# ON THE ORDER AND THE TYPE OF ENTIRE CHARACTERISTIC FUNCTIONS[1]

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**1. Introduction and summary.** In the theory of analytic characteristic functions (c.f.'s), it is well-known that:

(1) the order of an entire c.f. cannot be less than unity, unless the function is identically equal to one (P. Lévy); and

(2) a necessary and sufficient condition (NASC) for a distribution function (d.f.) $F(x) \neq \epsilon(x)$ to be a "finite" d.f. is that its c.f. be an entire function of order one and of exponential type (G. Pólya). (Throughout this paper, $F(x)$ will invariably denote a d.f., and $f(t)$ the corresponding c.f.)

These two results lead us naturally to the investigation of conditions under which a d.f. will have an entire c.f. (i) of order one and of maximal type, or (ii) of given finite order greater than one, or (iii) of infinite order. *Sufficient* conditions for (ii) were obtained for absolutely continuous d.f.'s by D. Dugué [2], and for general d.f.'s by E. Lukacs (see [4], p. 142).

The scope of the present paper extends beyond the problems (i), (ii) and (iii) above. In Section 6 below, we obtain NASC's for a d.f. to have an entire c.f. of given finite order greater than one. Section 7 deals with NASC's for a d.f. to have an entire c.f. of given finite order greater than one *and* given type (maximal, intermediate, or minimal). In Section 8, we consider entire c.f.'s of order one: we first obtain NASC's for a d.f. to have an entire c.f. of order one and maximal type; next, we obtain a relation between the extremities of a "finite" d.f. $\neq \epsilon(x)$, and the type of the corresponding entire c.f. (which is of order one); and finally, we obtain as a corollary the fact that there cannot exist an entire c.f. of order one and of minimal type. These constitute the main results of this paper.

In addition to the above, we obtain in Section 3 a theorem on the moments of a d.f., and in Section 4 a theorem on the interval of existence of the integral defining the moment-generating function of a d.f., these theorems being closely related in form and in content to the main results mentioned above. Allied results on analytic c.f.'s which are not entire, and on entire c.f.'s of infinite order are given in Section 9.

**2. Certain pre-requisite definitions and results.** We collect in this Section certain definitions and known theorems which we will require.

2.1. *Order and Type of Entire Functions.* We assume familiarity with the concepts of analytic functions and entire functions of a complex variable, and with

Received January 15, 1962; revised April 18, 1962.

[1] This paper is based on the author's doctoral dissertation (1961) submitted to The Catholic University of America, Washington, D. C. It was supported by the National Science Foundation under grant NSF-G-9968.

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the maximum modulus principle for analytic functions. (See, for instance, [6], pp. 98, 295 and 163 respectively. We shall treat "regular" and "analytic" as synonymous phrases.)

Let $f(z)$ be an entire function. For every $r > 0$, $|f(z)|$ has a maximum for $|z| \leqq r$, which is attained *on* the circle $|z| = r$, by the maximum modulus principle. We shall denote this maximum by $M(r, f)$.

The *order* $\rho$ of $f(z)$ is then given by

$$\rho = \limsup_{r \to +\infty} [\ln \ln M(r, f)/\ln r]. \tag{2.1.1}$$

If $\rho > 0$ and is finite, then the *type* $\tau$ of $f(z)$ is given by

$$\tau = \limsup_{r \to +\infty} [\ln M(r, f)/r^{\rho}], \tag{2.1.2}$$

$f(z)$ is said to be of *maximal*, *intermediate*, or *minimal* type according as $\tau = +\infty$, $0 < \tau < +\infty$, or $\tau = 0$. It is said to be of *exponential type* if it is either of order one and of finite (that is, not maximal) type or of order less than one; in other words, there exists some constant $A > 0$ such that for all sufficiently large $r$

$$M(r, f) \leqq \exp(Ar). \tag{2.1.3}$$

The following theorem relates the order and the type of an entire function to the coefficients in its power-series expansion (cf. [6], p. 326):

THEOREM 2.1.1. *Let the series* $\sum_{n=0}^{+\infty} c_n z^n$ *represent an entire function of the complex variable* $z$ (*so that the radius of convergence of the series, given by* $\liminf_{n \to +\infty} |c_n|^{-(1/n)}$, *is infinite*). *Then the order* $\rho$ *and the type* $\tau$ (*in case* $\rho$ *is finite and positive*) *of the entire function are given successively by*

$$\begin{aligned} \rho &= \limsup_{n \to +\infty} [n \ln (n)/-\ln |c_n|], \\ \tau e \rho &= \limsup_{n \to +\infty} n|c_n|^{\rho/n}. \end{aligned} \tag{2.1.4}$$

2.2. *Analytic characteristic functions.* A c.f. $f(t)$ is said to be an *analytic* c.f. if it coincides in some (real) neighborhood of $t = 0$ with an analytic function of the complex variable $z = t + iv$. We cite below certain basic and well-known results on analytic c.f.'s, which will be required in the sequel. Proofs of these results can be found, for instance, in [4], Chapter 7.

THEOREM 2.2.1 (*P. Lévy-D. A. Raikov*). *If* $f(t)$, *the* c.f. *of* $F(x)$, *coincides in a* (*real*) *neighborhood of* $t = 0$ *with an analytic function of the complex variable* $z = t + iv$, *then the latter is analytic in a "horizontal" strip of the form* $-\alpha < \operatorname{Im}(z) < \beta (\alpha > 0, \beta > 0)$, *and admits the representation*

$$f(z) = \int_{-\infty}^{+\infty} \exp(izx)\, dF(x) \tag{2.2.1}$$

*in that strip. It is possible, depending on* $F$, *for* $\alpha$ *or* $\beta$ *or both to be infinite. In case both are infinite, we say that* $F$ *has an entire* c.f.

It is known further that the points $-i\alpha$ (if $\alpha$ is finite) and $i\beta$ (if $\beta$ is finite) are singularities for $f(z)$.

THEOREM 2.2.2. *If* $F(x)$ *has an entire* c.f., *then, for every* $r > 0$,

$$M(r, f) = \text{Max}\,[f(ir), f(-ir)] \tag{2.2.2}$$

*where* $M(r, f)$ *is as defined in Section* 2.1.

THEOREM 2.2.3. *If* $F(x)$ *has an entire* c.f., *then, for every* $r > 0$ *and every* $x \geqq 0$, *we have*

$$M(r, f) \geqq \tfrac{1}{2}[1 - F(x) + F(-x)]\cdot e^{rx}. \tag{2.2.3}$$

THEOREM 2.2.4 (*P. Lévy*). *If* $F(x) \not\equiv \epsilon(x)$ *has an entire* c.f., *then the order of the* c.f. *is not less than unity.* (Here, $\epsilon(x)$ denotes the degenerate d.f. having the origin as its discontinuity-point.)

We say that a d.f. $F(x)$ is *finite* (or *bounded on both sides*) if there exist real numbers $a$ and $b$ such that $F(a) = 0$ and $F(b) = 1$. We define the left and right extremities of such a d.f. in an obvious manner:

$$\text{lext}\, F = \sup\,\{a \mid F(a) = 0\}; \qquad \text{rext}\, F = \inf\,\{b \mid F(b) = 1\}.$$

The following is a well-known result on finite d.f.'s due to G. Pólya [5] (see also [4], p. 141):

THEOREM 2.2.5. *The* c.f. *of any "finite"* d.f. $F(x) \not\equiv \epsilon(x)$ *is an entire function of order one and of exponential type. Conversely, every entire* c.f. $f(z)$ *of order one and of exponential type corresponds to some "finite"* d.f. $F(x)$, *whose extremities are given by the formulas*

$$\begin{aligned} \text{lext}\, F &= -\limsup_{y\to+\infty}\,[\ln f(iy)/y], \\ \text{rext}\, F &= \limsup_{y\to+\infty}\,[\ln f(-iy)/y]. \end{aligned} \tag{2.2.4}$$

**3. A theorem on the moments of a d.f.** For the purposes of this and the following Sections, it is convenient to introduce the following notation:

$F(x)$ invariably denotes a d.f., as already indicated in Section 1,

$$\begin{aligned} T(x) &= 1 - F(x) + F(-x), \\ g(x) &= \{\ln \ln [1/T(x)]\}/\ln x, \\ h(x) &= \{-\ln T(x)\}/x^{1+\alpha}. \end{aligned} \tag{3.1}$$

The following result is closely related to a result of H. Cramér's giving a sufficient condition for the existence of the moment of a specified order (see [1], p. 71).

THEOREM 3.1. *Let* $F(x)$ *be an arbitrary* d.f. *which is not "finite", so that* $T(x) > 0$ *for every* $x > 0$. *Let*

$$\delta = \liminf_{x\to+\infty}\,[-\ln T(x)/\ln x]. \tag{3.2}$$

*Then* $\int_{-\infty}^{+\infty} |x|^k\, dF(x)$ (i) *is defined for all* $k$ *in* $[0, \delta)$, *and* (ii) *does not exist for any* $k > \delta$. (In general, nothing can be said about the existence of the integral for $k = \delta$.)

We remark that if $F(x)$ has a "finite" d.f., then, of course, absolute moments of all orders exist.

PROOF.

(i) It is clear that $\delta \geqq 0$. If $\delta = 0$, there is nothing to prove. Hence we need only consider the case $\delta > 0$. Let then $\delta > 0$, and let $k$ be any number in $[0, \delta)$. We choose any number $r$ such that

$$k < r < \delta. \tag{3.3}$$

By the definition of $\delta$, there exists $X = X(r) > 0$ such that for all $x \geqq X$, $-\ln T(x) \geqq r \ln x$, that is, $T(x) \leqq 1/x^r$ so that, for all $x \geqq X$, we have

$$1 - F(x) \leqq 1/x^r \quad \text{and} \quad F(-x) \leqq 1/x^r. \tag{3.4}$$

On integrating by parts, and using (3.3) and the first relation in (3.4), it is easy to see that

$$\begin{aligned}\int_X^{+\infty} x^k\, dF(x) &= \lim_{A\to+\infty} -\int_X^A x^k\, d[1 - F(x)] \\ &= X^k\,[1 - F(X)] + k\int_X^{+\infty} x^{k-1}\,[1 - F(x)]\, dx \\ &\leqq X^k\,[1 - F(X)] + k\int_X^{+\infty} x^{k-r-1}\, dx < +\infty.\end{aligned}$$

Similarly, using the second relation in (3.4) together with (3.3), we prove that $\int_{-\infty}^{-X} |x|^k\, dF(x) < +\infty$. Obviously, $\int_{-X}^{X} |x|^k\, dF(x) \leqq X^k$. The above three inequalities together yield (i).

(ii) Let now $k$ be such that $\int_{-\infty}^{+\infty} |x|^k\, dF(x) = C < +\infty$. Then, for any $x > 0$,

$$x^k[1 - F(x)] \leqq \int_x^{+\infty} y^k\, dF(y) \leqq C,$$

and

$$x^k F(-x) \leqq \int_{-\infty}^{-x} |\, y\,|^k\, dF(y) \leqq C.$$

Hence, for any $x > 0$,

$$x^k T(x) \leqq 2C$$

whence it follows that

$$\liminf_{x\to+\infty} [-\ln T(x)/\ln x] \geqq k.$$

Thus (ii) is also proved.

**4. An existence theorem for the integral defining the moment-generating function of a d.f.** Let $F(x)$ be an arbitrary d.f. Consider the function $M(t)$ of the real variable $t$ defined by the relation $M(t) = \int_{-\infty}^{+\infty} e^{tx}\, dF(x)$. In general, $M(t)$ may not be defined for any real value of $t$ other than $t = 0$. The following theorem concerns the existence of $M(t)$ and shows that there is always an existence-interval of values of $t$ associated with it, which, depending on $F$, may also degenerate into the single point $t = 0$, or consist of the entire real line. (If $M(t)$ exists in some $t$-interval containing the origin as an interior point, then we call it the *moment-generating function* of $F(x)$.)

THEOREM 4.1. *Let* $F(x)$ *be an arbitrary* d.f. *Then,* (i) *the integral* $M(t)$ *is defined for all points* $t$ *in the open interval* $(-\beta, \alpha)$, *where*

$$\beta = \liminf_{x \to +\infty} [-\ln F(-x)/x]$$

*and*

$$\alpha = \liminf_{x \to +\infty} [-\ln [1 - F(x)]/x],$$

*and where it is understood that* $\beta$ *(respectively* $\alpha$*) is to be taken as infinite if* $F(-x) = 0$ *(respectively if* $1 - F(x) = 0$*) for some* $x > 0$; *and* (ii) *the integral does not exist for* $t < -\beta$ *(if* $\beta$ *is finite) or for* $t > \alpha$ *(if* $\alpha$ *is finite)*. (In general, nothing can be said about the existence of the integral $M(t)$ for $t = -\beta$ or for $t = \alpha$.)

PROOF.

(i) We first note that $\alpha \geqq 0$, $\beta \geqq 0$. Consider the $t$-interval $[0, \alpha)$. If $\alpha = 0$, then, $M(0)$ being defined, there is nothing to prove. Hence we need only consider the case $\alpha > 0$, and show that $M(t)$ is defined for all $t$ in $[0, \alpha)$.

Let $0 < t < \alpha$. Choose any $r$ such that $t < r < \alpha$. By the definition of $\alpha$, we can find $X = X(r) > 0$ such that for all $x \geqq X$,

$$\{-\ln [1 - F(x)]/x\} \geqq r,$$

that is, for all $x \geqq X$,

$$1 - F(x) \leqq \exp(-rx). \tag{4.1}$$

On integrating by parts, and using (4.1), we obtain

$$\int_X^{+\infty} e^{tx}\, dF(x) = -\lim_{A \to +\infty} \int_X^A e^{tx}\, d[1 - F(x)]$$

$$\leqq e^{tX}[1 - F(X)] + t \int_X^{+\infty} e^{(t-r)x}\, dx,$$

from (4.1),

$$\leqq e^{tX}[1 - F(X) + \frac{t}{r - t}, \tag{4.2}$$

since $t <$ r. Also, obviously,

$$\int_{-\infty}^X e^{tx}\, dF(x) \leqq e^{tX} F(X). \tag{4.3}$$

From (4.2) and (4.3), we see that $M(t)$ exists for $0 < t < \alpha$.

We then consider the $t$-interval $(-\beta, 0]$. If $\beta = 0$, then, $M(0)$ being defined, there is nothing to prove. Hence we need only consider the case $\beta > 0$. We proceed exactly as before, *mutatis mutandis*, and show that for $0 < t < \beta$, $\int_{-\infty}^{+\infty} e^{-tx}\, dF(x) < +\infty$.

It is obvious that $M(t)$ is defined for all $t > 0$ if $1 - F(x) = 0$ for some $x > 0$, and for all $t < 0$ if $F(-x) = 0$ for some $x > 0$. These facts motivate, and agree with, the definitions of $\alpha$ and $\beta$ given above for these two cases respectively. Thus (i) is proved.

(ii) Let, for some $t > 0$, $\int_{-\infty}^{+\infty} e^{ty}\, dF(y) = C < +\infty$. Then, for any $x > 0$, we have $e^{tx}[1 - F(x)] \leqq \int_{x}^{+\infty} e^{ty}\, dF(y) \leqq C$, whence we derive at once that

$$\liminf_{x \to +\infty} \{-\ln [1 - F(x)]/x\} \geqq t, \qquad i.\ e., t \leqq \alpha.$$

Similarly, if $\int_{-\infty}^{+\infty} e^{-ty}\, dF(y) < +\infty$ for some $t > 0$, then $t \leqq \beta$. Thus, (ii) is proved.

From Theorem 4.1, we derive the following consequences.

COROLLARY 1. *A* NASC *for $F(x)$ to have an analytic* c.f. *is that* $\alpha > 0$, $\beta > 0$.

PROOF. If $\alpha > 0$, $\beta > 0$, then $\int_{-\infty}^{+\infty} \exp(izx)\, dF(x)$ is defined and analytic in the open neighborhood $-\alpha < \operatorname{Im}(z) < \beta$ of the origin, and coincides with the c.f. of $F(x)$ on the real line. Hence, by definition, $F(x)$ has an analytic c.f.

Conversely, if $F(x)$ has an analytic c.f., say $f(z)$, then it is known from the proof of the Lévy-Raikov Theorem (Theorem 2.2.1; see, for instance, [4], p. 132) that $f(z)$ admits the representation

$$f(z) = \int_{-\infty}^{+\infty} \exp(izx)\, dF(x) \tag{4.4}$$

in a strip of the form $-\gamma < \operatorname{Im}(z) < \delta$ $(\gamma > 0, \delta > 0)$, which is also the strip of convergence of the integral (that is, the integral converges for $-\gamma < \operatorname{Im}(z) < \delta$, but not for $\operatorname{Im}(z) < -\gamma$ or for $\operatorname{Im}(z) > \delta$). Since $-\beta < t < \alpha$ is the interval of convergence of $M(t) = f(-it)$, it follows that we must have $\gamma = \alpha$, $\delta = \beta$.

It is also known (see, for instance, [4], p. 132) that $-i\gamma$, $i\delta$ are singularities for the function (4.4) above, so that $-i\alpha$, $i\beta$ are singularities for $f(z)$.

The above considerations further yield the following

COROLLARY 2. *A* NASC *for $F(x)$ to have an entire* c.f. *is that* $\alpha = \beta = +\infty$.

**5. Two basic lemmas.** In this Section, we establish two lemmas which are essential for the proofs of the main results in the sections that follow.

LEMMA 5.1. *The integral*

$$I(z) = \int_0^{+\infty} \exp(izx - kx^{1+\alpha})\, dx, \qquad \alpha > 0,$$

*defines an entire function of the complex variable z, of order* $1 + (1/\alpha)$, *and of type* $(c/k)^{(1/\alpha)}$, *where*

$$c = \alpha^{\alpha}/(1 + \alpha)^{1+\alpha}. \tag{5.1}$$

PROOF. Let $z$ be an arbitrary, but fixed, complex number. Since, for $x \geqq 0$,

$$\exp(izx - kx^{1+\alpha}) = \sum_{n=0}^{+\infty} f_n(x),$$

where $f_n(x) = (1/n!)(iz)^n x^n \exp(-kx^{1+\alpha})$, and the partial sums of the series on the right are dominated by the function $\exp(|z|\, x - kx^{1+\alpha})$ which is integrable over $[0, +\infty)$, we have, by the Dominated Convergence Theorem of Lebesgue's (see, for instance, [1], p. 47), that

$$I(z) = \sum_{n=0}^{+\infty} \int_0^{+\infty} f_n(x)\, dx.$$

But,

$$\int_0^{+\infty} f_n(x)\, dx = (1/n!)\, (iz)^n \int_0^{+\infty} x^n \exp(-kx^{1+\alpha})\, dx.$$

Making the substitution $kx^{1+\alpha} = y$, and setting $\delta = 1/(1 + \alpha)$, we have (with the usual notation as to the Gamma-function)

$$\int_0^{+\infty} f_n(x)\, dx = (1/n!)\, (iz)^n \frac{\delta\Gamma\{\delta(n+1)\}}{k^{\delta(n+1)}} = c_n z^n, \quad \text{say,}$$

so that $I(z) = \sum_{n=0}^{+\infty} c_n z^n$. We now invoke Stirling's approximation to the Gamma-function, namely, that, as $x \to +\infty$,

$$\Gamma(x) \sim (2\pi)^{\frac{1}{2}} e^{-x} x^{x-\frac{1}{2}},$$

where the symbol $\sim$ denotes that the ratio of the two members of the above relation tends to unity as $x \to +\infty$ (see, for instance, [6], pp. 420–421). By applying this to $\Gamma\{\delta(n+1)\}$ and to $n! = \Gamma(n+1)$, we have that, as $n \to +\infty$,

$$|c_n| \sim \delta^{\frac{1}{2}}\{[e/(n+1)]^{1-\delta}(\delta/k)^{\delta}\}^{(n+1)}. \tag{5.2}$$

It follows from (5.2) that the radius of convergence of the power-series $\sum_{n=0}^{+\infty} c_n z^n$—given by $\liminf_{n\to+\infty} |c_n|^{-(1/n)}$—is infinite, and so the series represents an entire function of $z$. (That $I(z)$ represents an entire function of $z$ is also obvious from straightforward considerations. We utilize the power-series expansion for $I(z)$ primarily in order to determine the order and the type of the entire function.)

From (5.2), it follows that $-\ln |c_n| \sim (1 - \delta)n \ln (n)$, so that, by relations (2.1.4) of Theorem 2.1.1, the entire function $I(z)$ is of order $\rho = 1 + (1/\alpha)$.

Again, from (5.2), it follows that $n\, |c_n|^{\rho/n} \sim n[e/(n+1)]^{\rho(1-\delta)}(\delta/k)^{\rho\delta}$. Noting that $\rho(1-\delta) = 1$ and that $\rho\delta = \rho - 1 = 1/\alpha$, we then have $n\, |c_n|^{\rho/n} \sim e(\delta/k)^{(1/\alpha)}$. By relations (2.1.4) of Theorem 2.1.1, it then follows that the type $\tau$ of $I(z)$ is $(c/k)^{(1/\alpha)}$, where $c$ is given by (5.1). This completes the proof of the lemma.

We recall Definitions 3.1 and 5.1 before proceeding to

LEMMA 5.2. *If for all $x \geqq X(>0)$ we have*

$$T(x) \leqq \exp(-kx^{1+\alpha}), \qquad \alpha > 0, \tag{5.3}$$

*then* $F(x)$ *has an entire* c.f. $f(z)$ *which is either of order* $1 + (1/\alpha)$ *and of type* $\leqq \tau$, *or of order less than* $1 + (1/\alpha)$, *where* $\tau = (c/k)^{(1/\alpha)}$.

PROOF: Let $A > X$, and $r > 0$; on integration by parts,

$$\int_X^A e^{rx}\, dF(x) = -\int_X^A e^{rx}\, d[1 - F(x)]$$

$$= [-e^{rx}\{1 - F(x)\}]_X^A + r\int_X^A [1 - F(x)]\, e^{rx}\, dx$$

$$= e^{rX}[1 - F(X)] - e^{rA}[1 - F(A)] + r\int_X^A [1 - F(x)]\, e^{rx}\, dx,$$

whence, using (5.3), and letting $A \to +\infty$, we have

$$\int_X^{+\infty} e^{rx}\, dF(x) = e^{rX}[1 - F(X)] + r\int_X^{+\infty} [1 - F(x)]\, e^{rx}\, dx$$

$$\leqq e^{rX}[1 - F(X)] + r\int_X^{+\infty} \exp(rx - kx^{1+\alpha})\, dx$$

again by virtue of (5.3),

$$\leqq e^{rX}[1 - F(X)] + r\int_0^{+\infty} \exp(rx - kx^{1+\alpha})\, dx.$$

Also, obviously, $\int_{-\infty}^X e^{rx}\, dF(x) \leqq e^{rX}F(X)$, so that

$$\int_{-\infty}^{+\infty} e^{rx}\, dF(x) = \int_{-\infty}^{X} e^{rx}\, dF(x) + \int_X^{+\infty} e^{rx}\, dF(x)$$
$$\leqq e^{rX} + r\int_0^{+\infty} \exp(rx - kx^{1+\alpha})\, dx \tag{5.4}$$

Exactly in the same way, we derive, using (5.3), that

$$\int_{-\infty}^{+\infty} e^{-rx}\, dF(x) = \int_{-\infty}^{-X} e^{-rx}\, dF(x) + \int_{-X}^{+\infty} e^{-rx}\, dF(x)$$
$$\leqq r\int_0^{+\infty} \exp(rx - kx^{1+\alpha})\, dx + e^{rX}. \tag{5.5}$$

Hence, from Theorem 2.2.2 and relations (5.4) and (5.5) above, it follows that $F(x)$ has an entire c.f. $f(z)$ such that

$$M(r, f) = \operatorname{Max}[f(ir), f(-ir)]$$

$$\leqq r\int_0^{+\infty} \exp(rx - kx^{1+\alpha})\, dx + e^{rX}.$$

The statement of Lemma 5.2 is an immediate consequence of the above inequality and Lemma 5.1.

**6. NASC's for an entire c.f. of given finite order greater than one.** We first recall the notations (3.1).

THEOREM 6.1. *A* NASC *for* $F(x)$ *to have an entire* c.f. *of order* $1 + (1/\alpha)$ $(\alpha > 0)$ *is that*

(i) $$T(x) > 0 \quad \textit{for every} \quad x > 0;$$

*and*

(ii) $$\liminf_{x\to+\infty} g(x) = 1 + \alpha.$$

PROOF. The necessity of (i) is clear from Theorem 2.2.5. $g(x)$ is therefore defined for all sufficiently large $x$. We turn to the necessity of (ii). In view of our assumption as to the order of $f(z)$, corresponding to any given $\epsilon > 0$, there exists an $R = R(\epsilon)$ such that for all $r \geqq R$

$$M(r, f) \leqq \exp r^{1+(1/\alpha)+\epsilon}. \tag{6.1}$$

Hence, from (2.2.3), we have that for any $x \geqq 0$, $r \geqq R$,

$$T(x) \leqq 2 \exp [-rx + r^{1+(1/\alpha)+\epsilon}].$$

Choosing $x \geqq X = 2R^{(1/\alpha)+\epsilon}$, and defining $r = (\frac{1}{2}x)^{\alpha/(1+\alpha\epsilon)}$—so that $r \geqq R$, we have, for any $x \geqq X = X(\epsilon)$, $T(x) \leqq 2 \exp [-(\frac{1}{2}x)^{1+\alpha-\epsilon'}]$, where $\epsilon' = \alpha^2\epsilon/(1 + \alpha\epsilon)$. It follows that

$$\liminf_{x\to+\infty} g(x) \geqq 1 + \alpha - \epsilon'. \tag{6.2}$$

Since $\epsilon(>0)$ and consequently $\epsilon'$ are arbitrary, we have

$$\liminf_{x\to+\infty} g(x) \geqq 1 + \alpha. \tag{6.3}$$

We proceed to show that the inequality sign cannot hold here. Suppose indeed that it does. Then we can find a $\gamma > \alpha$ and an $X = X(\gamma)$ such that, for all $x \geqq X, g(x) \geqq 1 + \gamma$. And this implies that, for all $x \geqq X$, $T(x) \leqq \exp(-x^{1+\gamma})$. By Lemma 6.2, it then follows that $F(x)$ has an entire c.f. of order $\leqq 1 + (1/\gamma) < 1 + (1/\alpha)$. This contradiction to our assumption establishes the necessity of condition (ii).

We proceed to the sufficiency part of the proof. Condition (i) implies not only that the c.f. of $F(x)$ is not an entire function of order one and of exponential type—by virtue of Theorem 2.2.5—but also that $g(x)$ is defined for all sufficiently large $x$. We turn to condition (ii).

If $\liminf_{x\to+\infty} g(x) = 1 + \alpha$, then, corresponding to any $\epsilon > 0$, we can find $X = X(\epsilon)$ such that, for all $x \geqq X$, $g(x) \geqq 1 + \alpha - \epsilon$. This implies that for all such $x, T(x) \leqq \exp(-x^{1+\alpha-\epsilon})$. Hence, by Lemma 5.2, $F(x)$ has an entire c.f. $f(z)$ of order $\leqq 1 + [1/(\alpha - \epsilon)]$. Since $\epsilon > 0$ is arbitrary, it follows that the order of $f(z)$ is $\leqq 1 + (1/\alpha)$. We proceed to show that the inequality sign cannot hold here. Suppose indeed that it does. Then we can find a $\gamma > \alpha$, and an $R = R(\gamma)$, such that, for all $r \geqq R$, $M(r, f) \leqq \exp(r^{1+(1/\gamma)})$. This being of the same form as (6.1) leads to a relation analogous to (6.2), namely,

$$\liminf_{x\to+\infty} g(x) \geqq 1 + \gamma > 1 + \alpha.$$

This contradiction establishes the sufficiency of the conditions given.

**7. NASC's for entire c.f.'s of given finite order greater than unity and given type.** The concept of "type" is a useful refinement to the notion of "order" in measuring the "growth" of an entire function. In this section, we establish results which exhaust all possible "types" for c.f.'s of given finite order greater than unity.

THEOREM 7.1. *A NASC for $F(x)$ to have an entire* c.f. *of order* $1 + (1/\alpha)(\alpha > 0)$ *and of (finite) type* $\tau > 0$ *is that*

(i) $$T(x) > 0 \quad \textit{for every} \quad x > 0;$$

*and*

(ii) $$\liminf_{x\to+\infty} h(x) = c/\tau^{\alpha},$$

*where $h(x)$ and $c$ are defined respectively by* (3.1) *and* (5.1). *We notice that $c$ is a function of $\alpha$.*

PROOF. Let $F(x)$ be assumed to have an entire c.f. of order $1 + (1/\alpha)$ and of finite type $\tau > 0$. The necessity of (i) follows from Theorems 2.2.5 and 6.1; $h(x)$ is therefore defined for all $x > 0$. We turn to the necessity of (ii); under the above assumption, corresponding to any $\epsilon > 0$, we can find $R = R(\epsilon)$ such that, for all $r \geqq R$,

(7.1) $$M(r, f) \leqq \exp\,[(\tau + \epsilon)r^{1+(1/\alpha)}].$$

Hence, by (2.2.3), we have that for any $x \geqq 0$, $r \geqq R$,

$$T(x) \leqq 2 \exp\,[-rx + (\tau + \epsilon)r^{1+(1/\alpha)}].$$

Let now $a$ be any positive number. Setting $r = ax^{\alpha}(r \geqq R)$ and defining $X$ by $R = aX^{\alpha}$, we have, for $x \geqq X$,

$$-\ln T(x) \geqq -\ln 2 + x^{1+\alpha}[a - (\tau + \epsilon)a^{1+(1/\alpha)}],$$

whence

$$\liminf_{x\to+\infty} h(x) \geqq a - (\tau + \epsilon)a^{1+(1/\alpha)}.$$

This being true for any $a > 0$ holds, in particular, for that value of $a$ which maximizes the right hand member of the above inequality, that is, for

$$a = \{\alpha/[(\tau + \epsilon)(1 + \alpha)]\}^{\alpha}.$$

Hence, with $c$ defined by (5.1),

(7.2) $$\liminf_{x\to+\infty} h(x) \geqq c/(\tau + \epsilon)^{\alpha}.$$

Since $\epsilon > 0$ is arbitrary, it follows that

(7.3) $$\liminf_{x\to+\infty} h(x) \geqq c/\tau^{\alpha}.$$

We assert that the inequality sign cannot hold in (7.3). Suppose indeed that it does. Then we can find $k > c/\tau^\alpha$ and $X = X(k)$ such that for $x \geqq X$, $h(x) \geqq k$. Hence, by Lemma 5.2, we see that, to any $\epsilon > 0$, there corresponds $R = R(\epsilon)$ such that, for $r \geqq R$,

$$M(r, f) \leqq \exp [(\tau' + \epsilon)r^{1+(1/\alpha)}],$$

where

$$\tau' = (c/k)^{(1/\alpha)} < \tau.$$

Since we have assumed that the order of $f(z)$ is $1 + (1/\alpha)$, the above inequality can only mean that the type of $f(z)$ is $\leqq \tau' < \tau$. This contradiction establishes the necessity of condition (ii).

Turning to the sufficiency part, we note that (i) ensures not only that $F(x)$ cannot have an entire c.f. of order one and of exponential type, but also that $h(x)$ is defined for all $x > 0$. By (ii), we then have that corresponding to any $k < c/\tau^\alpha$, we can find $X = X(k)$ such that for all $x \geqq X$, $h(x) \geqq k$; whence, by Lemma 5.2, we see as above that $f(z)$ is an entire function with the following property: for any $\epsilon > 0$, there exists $R = R(\epsilon)$ such that, for all $r \geqq R$,

$$M(r, f) \leqq \exp [(\tau' + \epsilon)r^{1+(1/\alpha)}],$$

where

$$\tau' = (c/k)^{(1/\alpha)}.$$

Now it is easy to verify that (ii) implies that $\liminf_{x\to+\infty} g(x)$ is equal to $1 + \alpha$, so that it follows from Theorem 6.1 that the order of $f(z)$ is precisely $1 + (1/\alpha)$. Hence, from the above, we see that the type of $f(z)$ is $\leqq\tau'$. Since $k < c/\tau^\alpha$ is arbitrary, it follows that the type of $f(z)$ is $\leqq\tau$. Again, we assert that the inequality sign cannot hold here. Suppose indeed that it does. Then we can find $\tau'' < \tau$ and $R = R(\tau'')$ such that, for all $r \geqq R$, $M(r, f) \leqq \exp (\tau'' r^{1+(1/\alpha)})$. Hence we derive in the same way as (7.2) from (7.1) that

$$\liminf_{x\to+\infty} h(x) \geqq c/(\tau'')^\alpha > c/\tau^\alpha.$$

This contradiction establishes the sufficiency of the given conditions.

THEOREM 7.2. *A* NASC *for $F(x)$ to have an entire* c.f. *of order* $1 + (1/\alpha)$ $(\alpha > 0)$ *and of minimal type* $(\tau = 0)$ *is that*

(i) $\quad T(x) > 0$ *for every* $x > 0$;

(ii) $\quad \liminf_{x\to+\infty} g(x) = 1 + \alpha$;

*and*

(iii) $\quad \lim_{x\to+\infty} h(x)$ (*exist and be*) $= +\infty$.

(*Note*. We say that $\lim_{x\to+\infty} f(x)$ exists if $\liminf_{x\to+\infty} f(x) = \limsup_{x\to+\infty} f(x)$ whether this limit be a real number or $+\infty$ or $-\infty$.)

PROOF. The necessity of (i) and (ii) follows from Theorems 2.2.5 and 6.1.

Since $\tau = 0$, relation (7.1) with $\tau = 0$ holds in this case, and we obtain a relation corresponding to (7.2) with $\tau = 0$, in the same way, giving $\liminf_{x\to+\infty} h(x) \geqq c/\epsilon^{\alpha}$ for any $\epsilon > 0$. Hence (iii) is necessary.

Turning to the sufficiency part, we notice that (i) ensures that: $g(x)$ is defined for all sufficiently large $x > 0$, and $h(x)$ for all $x > 0$; and that the c.f. of $F(x)$ is not an entire function of order one and of exponential type. (ii) then ensures that the c.f. is an entire function of order $1 + (1/\alpha)$, by Theorem 6.1. Finally, by (iii), if $k$ be any arbitrary positive number, however large, there exists $X = X(k)$ such that $h(x) \geqq k$ when $x \geqq X$, whence we derive, in view of Lemma 5.2, that, since $f(z)$ is of order $1 + (1/\alpha)$, the type of $f(z)$ should be less than or equal to $(c/k)^{(1/\alpha)}$. Since $k$ is arbitrary, we see that the type must be zero.

THEOREM 7.3 *A* NASC *for* $F(x)$ *to have an entire* c.f. *of order* $1 + (1/\alpha)$ *and of maximal type* ($\tau = +\infty$) *is that*

(i) $$T(x) > 0 \quad \textit{for every} \quad x > 0;$$

(ii) $$\liminf_{x\to+\infty} g(x) = 1 + \alpha;$$

*and*

(iii) $$\liminf_{x\to+\infty} h(x) = 0.$$

PROOF. The necessity of (i) and (ii) is obvious in the light of Theorems 2.2.5 and 6.1, as before. Since the type is maximal, $\liminf_{x\to+\infty} h(x)$ cannot be either finite and positive, or infinite, by virtue of Theorems 7.1 and 7.2. Hence it can only be zero, since $h(x) \geqq 0$.

As for the sufficiency of these conditions, we notice the obvious roles played by (i) and (ii), as in the proof of Theorem 7.2. Lastly, (iii) ensures that the type of the c.f. is not finite and positive, or zero, by virtue of Theorems 7.1 and 7.2. Hence it is maximal.

Thus, Theorems 7.1, 7.2 and 7.3, among themselves, exhaust all possible *types* for entire c.f.'s of given finite order greater than unity. It is to be noted that condition (ii) in the statements of Theorems 7.2 and 7.3 cannot be dropped; to this extent, these two theorems differ from Theorem 7.1, in which condition (ii) is sufficient to take care simultaneously of both the order and the type of the c.f.

**8. The type of an entire c.f. of order one.** We begin with a set of NASC's for a d.f. to have an entire c.f. of order one and of maximal type. The proof closely follows that of Theorem 6.1, with certain minor, but essential, modifications.

THEOREM 8.1. *A* NASC *for* $F(x)$ *to have an entire* c.f. *of order one and of maximal type is that*

(i) $$T(x) > 0 \quad \textit{for every} \quad x > 0;$$

*and*

(ii) $$\lim_{x\to+\infty} g(x) \ (\textit{exist and be}) = +\infty.$$

Proof. Let $F(x)$ have an entire c.f. of order one and of maximal type. Then, by Theorem 2.2.5, (i) holds, and therefore $g(x)$ is defined for all sufficiently large $x$. We turn to the necessity of condition (ii); by our assumption, given any $\epsilon > 0$, three exists $R = R(\epsilon)$ such that for all $r \geqq R$

$$\exp(r^{1+\epsilon}) \geqq M(r, f) \geqq \tfrac{1}{2}e^{rx}T(x)$$

for all $x > 0$, from relation (2.2.3), so that, for all such $r$ and $x$,

$$T(x) \leqq 2 \exp(-rx + r^{1+\epsilon}).$$

Choosing $x \geqq X = 2R^{\epsilon}$, and defining $r = (\frac{1}{2}x)^{(1/\epsilon)}$, we see that $T(x) \leqq 2 \exp[-(\frac{1}{2}x)^{1+(1/\epsilon)}]$ whence $\liminf_{x\to+\infty} g(x) \geqq 1 + (1/\epsilon)$. Since $\epsilon > 0$ is arbitrary, the necessity of (ii) is immediate.

To prove the sufficiency, we first note that (i) ensures that $F(x)$ cannot have an entire c.f. of order one and exponential type as well as that $g(x)$ is defined for all sufficiently large $x$. Turning to condition (ii), let

$$\lim_{x\to+\infty} g(x) \text{ (exist and be) } = +\infty.$$

Then, corresponding to any $\epsilon > 0$, we can find $X = X(\epsilon)$ such that for all $x \geqq X$,

$$g(x) \geqq 1 + (1/\epsilon).$$

This implies that for all such $x$

$$T(x) \leqq \exp(-x^{1+(1/\epsilon)}).$$

By Lemma 5.2, F$(x)$ has therefore an entire c.f. of order $\leqq 1 + \epsilon$. Since $\epsilon > 0$ is arbitrary, and since, by virtue of condition (i), $f(z)$ is not identically equal to one, it follows from Theorem 2.2.4 that the order of $f(z)$ is precisely equal to one. That the c.f. cannot be of exponential type and so can only be of maximal type then follows from Theorem 2.2.5 and condition (i) above.

We have seen that Theorem 2.2.5 gives a NASC for a d.f. $F(x) \not\equiv \epsilon(x)$ to have an entire c.f. of order one and of exponential type, namely, that $F(x)$ be "finite." Here, we establish a relation between the type of the c.f. and the extremities of the "finite" d.f. to which such a c.f. belongs.

Theorem 8.2. *Let* $F(x) \not\equiv \epsilon(x)$ *be a "finite"* d.f., *and let its left and right extremities be* $a$ *and* $b$ *respectively. Then the type of its* c.f. *is given by* Max $(b, -a)$.

Proof. By definition, the type of an entire function of order one is given by

$$\tau = \limsup_{r\to+\infty} [\ln M(r, f)/r] \tag{8.1}$$

Also, by Theorem 2.2.5, the left and right extremities of $F$ are given by

$$\begin{aligned} a &= \operatorname{lext} F = -\limsup_{r\to+\infty} [\ln f(ir)/r], \\ b &= \operatorname{rext} F = \limsup_{r\to+\infty} [\ln f(-ir)/r]. \end{aligned} \tag{8.2}$$

Hence, noting that by Theorem 2.2.2 $M(r, f) = \text{Max}\,[f(ir), f(-ir)]$, we see at once from (8.1) and (8.2) that

(8.3) $$\tau \geqq \operatorname{Max}(b, -a).$$

Again, we have, for $r > 0$, the elementary inequalities:

$$f(ir) = \int_a^b \exp(-rx)\, dF(x) \leqq \exp(-ar),$$

$$f(-ir) = \int_a^b \exp(\ \ rx)\, dF(x) \leqq \exp(br),$$

so that

(8.4) $$M(r, f) \leqq \operatorname{Max}[\exp(-ar), \exp(br)].$$

From (8.1) and (8.4), we have at once

(8.5) $$\tau \leqq \operatorname{Max}(b, -a);$$

(8.3) and (8.5) combine to give us the statement of the theorem.

COROLLARY. *There exists no entire* c.f. *of order one and of minimal type.*

PROOF. Suppose that there does exist such a c.f. Then, by Theorem 2.5.5, it should correspond to a "finite" d.f. If the extremities of this d.f. be $a$ and $b$, then we must have, by Theorem 8.2, that $a = b = 0$. But this means that $F(x) = \epsilon(x)$, that is, that $f(z)$ is identically equal to one, and so the c.f. is no longer of order one.

REMARK. It is of interest to note that both P. Lévy's theorem (Theorem 2.2.4) and the above corollary are immediate consequences of the following result from function-theory, which is itself derived from the Phragmén-Lindelöf extension of the maximum modulus principle. For details on the derivation of the result (quoted below), the reader is referred to Yu. V. Linnik [3], p. 32.

THEOREM. *If an entire function, which is not a constant, has order less than one, or order one and minimal type, then it cannot be bounded on any straight line (of the complex plane).*

To apply this result to our situation, we need only note that every c.f. is bounded on the real line ($|f(t)| \leqq 1$ for all real $t$).

**9. Analytic c.f.'s which are not entire, and entire c.f.'s of infinite order.** An almost immediate consequence of Theorems 6.1 and 8.1 is the following:

THEOREM 9.1. *In order that a* d.f. $F(x)$ *have either an analytic* c.f. *which is not entire, or an entire* c.f. *of infinite order, it is necessary (but not sufficient) that*

(i) $$T(x) > 0 \quad \textit{for every} \quad x > 0;$$

*and*

(ii) $$\liminf_{x\to+\infty} g(x) = 1.$$

PROOF. If $F(x)$ has an analytic c.f. (not necessarily entire), then, by Corollary 1 to Theorem 4.1, $\alpha > 0$, $\beta > 0$, where $\alpha$ and $\beta$ are as defined in the statement of Theorem 4.1. Let $r$ be any positive number strictly less than *both* $\alpha$ and $\beta$. Then, from the definitions of $\alpha$ and $\beta$, it follows that (cf. [4], p. 137)

$T(x) = O(e^{-rx})$ as $x \to +\infty$. Hence we have at once that $\liminf_{x\to+\infty} g(x) \geqq 1$. But, by virtue of Theorems 6.1 and 8.1, the inequality sign cannot hold here if $F(x)$ satisfies the conditions of the present theorem.

That the above conditions are not sufficient is seen by considering the example:

$$F'(x) = a \exp(-x/\ln x) \qquad \text{for } x \geqq e,$$
$$F'(x) = 0 \qquad \text{otherwise,}$$

where $a$ is a certain positive constant. $F(x)$ does not have an analytic c.f., though it satisfies conditions (i) and (ii) above. In proof of these assertions, we need only note that, for a suitable constant $c$, and for all $x \geqq e$, $c \exp(-\frac{1}{2}x/\ln x) \geqq T(x) \geqq a \exp(-x) > 0$ and that $M(t)$ is not defined for any $t > 0$. (Refer also to Corollary 1 to Theorem 4.1).

We can, however, obtain NASC's for $F(x)$ to have an analytic c.f. which is not entire, or an entire c.f. of infinite order, as immediate consequences of Corollary 1 to Theorem 4.1, and Theorem 9.1 and Corollary 2 to Theorem 4.1, respectively. The proofs are obvious and so we omit them.

THEOREM 9.2.1. *A* NASC *for* $F(x)$ *to have an analytic* c.f. *which is not entire is given by the following*:

(i)
$$\liminf_{x\to+\infty} [-\ln[1 - F(x)]/x] = \alpha > 0,$$
$$\liminf_{x\to+\infty} [-\ln F(-x)/x] = \beta > 0,$$

(*with the same understanding as in the statement of Theorem* 4.1); *and*

(ii) *at least one of* $\alpha$ *and* $\beta$ *is finite.*

*Then the* c.f. *is analytic in the strip* $-\alpha < \text{Im}(z) < \beta$.

THEOREM 9.2.2. *A* NASC *for* $F(x)$ *to have an entire* c.f. *of infinite order is given by conditions* (i) *and* (ii) *of Theorem* 9.1 *together with the following*:

(iii)
$$\lim_{x\to+\infty} [-\ln[1 - F(x)]/x] \text{ (exist and be) } = +\infty,$$
$$\lim_{x\to+\infty} [-\ln F(-x)/x] \text{ (exist and be) } = +\infty.$$

The following lemma (analogous to Lemma 5.2, but less informative) and theorems indicate how Theorems 6.1 and 8.1 can be extended to entire c.f.'s of infinite order, for various "rates of growth" of such functions. The proofs are similar to those in Sections 5, 6 and 8—no essentially new ideas being involved—and are given here for the sake of completeness.

LEMMA 9.1. *Suppose that a* d.f. $F(x)$ *is such that for all* $x \geqq x_0$,

$$T(x) \leqq L \exp[-\lambda x(\ln x)^{\delta}] \tag{9.1}$$

*where* $L, \lambda, \delta, x_0$ *are all positive constants. Then, the* c.f. *of* $F(x)$ *is an entire function* $f(z)$ *such that*

$$\limsup_{r\to+\infty} [\ln\ln\ln M(r,f)/\ln r] \leqq 1/\delta. \tag{9.2}$$

PROOF. Let $r_0 = \frac{1}{2}\lambda(\ln x_0)^{\delta}$. Consider any $r > r_0$, and define $X = X(r)$ by the relation

$$\lambda(\ln X)^{\delta} = 2r, \quad \text{or} \quad X = \exp[(2r/\lambda)^{(1/\delta)}]$$

so that $X \geqq x_0$. Then, proceeding as in the proof of Lemma 5.2, and using (9.1), we arrive at the relation

$$\int_X^{+\infty} e^{rx}\, dF(x) = e^{rX}[1 - F(X)] + r\int_X^{+\infty} e^{rx}[1 - F(x)]\, dx$$
$$\leqq e^{rX}[1 - F(X)] + L\, r\int_X^{+\infty} \exp[rx - \lambda x(\ln x)^{\delta}]\, dx.$$

But $\int_X^{+\infty} \exp[rx - \lambda x(\ln x)^{\delta}]\, dx = \int_X^{+\infty} \exp[-x\{\lambda(\ln x)^{\delta} - r\}]\, dx \leqq \int_X^{+\infty} \exp(-rx)\, dx \leqq 1/r$. Also, $\int_{-\infty}^{X} e^{rx}\, dF(x) \leqq e^{rX}F(X)$ so that

$$\int_{-\infty}^{+\infty} e^{rx}\, dF(x) \leqq e^{rX} + L \leqq \exp[r \exp\{(2r/\lambda)^{(1/\delta)}\}] + L.$$

Similarly it is found that

$$\int_{-\infty}^{+\infty} e^{-rx}\, dF(x) = \int_{-\infty}^{-X} e^{-rx}\, dF(x) + \int_{-X}^{+\infty} e^{-rx}\, dF(x)$$
$$\leqq \exp[r \exp\{(2r/\lambda)^{(1/\delta)}\}] + L.$$

Hence it follows that $F(x)$ has an entire c.f. $f(z)$ such that

$$M(r,f) = \text{Max}\,[f(ir), f(-ir)] \leqq \exp[r \exp\{(2r/\lambda)^{(1/\delta)}\}] + L.$$

Relation (9.2) is an immediate consequence of this inequality.

THEOREM 9.3. *A* NASC *for* $F(x)$ *to have an entire* c.f. $f(z)$ *such that*

$$\limsup_{r\to+\infty} [\ln\ln\ln M(r,f)/\ln r] = \alpha \qquad (\alpha > 0) \tag{9.3}$$

*is that*

(i) $T(x) > 0$ *for every* $x > 0$; *and*

(ii) $\liminf_{x\to+\infty} \phi(x) = 1/\alpha$,

*where*

$$\phi(x) = [\{\ln\ln[1/T(x)] - \ln x\}/\ln\ln x].$$

REMARK. The (standard) Poisson distribution gives us an example of such a d.f., with $\alpha = 1$.

PROOF. Let (9.3) be satisfied. The necessity of condition (i) follows from Theorem 2.2.5. Turning to the necessity of (ii), we have that, corresponding to any $\epsilon > 0$, there exists $R = R(\epsilon)$ such that for all $r \geqq R$,

$$M(r,f) \leqq \exp[\exp(r^{\alpha+\epsilon})] \tag{9.4}$$

whence it follows from (2.2.3) that for all such $r$ and $x \geqq 0$,

$$T(x) \leqq 2e^{-rx}M(r,f) \leqq 2\exp[\exp(r^{\alpha+\epsilon}) - rx].$$

Let $X = \exp(R^{\alpha+\epsilon})$. For any $x \geqq X$, take $r = (\ln x)^{1/(\alpha+\epsilon)}$ in the above inequality, so that $r \geqq R$. From the above, we then have for any $x \geqq X$, $T(x) \leqq 2 \exp [x - x(\ln x)^{1/(\alpha+\epsilon)}]$, whence it follows that

$$(9.5) \qquad \liminf_{x \to +\infty} \phi(x) \geqq 1/(\alpha + \epsilon).$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$(9.6) \qquad \liminf_{x \to +\infty} \phi(x) \geqq 1/\alpha.$$

We assert that the inequality sign cannot hold here. Suppose indeed that it does. Then we can find $\gamma < \alpha$ and $X = X(\gamma)$ such that for all $x \geqq X$,

$$(9.7) \qquad \phi(x) \geqq 1/\gamma$$

which implies that $T(x) \leqq \exp [-x(\ln x)^{(1/\gamma)}]$. By Lemma 9.1, it then follows that

$$(9.8) \qquad \limsup_{r \to +\infty} [\ln \ln \ln M(r, f)/\ln r] \leqq \gamma.$$

But, $\gamma < \alpha$, contradicting (9.3). Hence the necessity for (ii).

Turning to the sufficiency part, we see that condition (i) ensures that $F(x)$ does not have an entire c.f. of order one and of exponential type, and also that $\phi(x)$ is defined for all sufficiently large $x$. Then condition (ii) implies that, to any $\epsilon > 0$, there corresponds $X = X(\epsilon)$ such that for all $x \geqq X$, $T(x) \leqq \exp [-x(\ln x)^{(1/\alpha)+\epsilon}]$. Then, from Lemma 9.1, noting that $\epsilon > 0$ is arbitrary, it easily follows that

$$\limsup_{r \to +\infty} [\ln \ln \ln M(r, f)/\ln r] \leqq \alpha.$$

The sign of inequality cannot hold here; for, if it did, we can find $\gamma < \alpha$ and $R = R(\gamma)$ such that for all $r \geqq R$, $M(r, f) \leqq \exp [\exp (r^{\gamma})]$, whence it follows, in the same way as (9.5) from (9.4), that

$$\liminf_{x \to +\infty} \phi(x) \geqq 1/\gamma > 1/\alpha,$$

contradicting condition (ii). Hence (9.3) holds.

THEOREM 9.4. *A* NASC *for $F(x)$ to have an entire* c.f. *$f(z)$ such that*

$$\lim_{r \to +\infty} [\ln \ln \ln M(r, f)/\ln r] \ (\textit{exists and is}) = 0$$

*is that*

$$\text{(i)} \qquad T(x) > 0 \textit{ for every } x > 0;$$

*and*

$$\text{(ii)} \qquad \lim_{x \to +\infty} \phi(x) \ (\textit{exist and be}) = +\infty,$$

*where $\phi(x)$ is as defined in the statement of Theorem* 9.3.

PROOF. The necessity part follows from (9.5), on noting that the argument leading up to that relation is valid also if $\alpha = 0$, and from the fact that $\epsilon > 0$ is arbitrary. The sufficiency part also follows from the proof of Theorem 9.3, on noting that if (9.7) holds for arbitrary $\gamma > 0$, however small, then so does (9.8).

**10. Acknowledgments.** The author would like to take this opportunity of thanking Professor Eugene Lukacs for his interest in the author's work; and The Catholic University of America and the National Science Foundation for support during the writing of this paper.

## REFERENCES

[1] CRAMÉR, H. (1951). *Mathematical Methods of Statistics*. Princeton Univ. Press.

[2] DUGUÉ, D. (1939). Sur quelques propriétés analytiques des fonctions caractéristiques. *C. R. Acad. Sci. Paris* **208** 1778–1780.

[3] LINNIK, YU. V. (1960). *Factorization of Probability Laws* (in Russian). Leningrad Univ.

[4] LUKACS, E. (1960). *Characteristic Functions.* Charles Griffin, London.

[5] PÓLYA, G. (1949). Remarks on characteristic functions. *Proc. Berkeley Symp. Math. Statist. Prob.* (Ed. J. Neyman). Univ. of California Press, Berkeley. 115–123.

[6] SAKS, S. and ZYGMUND, A. (1952). *Analytic Functions.* **28** Monograffe Matematyczne, Warsaw.