EXAMPLES BEARING ON THE DEFINITION OF FIDUCIAL PROBABILITY WITH A BIBLIOGRAPHY[1]

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1. Introduction and summary. The history of fiducial probability dates back over thirty years and so is long by statistical standards; however thirty years have not proved long enough for agreement to be reached among statisticians as to the derivation, manipulation or interpretation of fiducial probability. The reason for this lack of agreement and the resulting controversy is possibly due to the fact that the fiducial method has been put forward as a general logical principle, but yet has been illustrated mainly by means of particular examples rather than broad requirements. This paper explores in two respects certain natural general requirements for the application of the fiducial argument in the bivariate case that have been proposed.

An essential step in the process of fiducial inference is the derivation of the fiducial distribution. Properties that have been emphasized by Fisher in order that a genuine fiducial distribution may be obtained include sufficiency, the absence of any *a priori* information and the existence of a pivotal quantity. About ten years ago examples were discovered demonstrating that the distribution induced by a pivotal, sufficient and smoothly invertible set of quantities is not necessarily unique; that is to say the induced distribution depends on the particular set of pivotal quantities chosen (see [50]). The pivotal quantities used in these examples were denounced as having been "artificially constructed" (see [15]), and additional requirements were proposed. In the bivariate case requirements reduce to either

$$f(x, y \mid \alpha, \beta) = f(x, \alpha)f(y \mid x, \alpha, \beta),$$

or

$$f(x, y \mid \alpha, \beta) = f(x, \alpha, \beta)f(y \mid x, \beta),$$

$f(x, y \mid \alpha, \beta)$ being the given density. However as will be seen in this paper, neither of these requirements is sufficient to ensure the uniqueness of the resulting distribution.

It may be argued (see [11]) that non-uniqueness of the induced distribution is no handicap. It is, however, more than intuitively appealing to feel that if one starts with a problem perfectly symmetrical in the parameters and the random variables, as the examples presented are, then the answer should be perfectly symmetrical as well.

Recently Lindley proved a theorem in the one-dimensional case to the effect

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that a fiducial distribution is Bayes' posteriori if and only if the c.d.f. is invariant under a continuous one-parameter group of transformations. It is natural to seek an extension of this theorem to the multi-parameter case, for if it can be shown that there is a group of a certain form, then there exists an essentially unique pivotal, a frequency interpretation and certain consistency properties (see [21]).

It is known in the multi-parameter case that if the c.d.f. is invariant under a particular type of group, then the fiducial distribution is Bayes' posteriori. However the final example presented in this paper demonstrates that a bivariate fiducial distribution may be Bayes' posteriori without possessing strong group invariance properties.

2. The one-dimensional case. The simplest situation to which the fiducial argument may be applied is the case of a single parameter θ, and a single sufficient statistic T, for that parameter. There is little controversy over the actual fiducial distribution obtained in this case, probably because of the following reasons: (i) one can show that any pivotal quantity must be a function of the c.d.f. $F(T, \theta)$, and consequently the fiducial distribution is unique, and (ii) any $\alpha\%$ fiducial interval derived from the fiducial distribution is actually an $\alpha\%$ confidence interval in the sense that if

$$\Pr_f(\theta_{\alpha_1}(T) \leqq \theta \leqq \theta_{\alpha_2}(T)) = \alpha_1 - \alpha_2 = \alpha$$

then,

$$\Pr_\theta(\theta_{\alpha_1}(T) \leqq \theta \leqq \theta_{\alpha_2}(T)) = \alpha_1 - \alpha_2 = \alpha,$$

$\Pr_f$ denoting probabilities derived from the fiducial distribution and $\Pr_\theta$ probabilities derived from the distribution of the random variable T. (Both of these results have been in the air a long time and the reader should be able to verify them easily.) However there certainly is controversy over the use and interpretation of the fiducial distribution in this case.

The accepted formula for the fiducial distribution in the one-dimensional case is that given by Fisher on p. 70 [15], namely,

$$-(\partial F/\partial \theta)\, d\theta$$

where (i) $F(T, \theta)$ denotes the c.d.f. of a sufficient statistic T, (ii) θ and T vary continuously over the same range, and (iii) F is a monotonic function of θ and T, a decreasing function of θ. (These are Fisher's conditions.) The above formula will be taken as the definition of the one-dimensional fiducial distribution in the rest of this paper.

3. The multi-dimensional case. A variety of methods and conditions have been proposed in order to derive multivariate fiducial distributions. The first method seems to have been to obtain the distribution by means of pivotal quantities, see [42] for example. However it was noticed by several people that if one defines a fiducial distribution simply as one induced by a set of pivotal quantities, then non-uniqueness may result, see [50]. The existence of such examples inspired various authors to propose additional conditions that must be satisfied in order to obtain a unique fiducial distribution.

Fisher in [15] gave examples of the derivation of multivariate fiducial distributions. If x, y are random variables with bivariate density $f(x, y, \alpha, \beta)$; these examples, (pp. 159–162, 169–171 in [15]), seem to indicate that Fisher would require the density to have the property

$$f(x, y, \alpha, \beta) = f(x, \alpha)f(y \mid x, \alpha, \beta). \tag{1}$$

The fiducial distribution is then derived as follows: from $f(x, \alpha)$ find the fiducial density for α, a univariate problem. Now considering α fixed, find a fiducial density for β from $f(y \mid x, \alpha, \beta)$, a univariate problem once again. The required joint fiducial density is the product of the above two densities.

Quenouille (see [37]) in contrast to Fisher, requires that the density should factor as follows:

$$f(x, y, \alpha, \beta) = f(x, \alpha, \beta)f(y \mid x, \beta). \tag{2}$$

He justifies this factorization by means of the following sufficiency arguments: x is sufficient for α; therefore for fixed β, $F(x, \alpha, \beta)$ may be used to obtain the fiducial distribution of α given β. Now with x fixed, y is quasi-sufficient for β; therefore $F(y \mid x, \beta)$ may be used to obtain the fiducial distribution of β. Once again the required joint fiducial distribution is the product of the two individual fiducial distributions.

In the case of independence, factorizations (1) and (2) reduce to the same thing. This is what happens if one seeks to determine the joint fiducial distribution of μ and σ^2 in the normal case for example.

Unfortunately in the case of non-independence, neither factorization (1) nor factorization (2) is sufficient for a unique fiducial distribution to be obtained. Examples of non-uniqueness will be presented, and these examples will be such that, (i) the statistics involved are sufficient, (ii) the statistics and parameters have the same range, (iii) the pivotal quantities are monotonic in statistics and parameters, and (iv) all the fiducial distributions involved are proper distributions.

Let (x, y) have the following c.d.f.,

$$F(x, y) = (1 - e^{-\alpha x})(1 - e^{-\beta y})(1 - \gamma e^{-\phi x - \mu y}),$$

for $x, y, \alpha, \beta > 0$; ϕ a function of α, μ a function of β, γ a constant to be specified later.

Let us now proceed to derive the joint fiducial distribution of α and β. Firstly the marginal distributions of x and y are

$$F(x) = 1 - e^{-\alpha x} \quad \text{and} \quad F(y) = 1 - e^{-\beta y} \tag{3}$$

respectively. Consequently the factorization (1) obtains, and more importantly it obtains for both of the combinations of random variables and parameters

One may easily show that

$$F(y \mid x) = (1 - e^{-\beta y})(1 - \gamma e^{-\phi x - \mu y}\{1 - (\phi/\alpha)(e^{\alpha x} - 1)\}). \tag{4}$$

The marginal fiducial density of α is easily derived from (3). It is $xe^{-\alpha x}$. The fiducial density of β given x, y, α is easily derived being $\partial F(y \mid x)/\partial\beta$.

The required joint fiducial density is the product of the above two densities. It is

$$\begin{aligned} xye^{-\alpha x - \beta y}(1 - \gamma e^{-(\phi-\alpha)x - (\mu-\beta)y}\{e^{-\alpha x} &- (\phi/\alpha)(1 - e^{-\alpha x})\} \\ &\cdot\{e^{-\beta y} - (\partial\mu/\partial\beta)(1 - e^{-\beta y})\}) \end{aligned} \tag{5}$$

ϕ, μ, γ may now be chosen such that conditions (i) to (iv) are satisfied. Examples of such a choice are

$$\phi = \alpha(1 + e^{-\alpha}) \qquad \mu = \beta(1 + e^{-\beta}) \qquad |\gamma| \leqq \tfrac{1}{4}$$

However with this choice, (5) is not a symmetric function of (x, α), (y, β)' whereas the c.d.f. was. The fiducial density obtained consequently depends on the marginal that one starts with and we have a counter-example to the assumption of uniqueness when the factorization (1) obtains.

One can easily see that a sufficient condition in order for a fiducial distribution, obtained in this case, to be unique is that the c.d.f. be a function of the marginals alone; however this condition is not necessary.

An example demonstrating that factorization (2) is not sufficient to ensure uniqueness has already appeared in the literature, see [50]. It is the following one, and is due to L. J. Savage: $0 \leqq x, y, \alpha, \beta < \infty$ and (x, y) has density

$$f(x, y, \alpha, \beta) = [\alpha^2\beta^2/(\alpha + \beta)](x + y)e^{-(\alpha x + \beta y)}.$$

(2) is satisfied because

$$f(y \mid x) = [\beta^2 e^{-\beta y}/(1 + \beta x)](x + y),$$

and it is seen that this example satisfies the rest of the stated properties (i) to (iv) as well, but yet does not lead to a unique fiducial distribution.

4. Multivariate fiducial distributions and Bayes' theorem. In a recent paper [30], Lindley proved the following theorem concerning x, a one-dimensional random variable, whose distribution depends on a single parameter θ.

THEOREM. *The necessary and sufficient condition for the fiducial distribution of θ, given x, to be a Bayes' distribution is that there exist transformations of x to u and of θ to τ, such that τ is a location parameter for u.*

Let us look for a multivariate generalization of this theorem. A natural generalization of the transformations described in Lindley's Theorem to the 2-dimensional case is the following: there exist transformations x to $a(x, y, \mu, \nu)$, y to $b(x, y, \mu, \nu)$, α to $c(\alpha, \beta, \mu, \nu)$ and β to $d(\alpha, \beta, \mu, \nu)$, forming a group for arbitrary parameters μ, ν, and such that the c.d.f. satisfies $F(x, y, \alpha, \beta) = F(a, b, c, d)$. If the transformations are continuous, they are said to form a

2-parameter continuous transformation group or a 2-parameter Lie transformation group.

An example will now be presented of a bivariate fiducial distribution, derived from a density satisfying both factorization (1) and factorization (2), that is Bayesian posteriori, but yet that is not invariant under any two-parameter continuous transformation group.

Consider the following bivariate c.d.f.

$$H(x, y) = 1/(1 + e^{-x-\alpha})(1 + e^{-\phi(\alpha)(y+\beta)}),$$

where $-\infty < x, y, \alpha, \beta < \infty$ and $\phi > 0$ will be specified later. This example satisfies both factorization (1) and factorization (2), and the joint fiducial density of α and β is easily found to be

$$\phi(\alpha)e^{-(x+\alpha)-\phi(\alpha)(y+\beta)}H^2. \tag{6}$$

Providing $\phi(\alpha)$ is selected such that the above density is integrable in α and β, it is easily seen that (6) equals

$$h(x, y, \alpha, \beta)/\int h(x, y, \alpha, \beta)\, d\alpha\, d\beta,$$

i.e., the fiducial density is Bayes' posteriori with uniform prior density. An example of a function ϕ of the required type is $\phi(\alpha) = 2 + [\alpha/(1 + \alpha^2)^{\frac{1}{2}}]$.

To prove that $H(x, y)$ is not invariant under any two-parameter continuous transformation group the following theorem proved in [8] is required.

THEOREM. *The* c.d.f. $H(x, y, \alpha, \beta)$ *is invariant under a local two-parameter continuous transformation group, if and only if there exist linearly independent*

$$(\epsilon(x, y), \eta(x, y), a(\alpha, \beta), b(\alpha, \beta)), \qquad (\bar{\epsilon}(x, y), \bar{\eta}(x, y), \bar{a}(\alpha, \beta), \bar{b}(\alpha, \beta)),$$

each component being an analytic function, such that

$$\epsilon H_x + \eta H_y + aH_\alpha + bH_\beta = 0, \qquad \bar{\epsilon}H_x + \bar{\eta}H_y + \bar{a}H_\alpha + \bar{b}H_\beta = 0.$$

Applying this theorem to the particular case under consideration, $H(x, y)$ is invariant under some two-parameter group of transformations only if there exist $\epsilon(x, y)$, $\eta(x, y)$, $a(\alpha, \beta)$ and $b(\alpha, \beta)$ such that

$$(\epsilon + a)/(e^{x+\alpha} + 1) = -\{\eta + b + a(\phi'/\phi)[(y + \beta)\phi/(e^{\phi(y+\beta)} + 1)]\}.$$

Consider the implications of this relation by differentiating it with respect to x, then with respect to β twice. Combining the relations obtained in this fashion one sees first of all that $a_\beta = 0 = \eta_x$. Using these results one sees that $\epsilon = Ke^x + R$, $a = Ke^{-\alpha} - R$, K and R being constants. Substituting in the original relations one finds that $K = 0 = R$ implying $\eta + b = 0$, each being a constant. We have found therefore, $\epsilon = 0$, $\eta = C$, $a = 0$, $b = -C$ are the only functions compatible with the given relation. $H(x, y)$ is therefore invariant only under the one-parameter transformation group $y \to y + A$, $\beta \to \beta - A$.

The given example therefore provides a counter-example to the proposed extension of Lindley's Theorem.

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