf_{H \varepsilon \mathcal{K}_o} d_2(F_n, H)(\omega) < r$, i.e., for at least one (μ_0, σ_0), $\sigma_0 > 0$

$$\sup_x\{F_c([x - \mu_0]/\sigma_0) - F_n(x)(\omega)\} + \sup_x\{F_n(x)(\omega) - F_c([x - \mu_0]/\sigma_0)\} < r. \tag{4.10}$$

Choose m_1 to be the smallest integer exceeding

$$\sup_x\{F_n(x)(\omega) - F_c([x - \mu_0]/\sigma_0)\}/\epsilon.$$

(Note that due to (4.10), $0 < m_1 \leqq r/\epsilon$.) Then

$$(m_1 - 1)\epsilon \leqq \sup_x\{F_n(x)(\omega) - F_c([x - \mu_0]/\sigma_0)\} < m_1\epsilon. \tag{4.11}$$

Utilizing the left hand inequality and (4.10) we obtain

$$\sup_x\{F_c([x - \mu_0]/\sigma_0) - F_n(x)(\omega)\} < r - (m_1 - 1)\epsilon. \tag{4.12}$$

From (4.11) we have that for all x, $F_n(x)(\omega) < F_c([x - \mu_0]/\sigma_0) + m_1\epsilon$, while from (4.12) we have that for all x, $F_c([x - \mu_0]/\sigma_0) - [r + (1 - m_1)\epsilon] < F_n(x)(\omega)$; i.e., for $m = m_1$ (4.6) is satisfied, and $0 < m_1 \leqq r/\epsilon$.

Thus if $F_n(\ )(\omega)\ \varepsilon\ \mathcal{K}(r)$ *we accept* $\mathcal{K}_o^*$. Therefore $\mathcal{P}_o(r, \epsilon)$ satisfies (4.2) and the theorem is proved.

We now show that $\mathcal{P}_o(r, \epsilon)$ can be applied in a finite number of steps.

DEFINITION. Let $a_{j,m,\epsilon}$ and $b_{j,m,\epsilon}$ be any numbers for which

$$F_c(a_{j,m,\epsilon}) = j/n - m\epsilon, \qquad F_c(b_{j,m,\epsilon}) = r + (1 - m)\epsilon + (j - 1)/n, \qquad \text{for } m = 0, \cdots, r/\epsilon.$$

THEOREM 4.4. (4.6) *is satisfied by some* (μ, σ), $\sigma > 0$, *if and only if for some* $m = 0, \cdots, r/\epsilon$, *for all* $j = 1, \cdots, n$,

$$(X_{[j]}(\omega) - \mu)/\sigma > a_{j,m,\epsilon}, \qquad (X_{[j]}(\omega) - \mu)/\sigma < b_{j,m,\epsilon}. \tag{4.13}$$

The proof is obvious from the definition of $X_{[j]}(\omega)$ as that $X_i(\omega)$ which is jth in order of magnitude.

The significance of this theorem is the same as that of Theorem 3.1, namely that determining whether or not there is a (μ, σ), $\sigma > 0$, for which (4.13) can be satisfied, (for all j and at least one m), can be done geometrically. This can also be programmed on a digital computer, where it takes $r/\epsilon + 1$ times as much computation as in the d_1 case. Given today's high speed computers, we can keep ϵ small enough so as not to substantially increase the sample size, yet not so small as to make the computation too lengthy.

The question of the effects of various round-off errors arises here too. Suppose these errors introduce an error of ϵ', i.e., we do not really reject $\mathcal{H}_o^*$ for observed ω if $\mathcal{P}_o(r, \epsilon)(F_n[\,](\omega)) =$ "rej $\mathcal{H}_o^*$", but rather if $\mathcal{C}[\mathcal{P}_o(r, \epsilon)](F_n[\,](\omega)) =$ "rej $\mathcal{H}_o^*$", where $\mathcal{C}[\mathcal{P}_o(r, \epsilon)]$, the computed mapping is really a $\mathcal{P}_o(r - \epsilon', \epsilon + \epsilon')$, with ϵ' fixed. Then it is seen that in our computations we should use

$$\mathcal{C}[\mathcal{P}_o(r + \epsilon', \epsilon - \epsilon')],$$

where n and r satisfy (4.3) (4.4) but δ may have to be decreased sufficiently so that $h_{2,\alpha-\delta,n}$ is smaller, in order that $r + \epsilon'$ be a multiple of $\epsilon - \epsilon'$. Note that in this case the sample size is unchanged, but the amount of computation is greater. As usual, if $h_{2,\alpha,n} \leqq h_{2,\alpha}$ for all $\alpha \varepsilon (0, \frac{1}{2}]$ we may replace $h_{2,\alpha-\delta,n}$, $h_{2,\beta,n}$, $h_{2,\alpha,n}$ by $h_{2,\alpha-\delta}$, $h_{2,\beta,}$, $h_{2,\alpha}$ respectively and all of the theorems proved remain valid.

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