EXACT LOWER MOMENTS OF ORDER STATISTICS IN SAMPLES FROM THE CHI-DISTRIBUTION (1 d.f.)[1]

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0. Introduction and summary. Numerous contributions have been made to the problem of order statistics in samples from normal and exponential populations. For the problem of location with symmetry Fraser [1] derived a locally most powerful rank test against normal alternatives. It is the Wilcoxon test statistic with the ranks replaced by the corresponding expected values of order statistics in a sample from the chi-distribution with one degree of freedom. Gupta [5] considered the order statistics from the standardized gamma distribution with the parameter r defined on the positive integers (that is, from the chi-distribution with even degrees of freedom) and derived expressions for the kth moments of an order statistic and the covariance between two order statistics. He also presented a table of numerical values of the kth moments of an order statistic accurate to six magnificant digits for $k = 1(1)N$, $N \leqq 15$ and $r = 1(1)5$, where N is the sample size. It might be of interest to consider the problem of order statistics in samples from chi-populations with odd degrees of freedom. However, this problem seems to be more difficult than the one considered by Gupta [5].

In the present paper, the expected values for samples to size four and the mixed and second moments (about the origin) for samples to size five, drawn from the chi-population (1 d.f.) have been evaluated. Numerical values of these to eight decimal places are computed. Section 2 contains general formulae and some definite integrals used in the computation. The results in Section 3 have theoretical interest in showing the relationships between moments of order statistics from chi (1 d.f.) and the standard normal distributions. In Section 4, there is a discussion about the number of integrals required to evaluate the first, second and mixed moments of order statistics for each N, given these moments to $N - 1$ and the existence of the tables for the normal distribution. There is also a discussion about the cumulative rounding error involved in using the formulae recurrently.

1. Notation. Let $X_{1,N} < X_{2,N} < \cdots < X_{N,N}$ be the order statistics in a random sample of size N from the chi-population (1 degree of freedom) having the probability density function (pdf)

$$(1.1)\qquad f(x) = \begin{cases} (2/\pi)^{\frac{1}{2}} e^{-\frac{1}{2}x^2} & x \geqq 0 \\ = 0 & x < 0, \end{cases}$$

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[1] Some of these results were found during the 1958 Summer Statistical Institute sponsored by the National Science Foundation.

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and cumulative distribution function (cdf) $F(x)$. Let $Z_{1,N} < Z_{2,N} < \cdots < Z_{N,N}$ be the order statistics in a random sample of size N drawn from the standard normal population with *pdf* and *cdf* φ and Φ respectively. Note that when $x \geqq 0$, $F(x) = 2\Phi(x) - 1$ and $f(x) = 2\varphi(x)$. Setting $\nu_{i,N}^{(k)} = E(X_{i,N}^k)$, we have the well known integral

$$\nu_{i,N}^{(k)} = \frac{N!}{(i-1)!(N-i)!} \cdot \int_0^\infty x^k F^{i-1}(x)[1 - F(x)]^{N-i}\, dF(x), \quad i = 1, \cdots, N, k = 1, 2, \cdots. \tag{1.2}$$

For simplicity, we shall write $\nu_{i,N} = \nu_{i,N}^{(1)}$. Similarly, setting $\nu_{i,j,N} = E(X_{i,N}X_{j,N})$, we have

$$\nu_{i,j,N} = \frac{N!}{(i-1)!(j-i-1)!(N-j)!} \iint_{0<x<y<\infty} xyF^{i-1}(x) \cdot [F(y) - F(x)]^{j-i-1}[1 - F(y)]^{N-j}\, dF(x)\, dF(y); \quad 1 \leqq i < j \leqq N. \tag{1.3}$$

Consequently, $\nu_{i,N}^{(2)} = \nu_{i,i,N}$. Also set $\mu_{i,N}^{(k)} = E(Z_{i,N}^k)$ and $\mu_{i,j,N} = E(Z_{i,N}Z_{j,N})$; $1 \leqq i \leqq j \leqq N$, $k = 1, 2, \cdots$. As before, we write $\mu_{i,N} = \mu_{i,N}^{(1)}$ and $\mu_{i,N}^{(2)} = \mu_{i,i,N}$.

2. Basic formulae. Following is a collection of formulae that are useful in the computation of the desired expected values. Formulae (2.1)–(2.6), (2.10) and (2.11) are true for an arbitrary distribution, for which the corresponding integrals converge. Formulae (2.1)–(2.5) can be derived by writing every term on the left side of each formula as an integral and summing underneath the integral sign. Formula (2.6) follows by considering the variance of $(X_{1,N} + \cdots + X_{N,N})/N$.

$$i\nu_{i+1,N}^{(k)} + (N - i)\nu_{i,N}^{(k)} = N\nu_{i,N-1}^{(k)}; \quad i = 1, 2, \cdots, N-1, \quad k = 1, 2, \cdots. \tag{2.1}$$

$$(i-1)\nu_{i,j,N} + (j-i)\nu_{i-1,j,N} + (N-j+1)\nu_{i-1,j-1,N} = N\nu_{i-1,j-1,N-1}, \qquad 1 < i \leqq j \leqq N. \tag{2.2}$$

$$\sum_{i=1}^N \nu_{i,N} = N\, E(X). \tag{2.3}$$

$$\sum_{i=1}^N \nu_{i,N}^{(2)} = N\, E(X^2). \tag{2.4}$$

$$\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \nu_{i,j,N} = \tfrac{1}{2}N(N-1)\,[E(X)]^2. \tag{2.5}$$

$$\sum_{i=1}^N \sum_{j=1}^N \operatorname{cov}\,(X_{i,N}\,; X_{j,N}) = N\, E[X - E(X)]^2. \tag{2.6}$$

Some definite integrals which have been used in evaluating the moments of chi order statistics will be given below. In evaluating these integrals, integration by parts has been employed wherever possible. The limits of integration are $(0, \infty)$ unless otherwise specified. f and F respectively denote $f(x)$ and $F(x)$.

$$\int_b^a xf\,dx = f(b) - f(a), \qquad a > b > 0. \tag{2.7}$$

$$\int f^2\,dx = \pi^{-\frac{1}{2}}. \tag{2.8}$$

$$\int f^3\,dx = 2/(3^{\frac{1}{2}}\pi). \tag{2.9}$$

$$\int F^n f^m\,df = -n(m+1)^{-1} \int F^{n-1} f^{m+2}\,dx, \tag{2.10}$$

for all positive integral m and n.

$$\int F^n\,dF = (n+1)^{-1}. \tag{2.11}$$

$$\int Ff^2\,dx = 2\pi^{-3/2} \text{ Arc tan } 2^{-\frac{1}{2}}. \tag{2.12}$$

$$\int Ff^3\,dx = 2/(3^{3/2}\pi). \tag{2.13}$$

$$\int F^2 f^2\,dx = 2\pi^{-3/2} \text{ Arc tan } 8^{-\frac{1}{2}} = \pi^{-\frac{1}{2}}(1 - 4\pi^{-1} \text{ Arc tan } 2^{-\frac{1}{2}}), \tag{2.14}$$

since $\text{Arc tan } 8^{-\frac{1}{2}} + 2 \text{ Arc tan } 2^{-\frac{1}{2}} = \pi/2$.

$$\int F^2 f^3\,dx = [4/(3^{\frac{1}{2}}\pi^2)] \text{ Arc tan } 15^{-\frac{1}{2}}. \tag{2.15}$$

$$\int f^2(y) \left[\int_0^y f^2(x)\,dx\right] dy = (2\pi)^{-1}. \tag{2.16}$$

$$\int f^2(y) F(y) \left[\int_0^y f^2(x)\,dx\right] dy = 2\pi^{-2} \text{ Arc tan } 5^{-\frac{1}{2}}. \tag{2.17}$$

When an integral contains a power of F, the integral can be thought of as a multiple integral by using $F(u) = \int_0^u f(t)\,dt$. (2.12) and (2.13) have been evaluated using polar coordinate transformation and (2.14) to (2.17) have been evaluated using the transformation $u = \rho$, $v = \rho x$, $w = \rho y$ and well-known integrals.

Using (2.2) recurrently, one can generate the $\nu_{i,N}^{(k)}$ $(i = 1, 2, \cdots, N)$ if the $\nu_{i,N-1}^{(k)}$ $(i = 1, 2, \cdots, N-1)$ and any one of the $\nu_{i,N}^{(k)}$ are available. Similarly, using (2.3) recurrently, one can generate the $\nu_{i,j,N}$ $(i < j,\ i, j = 1, 2, \cdots, N)$ if the $\nu_{i,j,N-1}$ $(i < j,\ i, j = 1, 2, \cdots, N-1)$ and any $N - 1$ of the $\nu_{i,j,N}$ are available. Formulae (2.3) to (2.5) can be used for checking the computations.

3. Certain relationships. In the formulae of this section we find some relationships among the moments of order statistics in samples drawn from the chi-population (1 d.f.). One can also find some relationships between the moments of order statistics from the chi (1 d.f.) and the standard normal distribution. These formulae can be used for checking numerical values from existing tables, for computing some in terms of others and for obtaining some values from existing values for the normal distribution. Formulae (3.3), (3.4), (3.5) and (3.6) will be used for the discussion in Section 4. All the formulae of this section will be listed below; the proofs will be given later.

When N is even,[3]

$$\nu_{N,N} = \sum_{i=0}^{N-2} (-1)^i 2^{N-1-i} \binom{N}{i} \mu_{N-i,N-i}. \tag{3.1}$$

If $N + k$ is odd,[4]

$$\nu_{N,N}^{(k)} = (\tfrac{1}{2}) \sum_{i=0}^{N-1} (-1)^i 2^{N-i} \binom{N}{i} \mu_{N-i,N-i}^{(k)}. \tag{3.1.1}$$

$$\nu_{1,N} = \sum_{i=1}^{N} (-1)^{i-1} \binom{N}{i} \nu_{i,i}. \tag{3.2}$$

$$\nu_{N,N}^{(2)} = 1 + \nu_{N-1,N,N}. \tag{3.3}$$

$$\nu_{1,N}^{(2)} = 1 + \nu_{1,2,N} - N(2/\pi)^{\frac{1}{2}}\nu_{1,N-1}. \tag{3.4}$$

When N is even,

$$\nu_{1,N,N} = \sum_{i=1}^{(N-2)/2} (-1)^{i-1} \binom{N}{i} \nu_{i,i}\, \nu_{N-i,N-i} + (-1)^{(N-2)/2} \left(\frac{1}{2}\right)\binom{N}{N/2} \nu_{N/2,N/2}^2. \tag{3.5}$$

When N is odd,

$$\nu_{N-1,N,N} = \sum_{i=0}^{N-2} (-1)^i\, 2^{N-1-i} \binom{N}{i} \mu_{N-i-1,N-i,N-i}. \tag{3.6}$$

$$\nu_{1,2,N} = \sum_{i=2}^{N} (-1)^{i-1} \binom{N}{i} \nu_{i-1,i,i} + N(2/\pi)^{\frac{1}{2}} \sum_{j=1}^{N-1} (-1)^{j-1} \binom{N}{j} (N-j)\nu_{j,j}. \tag{3.7}$$

When N is even,

$$2\nu_{1,N,N} = 2^N \mu_{1,N,N} + \sum_{i=1}^{N-1} \binom{N}{i} (-1)^{i+1} \nu_{1,N-i} \left[\sum_{j=1}^{i} \binom{i}{j} \nu_{j,j}\right]. \tag{3.8}$$

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[3], [4] Formulae (3.1) and (3.1.1) were found by Professor Milton Sobel at the 1958 Summer Statistical Institute, sponsored by the National Science Foundation.

(3.9)
$$\sum_{j=2}^{N} \nu_{1,j,N} = 1 - \nu_{1,N}^{(2)} .$$

(3.10)
$$\sum_{j=1}^{N-1} \nu_{j,N,N} = N(2/\pi)^{\frac{1}{2}}\nu_{N-1,N-1} - \nu_{N,N}^{(2)} + 1.$$

(3.11)
$$\sum_{j=i+1}^{N} \nu_{i,j,N} - \sum_{j=1}^{N} \nu_{i-1,j,N} = 1 - \nu_{i,N}^{(2)}, \qquad 1 < i \leqq N - 1.$$

For all positive integral N

(3.12)
$$\begin{aligned} \nu_{i,j,N} = {} & 2^N \sum_{m=0}^{i-1} (-1)^m 2^{-m} \binom{N}{m} \mu_{i-m,j-m,N-m} \\ & + (-1)^{i+1} \binom{N}{i} \sum_{m=1}^{j-i} i(j-m)^{-1} \binom{N-i}{j-i-m} \nu_{j-m,j-m}\nu_{m,N-j+m} \\ & + (-1)^i \sum_{m=1}^{i} \sum_{n=1}^{N-j+1} (-1)^{m-1} \binom{N}{j+n-1} \\ & \cdot \binom{j+n-1}{i-m} \nu_{n,j-i+n,j-i+m+n-1} . \end{aligned}$$

Proof of (3.1). Consider the expression

(A)
$$2^N N \int_0^\infty x\varphi(x)[\Phi(x) - \tfrac{1}{2}]^{N-1}\, dx.$$

If we let $2\Phi(x) - 1 = F(x)$, so that $2\varphi(x) = f(x)$ then (A) becomes

$$N \textstyle\int_0^\infty xf(x)[F(x)]^{N-1}\, dx = \nu_{N,N} .$$

On the other hand, if N is even, the integrand in (A) is an even function and we can write (A) as

$$\begin{aligned} & N2^{N-1} \int_{-\infty}^{+\infty} x\varphi(x)[\Phi(x) - \tfrac{1}{2}]^{N-1}\, dx \\ & \qquad = N2^{N-1} \sum_{i=0}^{N-1} (-1)^i \binom{N-1}{i} 2^{-i} \int_{-\infty}^{+\infty} x\varphi(x)\Phi^{N-1-i}(x)\, dx, \end{aligned}$$

which simplifies to the form given in (3.1).

Proof of (3.1.1). If $N + k$ is odd, then symbolically $\nu_{N,N}^{(k)} = (\frac{1}{2})(2\mu_{1,1}^{(k)} - 1)^N$, if powers of $\mu_{1,1}^{(k)}$ say $(\mu_{1,1}^{(k)})^i$ are replaced by $\mu_{i,i}^{(k)}$ for $i \geqq 1$; for $i = 0$, we define $\mu_{0,0}^{(k)}$ as zero.

Proof of 3.2. Writing $\nu_{1,N}$ as an integral, expanding $[1 - F(x)]^{N-1}$ as a binomial series and integrating termwise, one gets the desired result. Note that (3.2) is true for an arbitrary $F(x)$.

Proof of (3.3). $\nu_{N-1,N,N} = N(N-1) \int\int_{0<x<y<\infty} xyf(x)f(y)[F(x)]^{N-2}\, dx\, dy$. Integrating with respect to y one gets

$$\nu_{N-1,N,N} = N(N-1) \textstyle\int_0^\infty xf^2(x)[F(x)]^{N-2}\, dx.$$

Writing $f(x)[F(x)]^{N-2} = d/dx[F^{N-1}(x)/(N-1)]$ and integrating by parts we get the desired result.

Proof of (3.4). Use the method of (3.3).

Proof of (3.5).

$$\nu_{1,N,N} = N(N-1) \iint_{0<x<y<\infty} xyf(x)f(y)[F(y) - F(x)]^{N-2}\, dx\, dy.$$

Since N is even,

$$\nu_{1,N,N} = [N(N-1)/2] \int_0^\infty\int_0^\infty xyf(x)f(y)[F(y) - F(x)]^{N-2}\, dx\, dy.$$

Now, expanding $[F(y) - F(x)]^{N-2}$ and integrating termwise, we get the result.

Proof of (3.6).

$$\nu_{N-1,N,N} = N(N-1) \int_0^\infty xf^2(x)F^{N-2}(x)\, dx \qquad \text{(See the proof of 3.3)}$$

$$= 2^N N(N-1) \int_0^\infty x\varphi^2(x)[\Phi(x) - \tfrac{1}{2}]^{N-2}\, dx$$

$$= 2^{N-1} N(N-1) \int_{-\infty}^{+\infty} x\varphi^2(x)[\Phi(x) - \tfrac{1}{2}]^{N-2}\, dx,$$

when N is odd. Now expanding $[\Phi(x) - \frac{1}{2}]^{N-2}$ and integrating term by term, one gets the result.

Proof of (3.7). $\nu_{1,2,N} = N(N-1) \iint_{0<y<x<\infty} xyf(x)f(y)[1 - F(x)]^{N-2} dx\, dy$. Integrating with respect to y one gets

$$\nu_{1,2,N} = -N(N-1) \int_0^\infty xf^2(x)[1 - F(x)]^{N-2}\, dx$$

$$+ N(N-1)(2/\pi)^{\frac{1}{2}} \int_0^\infty xf(x)[1 - F(x)]^{N-2}\, dx.$$

The result follows after expanding $[1 - F(x)]^{N-2}$ and integrating termwise.

Proof of (3.8).

$$\nu_{1,N,N} = 2^N N(N-1) \iint_{0<x<y<\infty} xy\varphi(x)\varphi(y)[\Phi(y) - \Phi(x)]^{N-2}\, dx\, dy.$$

The integrand is symmetrical with respect to origin and in x and y. Consider

$$\mu_{1,N,N} = N(N-1) \iint_{-\infty<x<y<\infty} xy\varphi(x)\varphi(y)[\Phi(y) - \Phi(x)]^{N-2}\, dx\, dy$$

$$= 2^{-(N-1)}\nu_{1,N,N} + N(N-1) \int_{x=-\infty}^0 \int_{y=0}^\infty xy\varphi(x)\varphi(y)[\Phi(y) - \Phi(x)]^{N-2}\, dx\, dy.$$

Expanding $[\Phi(y) - \Phi(x)]^{N-2}$ and changing x to $-z$, one gets

$$\mu_{1,N,N} = 2^{-(N-1)}\nu_{1,N,N} + \sum_{i=0}^{N-2} (-1)^{i+1} N(N-1)\binom{N-2}{i} \cdot \left[\int_0^\infty z\varphi(z)\{1 - \Phi(z)\}^{N-2-i}\,dz\right]\left[\int_0^\infty y\varphi(y)\Phi^i(y)\,dy\right]$$

$$= 2^{-(N-1)}\nu_{1,N,N} + \sum_{i=0}^{N-2} (-1)^{i+1}\binom{N-2}{i} N(N-1)2^{-N} \cdot \left[\int_0^\infty zf(z)\{1 - F(z)\}^{N-2-i}\,dz\right]\left[\int_0^\infty yf(y)\{1 + F(y)\}^i\,dy\right].$$

Now, expanding $[1 \pm F)^\alpha$ in powers of F and integrating termwise we obtain the result.

Proof of (3.9).

$$\text{L.H.S.} = \sum_{j=2}^{N} \frac{N!}{(j-2)!(N-j)!} \cdot \iint_{0<x<y<\infty} xyf(x)f(y)[F(y) - F(x)]^{j-2}[1 - F(y)]^{N-j}\,dx\,dy.$$

Taking the summation underneath the integral sign, we get

$$\text{L.H.S.} = N(N-1) \iint_{0<x<y<\infty} xyf(x)f(y)[1 - F(x)]^{N-2}\,dx\,dy$$

$$= N(N-1) \int_0^\infty xf^2(x)[1 - F(x)]^{N-2}\,dx.$$

Now the result follows after one integration by parts.

Proof of (3.10). Following the same method as in (3.9), we get

$$\text{L.H.S.} = N(N-1) \iint_{0<x<y<\infty} xyf(x)f(y)[F(y)]^{N-2}\,dx\,dy.$$

$$\nu_{N-1,N,N} + \text{L.H.S. of (3.10)} = N(N-1)\int_0^\infty \int_0^\infty xyf(x)f(y)[F(x)]^{N-2}\,dx\,dy$$

$$= N(2/\pi)^{\frac{1}{2}}\nu_{N-1,N-1}.$$

The result follows after using (3.3).

Proof of (3.11). Following the method of (3.9), we have

$$\sum_{j=i+1}^{N} \nu_{i,j,N}$$

$$= \frac{N(N-1)\cdots(N-i)}{(i-1)!} \iint_{0<x<y<\infty} xyf(x)f(y)F^{i-1}(x)[1 - F(x)]^{N-i-1}\,dx\,dy$$

$$= \frac{N!}{(N-i-1)!(i-1)!}\int_0^\infty xf^2(x)F^{i-1}(x)[1 - F(x)]^{N-i-1}\,dx.$$

Write $(N - i)f(x)[1 - F(x)]^{N-i-1} = -d[1 - F(x)]^{N-i}$ and integrate by parts. The result follows.

Proof for 3.12. $\nu_{i,j,N}$ is given by the integral

$$\nu_{i,j,N} = C_{i,j,N}\, 2^{N-i+1} \cdot \iint_{0<x<y<\infty} xy[2\Phi(x) - 1]^{i-1}[\Phi(y) - \Phi(x)]^{j-i-1}[1 - \Phi(y)]^{N-j}\varphi(x)\varphi(y)\, dx\, dy$$

where $C_{i,j,N} = N![(i-1)!(j-i-1)!(N-j)!]^{-1}$.

$$\nu_{i,j,N} = C_{i,j,N}\, 2^{N-i+1}\left[\iint_{-\infty<x<y<\infty} - \int_{x=-\infty}^{0}\int_{y=0}^{\infty} - \iint_{-\infty<x<y<0}\right].$$

Writing $[2\Phi(x) - 1]^{i-1} = \sum_{m=0}^{i-1}(-1)^m\binom{N}{m}[2\Phi(x)]^{i-1-m}$ and integrating termwise one obtains

$$2^{N-i+1}C_{i,j,N}\iint_{-\infty<x<y<\infty} = 2^N\sum_{m=0}^{i-1}(-1)^m 2^{-m}\binom{N}{m}\mu_{i-m,j-m,N-m}\,.$$

$$C_{i,j,N}\, 2^{N-i+1}\int_{x=-\infty}^{0}\int_{y=0}^{\infty} = C_{i,j,N}\, 2^{N-i+1}(-1)^i\sum_{m=0}^{j-i-1}\binom{j-i-1}{m} \cdot\left[\int_0^\infty x(2\Phi - 1)^{i-1}(\Phi - \tfrac{1}{2})^{j-i-1-m}\varphi(x)\, dx\right] \cdot\left[\int_0^\infty y(\Phi - \tfrac{1}{2})^m(1 - \Phi)^{N-j}\varphi(y)\, dy\right].$$

Express the integrands in terms of F's and f's, and integrate. Change $m + 1$ to m and obtain

$$C_{i,j,N}\, 2^{N-i+1}\int_{x=-\infty}^{0}\int_{y=0}^{\infty} = (-1)^i\binom{N}{i}\sum_{m=1}^{j-i} i(j-m)^{-1} \cdot\binom{N-i}{j-i-m}\nu_{j-m,j-m}\,\nu_{m,N-j+m}\,.$$

In the third integral put $x = -x$, $y = -y$, use and relationships $\Phi(-x) = 1 - \Phi(x)$ and $F(x) = 2\Phi(x) - 1$ and obtain

$$C_{i,j,N}\, 2^{N-i+1}\iint_{-\infty<x<y<0}$$
$$= (-1)^{i-1}C_{i,j,N}\iint_{0<y<x<\infty} xyF^{i-1}(x)[F(x) - F(y)]^{j-i-1}[1 + F(y)]^{N-j}f(x)f(y)\, dx\, dy.$$

Now, write

$$F^{i-1}(x) = \sum_{m=0}^{i-1}\binom{i-1}{m}(-1)^m[1 - F(x)]^m,$$

$$[1 - F(y)]^{N-j} = \sum_{n=0}^{N-j}\binom{N-j}{n}[F(y)]^n,$$

integrate termwise and obtain

$$C_{i,j,N}\, 2^{N-i+1} \iint_{-\infty<x<y<0}$$

$$= (-1)^{i-1} \sum_{m=1}^{i} \sum_{n=1}^{N-j+1} (-1)^{m-1} \binom{N}{j+n-1} \binom{j+n-1}{i-m} \nu_{n,j-i+n,j-i+m+n-1} \, .$$

If we combine the three preceding results, (3.12) readily follows. Formula (3.12) expresses the relationship between $\nu_{i,j,N}$ and $\nu_{N-j+1,N-i+1,N}$ given a table of values of the μ's up to N and the ν's up to $N - 1$.

4. Minimum number of integrals to be evaluated. It will be useful, especially when we are interested in exact lower moments, to find out the number of integrals required in order to evaluate the first, second and mixed moments of order statistics in a sample of size N, given these moments in samples to size $N - 1$ and a table of these moments of normal order statistics to size N. Hence the following theorem.

THEOREM 4.1. *In order to obtain the first, second and the mixed moments of order statistics in a sample of size N, one has to evaluate at most one single integral (second moment) and* $(N - 4)/2$ *double integrals when N is even, for example, evaluate* $\nu_{1,N}^{(2)}$ *and* $\nu_{1,j,N}$, *for* $j = 3, 4, \cdots, N/2$; *and one single integral (first moment) and* $(N - 3)/2$ *double integrals when N is odd, for example, evaluate* $\nu_{1,N}$ *and* $\nu_{1,j,N}$ *for* $j = 3, 4, \cdots, (N + 1)/2$.

PROOF. If N is even, $\nu_{N,N}$ can be computed from the normal tables (see (3.1)) and by use of (2.1) the rest of $\nu_{i,N}$ can be found. Compute $\nu_{1,N}^{(2)}$ and find the rest of $\nu_{i,N}^{(2)}$ by use of (2.1). Since (3.12) expresses $\nu_{i,j,N}(i < j)$ in terms of $\nu_{N-j+1,N-i+1,N}$ and the μ's up to size N and the ν's up to $N - 1$, it is enough if we consider just the $\nu_{i,j,N}(i < j)$ that lie in the upper wedge shaped portion of the matrix $(\nu_{i,j,N})$ (See Figure 1). It is easily seen that the number of $\nu_{i,j,N}(i < j)$ that lie in the aforesaid portion of the matrix $(\nu_{i,j,N})$ is $N^2/4$ and that (2.2) gives $N(N - 2)/4$ independent constraints among the $\nu_{i,j,N}(i < j)$ lying in the upper wedge shaped portion of the matrix $(\nu_{i,j,N})$. Moments $\nu_{1,2,N}$ and $\nu_{1,N,N}$ are respectively known from (3.4) and (3.5). Hence the number of $\nu_{i,j,N}(i < j)$ to be evaluated is

$$[N^2/4] - [N(N - 2)/4] - 1 - 1 = (N - 4)/2.$$

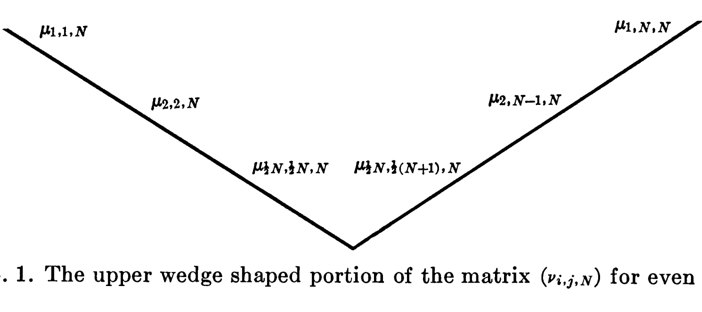

FIG. 1. The upper wedge shaped portion of the matrix $(\nu_{i,j,N})$ for even N.

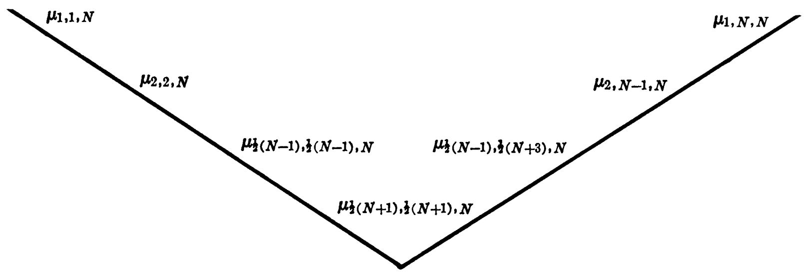

FIG. 2. The upper wedge shaped portion of the matrix $(\nu_{i,j,N})$ for odd N.

Thus, it is sufficient to evaluate $\nu_{1,j,N}$ for $j = 3, 4, \cdots, N/2$. In order to find the rest of $\nu_{i,j,N}$ use (2.2) with $i = 1, j = 2, 3, \cdots, N - 1$; $i = 2, j = 2, 3, \cdots, N - 2$; etc. until the total number of these relationships is $N(N - 2)/4$. While applying (2.2) with $i = 1$ and any j less than N, write $i = N - j + 1$ and $j = N$. Solve for $\nu_{N-j+1,N,N}$, $j = 2, 3, \cdots, (N - 2)/2$ using (3.12).

When N is odd, evaluate one $\nu_{i,N}$; for example, evaluate $\nu_{1,N}$. By use of (2.1), solve for the rest of $\nu_{i,N}$. Also, from (3.1.1) we can obtain $\nu_{N,N}^{(2)}$ in terms of second moments of normal order statistics. Using (2.1), one can solve for the rest of $\nu_{i,N}^{(2)}$. As remarked earlier, since (3.12) relates $\nu_{i,j,N}$ and $\nu_{N-j+1,N-i+1,N}(i < j)$, we consider only those $\nu_{i,j,N}(i < j)$ that lie in the upper wedge shaped portion of the matrix $(\nu_{i,j,N})$ (See Figure 2). It is clear that the number of $\nu_{i,j,N}(i < j)$ lying in the aforesaid portion of the matrix $(\nu_{i,j,N})$ is $(N^2 - 1)/4$ and that the number of independent constraints among these $\nu_{i,j,N}(i < j)$ lying in the upper wedge shaped portion of the matrix is $(N - 1)^2/4$. Moreover, (3.4) gives $\nu_{1,2,N}$. Thus the number of $\nu_{i,j,N}$ $(i < j)$ to be evaluated $= [(N^2 - 1)/4] - [(N - 1)^2/4] - 1 = (N - 3)/2$. Hence, it is sufficient to evaluate $\nu_{1,j,N}$ for $j = 3, 4, \cdots, (N + 1)/2$. Solve for $\nu_{N-j+1,N,N}$, $j = 2, 3, \cdots, (N - 3)/2$ using (3.12). By use of (2.2) as stated in the case N is even, until the number of relationships is $(N - 1)^2/4$, the rest of $\nu_{i,j,N}$ can be solved for.

REMARK 4.1. It will be easier to find $\nu_{N-1,N,N}$ from (3.3) rather than from (3.12) with known $\nu_{1,2,N}$. Further, it might be convenient to write the constraints given by (2.2) with $i = 2, j = 3, 4, \cdots, N$; $i = 3, j = 3, 4, \cdots, N$, etc. until the total number of these is $(N - 1)(N - 2)/2$ and use (3.12) to obtain the rest of the constraints needed.

REMARK 4.2. Formulae (2.3), (2.4) and (2.5) do not constitute constraints independent of those given by (2.1) and (2.2). Formulae (2.3) and (2.4) can be obtained by summing (2.1) on $i = 1, 2, \cdots, N - 1$, with $k = 1$ and 2 respectively. Similarly, (2.5) can be obtained by summing (2.2) on $i = 1, 2, \cdots, N - 1$ and $j = i + 1, i + 2, \cdots, N$.

Let us demonstrate how one obtains the moments of order statistics in samples of sizes three and four. Assume that the table of these moments for $N = 2$ and the table of these for the standard normal distribution are available.

$N = 3$: Evaluate $\nu_{1,3}$. Solve for $\nu_{2,3}$ and $\nu_{3,3}$ from the following equations obtained from (2.1).

TABLE I

Low moments of order statistics from the chi-distribution (1 d.f.)*

Statistic	N = 2	N = 3	N = 4	N = 5
$X_{1,N}$	$\nu_{1,2} = 2(2^{1/2} - 1)\pi^{-1/2}$ 0.46738996	$\nu_{1,3} = 3(2^{1/2} - 2 + 4\alpha/\pi)\pi^{-1/2}$ 0.33490293	$\nu_{1,4} = 4(2^{1/2} - 6 + 24\alpha/\pi)\pi^{-1/2}$ 0.26208228	where $\nu_{1,1} = (2/\pi)^{1/2} = 0.79788456$ $\nu_{1,1}^{(2)} = 1$ $\text{Var}(X_{1,1}) = 0.36338023$ $\alpha = \text{Arc tan } 2^{-1/2} = 0.61547971$ $\beta = \text{Arc tan } 5^{-1/2} = 0.42053434$ and $\delta = \text{Arc tan } 15^{-1/2} = 0.25268025$
$X_{2,N}$	$\nu_{2,2} = 2\pi^{-1/2}$ 1.12837917	$\nu_{2,3} = 6(\pi - 4\alpha)\pi^{-3/2}$ 0.73236399	$\nu_{2,4} = 48(\pi - 5\alpha)\pi^{-3/2}$ 0.55336491	
$X_{3,N}$		$\nu_{3,3} = 12\alpha\pi^{-3/2}$ 1.32638676	$\nu_{3,4} = 12(16\alpha - 3\pi)\pi^{-3/2}$ 0.91136308	
$X_{4,N}$			$\nu_{4,4} = 12(\pi - 4\alpha)\pi^{-3/2}$ 1.46472798	
$X_{1,N}^2$	$\nu_{1,2}^{(2)} = 1 - 2\pi^{-1}$ 0.36338023 Var = 0.14492686	$\nu_{1,3}^{(2)} = 1 - 2(3 - 3^{1/2})\pi^{-1}$ 0.19279847 Var = 0.08063850	$\nu_{1,4}^{(2)} = 1 - 4(3 - 3^{-1/2}4)\pi^{-1}$ 0.12070214 Var = 0.05201502	$\nu_{1,5}^{(2)} = 1 - 20(3 - 3^{1/2} - 3^{1/2}6\delta\pi^{-1})(3\pi)^{-1}$ 0.08307731
$X_{2,N}^2$	$\nu_{2,2}^{(2)} = 1 + 2\pi^{-1}$ 1.63661977 Var = 0.36338023	$\nu_{2,3}^{(2)} = 1 - 2(3^{1/2}2 - 3)\pi^{-1}$ 0.70454374 Var = 0.16818672	$\nu_{2,4}^{(2)} = 1 - 4(3^{1/2}2 - 3)\pi^{-1}$ 0.40908747 Var = 0.10287475	$\nu_{2,5}^{(2)} = 1 - 20(3^{1/2}8\delta - \pi)\pi^{-2}$ 0.27120146
$X_{3,N}^2$		$\nu_{3,3}^{(2)} = 1 + 3^{1/2}2\pi^{-1}$ 2.10265779 Var = 0.34335597	$\nu_{3,4}^{(2)} = 1$ Var = 0.16941735	$\nu_{3,5}^{(2)} = 1 - 3^{1/2}20(\pi - 12\delta)\pi^{-2}$ 0.61591648
$X_{4,N}^2$			$\nu_{4,4}^{(2)} = 1 + 3^{-1/2}8\pi^{-1}$ 2.47021039 Var = 0.32478233	$\nu_{4,5}^{(2)} = 1 + 3^{-1/2}40(\pi - 12\delta)\pi^{-2}$ 1.25605568
$X_{5,N}^2$				$\nu_{5,5}^{(2)} = 1 + 3^{1/2}40\delta\pi^{-2}$ 2.77374906
$X_{1,N}X_{2,N}$	$\nu_{1,2,2} = 2/\pi$ 0.63661977 Cov = 0.10922668	$\nu_{1,2,3} = 2(3 + 3^{1/2} - 2^{1/2}3)\pi^{-1}$ 0.31156816 Cov = 0.06629731	$\nu_{1,2,4} = 12(1 - 2^{1/2}2 + 3^{-3/2}4 + 2^{1/2}4\alpha\pi^{-1})\pi^{-1}$ 0.18955768 Cov = 0.04453054	$\nu_{1,2,5} = 20(1 + 3^{-1/2} - 2^{1/2}6 + 2^{1/2}24\alpha\pi^{-1} + 3^{1/2}2\delta\pi^{-1})\pi^{-1}$ 0.12863436

$X_{1,N}X_{3,N}$		$\nu_{1,3,3} = 2(2^{\frac{1}{2}}3 - 3^{\frac{1}{2}}2)\pi^{-1}$ 0.49563337 Cov = 0.05142255	$\nu_{1,3,4} = 4(3 + 2^{\frac{1}{2}}6 - 3^{-\frac{1}{2}}8 - 2^{\frac{1}{2}}24\alpha\pi^{-1})\pi^{-1}$ 0.27624476 Cov = 0.03739265	$\nu_{1,3,5} = 20[3 + 2^{\frac{1}{2}}12 - 3^{-\frac{1}{2}}2 - \pi^{-1}(2^{\frac{1}{2}}60\alpha + 12\beta + 3^{\frac{1}{2}}4\delta)]\pi^{-1}$ 0.18148084
$X_{2,N}X_{3,N}$		$\nu_{2,3,3} = 3^{\frac{1}{2}}2\pi^{-1}$ 1.10265779 Cov = 0.13125989	$\nu_{2,3,4} = 4(3^{\frac{1}{2}}2 - 3)\pi^{-1}$ 0.59091253 Cov = 0.08659619	$\nu_{2,3,5} = 20[(3^{\frac{1}{2}} - 3)\pi + 12\beta - 3^{\frac{1}{2}}2\delta]\pi^{-2}$ 0.38040446
$X_{1,N}X_{4,N}$			$\nu_{1,4,4} = 12(2^{\frac{1}{2}}4\alpha - \pi)\pi^{-2}$ 0.41349542 Cov = 0.02961618	$\nu_{1,4,5} = 60[4(2^{\frac{1}{2}}4\alpha - \alpha + 3\beta) - (1 + 2^{\frac{1}{2}}3)\pi]\pi^{-2}$ 0.24905402
$X_{2,N}X_{4,N}$			$\nu_{2,4,4} = 4(3 - 3^{-\frac{1}{2}}4)\pi^{-1}$ 0.87929786 Cov = 0.06876880	$\nu_{2,4,5} = 20(3\pi - 3^{-\frac{1}{2}}4\pi + 24\alpha - 48\beta + 3^{\frac{1}{2}}8\delta)\pi^{-2}$ 0.52015408
$X_{3,N}X_{4,N}$			$\nu_{3,4,4} = 3^{-\frac{1}{2}}8\pi^{-1}$ 1.47021039 Cov = 0.13531139	$\nu_{3,4,5} = 40(3^{-\frac{1}{2}}\pi - 6\alpha + 6\beta - 3^{\frac{1}{2}}\delta)\pi^{-2}$ 0.83679982
$X_{1,N}X_{5,N}$				$\nu_{1,5,5} = 120[2^{-\frac{1}{2}}\pi - 2(2^{\frac{1}{2}} - 1)\alpha - 4\beta]\pi^{-2}$ 0.35775347
$X_{2,N}X_{5,N}$				$\nu_{2,5,5} = 240(3\beta - 2\alpha)\pi^{-2}$ 0.74516270
$X_{3,N}X_{5,N}$				$\nu_{3,5,5} = 80(3\alpha - 3\beta - 3^{\frac{1}{2}}\delta)\pi^{-2}$ 1.19300492
$X_{4,N}X_{5,N}$				$\nu_{4,5,5} = 3^{\frac{1}{2}}40\delta\pi^{-2}$ 1.77374907

* The Arc tangents have been computed using tables of Arc tan x and the interpolation formula [2] given by

$$\text{Arc tan } x = \text{Arc tan } x_0 + ph(1 + xx_0)^{-1} - 3^{-1}(ph)^3(1 + xx_0)^{-3},$$

where $x = x_0 + ph$.

$$\nu_{2,3} + 2\nu_{1,3} = 3\nu_{1,2} \quad \text{and} \quad 2\nu_{3,3} + \nu_{2,3} = 3\nu_{2,2}\,.$$

From (3.1.1), we can compute $\nu_{3,3}^{(2)}$ which is given by the equation $\nu_{3,3}^{(2)} = 4\mu_{3,3}^{(2)} - 6\mu_{(2)}^{2,2} + 6$. Using (2.1) with $k = 2$, we can solve for $\nu_{1,3}^{(2)}$ and $\nu_{2,3}^{(2)}$. Also, (3.3) and (3.4) respectively give $\nu_{2,3,3} = \nu_{3,3}^{(2)} - 1$ and $\nu_{1,2,3} = \nu_{1,3}^{(2)} + 3(2/\pi)^{\frac{1}{2}}\nu_{1,2} - 1$. Formula (2.2) with $i = 2, j = 3$, and $N = 3$ gives $\nu_{2,3,3} + \nu_{1,3,3} + \nu_{1,2,3} = 3\nu_{1,2,3}$, from which we can solve for $\nu_{1,3,3}$.

$N = 4$: From (3.1) we have $\nu_{4,4} = 4(2\mu_{4,4} - 4\mu_{3,3} + 3\mu_{2,2})$. We can solve for $\nu_{1,4}$, $\nu_{2,4}$ and $\nu_{3,4}$ from the following equations obtained from (2.1). $\nu_{2,4} + 3\nu_{1,4} = 4\nu_{1,3}$, $2\nu_{3,4} + 2\nu_{2,4} = 4\nu_{2,3}$ and $3\nu_{4,4} + \nu_{3,4} = 4\nu_{3,3}$. Now evaluate $\nu_{1,4}^{(2)}$ and solve for $\nu_{2,4}^{(2)}$, $\nu_{3,4}^{(2)}$ and $\nu_{4,4}^{(2)}$, using (2.1) with $k = 2$. Also,

(3.3) gives $\nu_{3,4,4} = \nu_{4,4}^{(2)} - 1$,
(3.4) gives $\nu_{1,2,4} = \nu_{1,4}^{(2)} + 4(2/\pi)^{\frac{1}{2}}\nu_{1,3} - 1$,
(3.5) gives $\nu_{1,4,4} = 4(2/\pi)^{\frac{1}{2}}\nu_{3,3} - 3\nu_{2,2}^{2}$.

Formula (2.2) with

$i = 2, j = 3$ gives $\nu_{2,3,4} + \nu_{1,3,4} + 2\nu_{1,2,4} = 4\nu_{1,2,3}$,
$i = 2, j = 4$ gives $\nu_{2,4,4} + 2\nu_{1,4,4} + \nu_{1,3,4} = 4\nu_{1,3,3}$,
$i = 3, j = 4$ gives $2\nu_{3,4,4} + \nu_{2,4,4} + \nu_{2,3,4} = 4\nu_{2,3,3}$,

from which we can solve for $\nu_{1,3,4}$, $\nu_{2,3,4}$ and $\nu_{2,4,4}$.

The above discussion is pertinent especially for the evaluation of exact moments of order statistics for each N. However, it will be difficult, if not impossible, to evaluate these exactly except for small values of N. For the numerical evaluation of normal order statistics, it has been pointed out by Pearson (see [6], p. 152) and Srikantan [7] that there will be a steady decrease in accuracy due to the repeated application of the recurrence formulae (2.1) and (2.2) for working "upwards." The same can be said in the case of the chi-order statistics because of the relationship of the chi and normal density functions. However, as noted in [6] and [7], by writing the formulae as

$$\nu_{i,N-1} = N^{-1}[i\nu_{i+1,N} + (N - i)\nu_{i,N}], \quad i = 1, 2, \cdots, N - 1$$

and

$$\nu_{i-1,j-1,N-1} = N^{-1}[(i - 1)\nu_{i,j,N} + (j - i)\nu_{i-1,j,N} + (N - j + 1)\nu_{i-1,j-1,N}],$$
$$1 < i \leqq j \leqq N - 1,$$

TABLE II

Expected values of the largest order statistic for even N, computed using (3.1) and Teichroew's Tables [8]

N	$\nu_{N,N}$	N	$\nu_{N,N}$
4	1.46472798	14	2.02252221
6	1.65399631	16	2.07705370
8	1.78336710	18	2.12406668
10	1.88071558	20	2.17252108
12	1.95829680		

they can be used for working "downwards" with no serious accumulation of rounding error.

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